Strengths of singularities in spherical symmetry.

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Abstract

Covariant equations characterizing the strength of a singularity in spherical symmetry are derived and several models are investigated. The difference between central and non-central singularities is emphasised. A slight modification to the definition of singularity strength is suggested. The gravitational weakness of shell crossing singularities in collapsing spherical dust is proven for timelike geodesics, closing a gap in the proof.

PACS: 04.20.Dw, 04.20.Jb, 04.70.-s

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I. INTRODUCTION

Over thirty years have passed since the appearance of the first of a series of theorems establishing that under very general circumstances, space-times must develop singularities. This first result, due to Penrose [1], appeared in 1965 and the body of work which grew up around the singularity theorems is contained in the book of Hawking and Ellis [2] which was first published in 1973. However, our understanding of the nature of these singularities remains far from complete. This lack is best exemplified by the absence of a proof, or definitive refutation, of the cosmic censorship conjecture (CCC) [3,4].

An important aspect of a singularity is its gravitational strength [5]. A singularity is termed gravitationally strong, or simply strong, if it destroys by crushing or stretching any object which falls into it. The most familiar example is the singularity at $r = 0$ in the Schwarzschild solution. (Throughout this paper, we will refer to a singularity at $r = 0$ as a central singularity, and to others as non-central.) A radially infalling object is infinitely stretched in the radial direction and crushed in the tangential directions, with the net result of crushing to zero volume. A singularity is termed weak if no object which falls into the singularity is destroyed in this way. The mathematical description of these ideas runs as follows [5,4].

Let $\gamma : [\tau_0, 0) \to M$ be a causal geodesic which approaches a singularity as $\tau \to 0^-$. Define $J_\nu(\gamma)$ for $\nu \in [\tau_0, 0)$ to be the set of maps $Z : [\nu, 0) \to TM$ satisfying the geodesic deviation equation along $\gamma$ such that $Z(\tau) \in T_{\gamma(\tau)}M$, $g_{ab}k^a(\tau)Z^b(\tau) = 0$ where $k^a$ is the tangent to $\gamma$ and

$$Z(\nu) = 0,$$

i.e. $J_\nu(\gamma)$ is the set of Jacobi fields along $\gamma$ which vanish at $\gamma(\nu)$. Along a timelike geodesic, three independent Jacobi fields define, via the exterior product, a volume element $V(\tau)$ along $\gamma$. Along a null geodesic, two such fields define an area element which we also denote $V(\tau)$. A timelike (null) geodesic terminates in a strong curvature singularity if for all $\nu \in [\tau_0, 0)$ and all independent triads (dyads) in $J_\nu(\gamma)$ we have

$$\lim_{\tau \to 0^-} V(\tau) = 0.$$

Then the singularity itself is said to be strong if every causal geodesic which approaches it terminates in a strong singularity. The geodesic terminates in a weak singularity if the limit above is finite and non-zero, and the singularity is weak if every causal geodesic approaching it is weak. We will argue below for a slight modification of this definition whereby the term strong is also attached to a singularity if the norms of the Jacobi fields themselves have zero or infinite limit.

The importance of the notion of the gravitational strength of a singularity for the CCC is that a statement of such possibly need not rule out the occurrence of naked weak singularities. This is based on the belief that one may extend the geometry of space-time through a weak singularity without traumatic effects [6,7]. A general description of this extension does not exist - indeed as far as the author can determine, only two examples of this procedure exist in the literature, one due to Papapetrou and Hamoui [8] and the other due to Clarke and O’Donnell [9]. Both deal with extending through a shell crossing singularity in collapsing spherical dust. However the fact that at a weak singularity one has, along any timelike
geodesic, a finite non-degenerate triad of Jacobi fields, from which it may be possible to construct a metric in a canonical way, lends support to the idea.

Our aim here is to give a comprehensive analysis of the strengths of singularities in spherical symmetry. This has been the arena of some of the most interesting developments in general relativity in recent years, and an understanding of what can and cannot occur in spherical symmetry may be a valuable guide for more general situations. Specifically, we study the Jacobi equations for arbitrary radial causal geodesics. This allows us to give covariant equations identifying the geometrical terms which control the strength of the singularity. As one would expect, the results are simpler than those obtained by Clarke and Krolak [10] which apply to general space-times. We study three different models which help illustrate the different situations which occur, and by way of application, demonstrate that (i) a non-central singularity is always weak along null directions and (ii) the shell-crossing singularities in collapsing spherical dust are weak. (This has only been demonstrated previously for radial null directions [11]; we complete the proof by showing that it is also true for radial timelike directions.)

II. RADIAL CAUSAL GEODESICS, JACOBI FIELDS AND THE VOLUME ELEMENT.

The line element of a spherically symmetric space-time may be written as

\[ ds^2 = -2e^{-2f}du dv + r^2d\omega^2, \quad (2.1) \]

where \( f = f(u, v) \), \( r = r(u, v) \) and \( d\omega^2 \) is the line element of the unit 2-sphere. The function \( r \) is an invariant of the space-time which we will, quite properly, refer to as the radius. The coordinates \( u \) and \( v \) are both null, labelling the null hypersurfaces generated by the two families of null geodesics orthogonal to the orbits of the \( SO(3) \) symmetry group of the space-time. The form (2.1) is invariant under the transformations \( u \to u'(u), v \to v'(v) \).

The non-vanishing Ricci tensor components are

\[ R_{uu} = -2r^{-1}(r_{uu} + 2ruf_u) \quad (2.2a) \]
\[ R_{vv} = -2r^{-1}(r_{vv} + 2rf_v) \quad (2.2b) \]
\[ R_{uv} = -2r^{-1}(r_{uv} - rf_{uv}) \quad (2.2c) \]
\[ R_{\theta\theta} = \csc^2 \theta R_{\phi\phi} = 1 + 2e^{2f}(ruv + rr_{uv}). \quad (2.2d) \]

We use the convention that subscripts attached to lower case letters refer to partial derivatives, but elsewhere refer to tensor components in the associated coordinate basis. The Misner-Sharp energy [12] is

\[ E = \frac{r}{2}(1 + 2e^{2f}ruv), \quad (2.3) \]

and the Weyl tensor is completely determined by the Newman-Penrose term \( \Psi_2 \), calculated on a null tetrad based on the principal null directions of the space time. Thus \( \Psi_2 \) is an invariant of the space-time and is given by
\[ \Psi_2 = \frac{e^{2f}}{3r} (r_{uv} + rf_{uv}) - \frac{E}{3r^3}. \]  

(2.4)

We note the further invariant,

\[ e^{2f} f_{uu} = E_r - \frac{R}{12}, \]  

(2.5)

where \( R \) is the Ricci scalar.

The radial geodesic equations are

\[ -2e^{-2f} \dot{u} \dot{v} = \epsilon \]  

(2.6a)

\[ \ddot{u} - 2f_u \dot{u}^2 = 0 \]  

(2.6b)

\[ \ddot{v} - 2f_v \dot{v}^2 = 0 \]  

(2.6c)

where \( \epsilon = 0 \) for null geodesics and \(-1\) for timelike; the overdot is respectively, differentiation with respect to affine parameter and proper time. Space-like geodesics will not concern us here.

We now look at the Jacobi fields along arbitrary radial causal geodesics, beginning with time-like geodesics.

The unit tangent to an arbitrary time-like curve \( \gamma(\tau) \) in the radial 2-space can be written in the form

\[ \vec{k} = h \frac{\partial}{\partial u} + \frac{1}{2} e^{2f} h^{-1} \frac{\partial}{\partial v}, \]

where \( h = h(u, v) \). The condition that \( \gamma \) be geodesic is then

\[ h_v + 2h^2 e^{-2f} (h_u - 2f_u h) = 0. \]  

(2.7)

This follows from the geodesic equations (2.6). The variation of any scalar quantity \( s \) along this geodesic is given by

\[ \dot{s} = k^a \nabla_a s = hs_u + \frac{1}{2} e^{2f} h^{-1} s_v, \]  

(2.8)

and using (2.7),

\[ \ddot{s} = h^2 (s_{uu} + 2f_u s_u) + \frac{1}{4} e^{4f} h^{-2} (s_{vv} + 2f_v s_v) + e^{2f} s_{uv}. \]  

(2.9)

A Jacobi field \( Z^a \) along \( \gamma \) satisfies the geodesic deviation equation

\[ \ddot{Z}^a + R_{cbd}^a Z^b k^c k^d = 0, \]  

(2.10)

which is a linear equation for \( Z^a \) and so a basis for the Jacobi fields may be found by obtaining all independent Jacobi fields in the radial 2-space and in the tangential 2-space.

We take

\[ \vec{\xi}_{(1)} = x(u, v) \frac{\partial}{\partial \theta} \quad \vec{\xi}_{(2)} = y(u, v) \csc \theta \frac{\partial}{\partial \phi} \]
as candidates for the Jacobi fields in the tangential 2-space. Note that the norms of $\xi^a_{(1)}$ and $\xi^a_{(2)}$ are $|x|$ and $|y|$ respectively. The geodesic deviation equation (2.10) applied to $\xi^a_{(1)}$ yields the following equation for $x$ (the same result applies to $y$):

$$r(4h^4(x_{uu} + 2f_u x_u) + e^{4f}(x_{vv} + 2f_v x_v) + 4h^2 e^{2f} x_{uv}) + 2(2h^2 x_u + e^{2f} x_v)(2h^2 r_u + e^{2f} r_v) = 0.$$ 

Using (2.8) and (2.9), this assumes the remarkably simple form

$$r\ddot{x} + 2\dot{x} \dot{r} = 0,$$

which can be integrated to obtain

$$x(\tau) = x_0 \int_{\tau_1}^{\tau} \frac{d\tau'}{r(\tau')},$$

(2.12a)

where $x_0$ is constant and we have included the initial condition $x(\tau_1) = 0$, so that $\xi^a_{(1)}(\tau_1) = 0$. The second linearly independent solution is $x(\tau) \equiv 0$. We obtain the same result for $\xi^a_{(2)}$:

$$y(\tau) = y_0 \int_{\tau_1}^{\tau} \frac{d\tau'}{r(\tau')}.$$ 

(2.12b)

Before dealing with the radial Jacobi fields along the time-like geodesics, we describe the situation for radial null geodesics. It turns out to be remarkably simple. From (2.6a), either $\dot{u}$ or $\dot{v}$ vanishes along a radial null geodesic. Take it to be the latter. We can then integrate (2.6b) to obtain $\dot{u} = ce^{2f}$ and so the tangent is

$$\vec{k} = ce^{2f} \frac{\partial}{\partial u}. $$

The variation of a scalar $s$ along the geodesic is $\dot{s} = ce^{2f} s_u$ and

$$\ddot{s} = c^2 e^{4f} (s_{uu} + 2f_u s_u).$$

In the null case, there are only tangential Jacobi fields. Candidates for such are given by $\vec{\xi}_{(1,2)}$ as above, and it turns out that the norms $x, y$ obey the same equation as in the time-like case, and thus the solutions are given by (2.12a).

We now treat the radial Jacobi fields along a time-like geodesic. A space-like vector in the radial 2-space orthogonal to $k^a$ has the form

$$\vec{\xi} = ah \frac{\partial}{\partial u} - \frac{1}{2} e^{2f} ah^{-1} \frac{\partial}{\partial v},$$

where $a = a(u, v)$. This has norm $|a|$. Using (2.9), we find that the condition for $\vec{\xi}$ to satisfy the geodesic deviation equation along $k^a$ is

$$\ddot{a} + 2e^{2f} f_{uu} a = 0.$$ 

(2.13)
Recall that according to (2.5), $e^{2f} f_{uv}$ is an invariant of the space-time. Thus equations (2.12) and (2.13) provide a covariant set of equations which will determined the strength of the singularity. To see how this comes about, we obtain the relationship between $V(\tau)$ and the quantities $a, x, y$.

Since each of $a, x$ and $y$ satisfy second order linear ordinary differential equations, there are six independent Jacobi fields along a time-like geodesic $\gamma$. An arbitrary triad of corresponding 1-forms is given by

$$z_\alpha = a_\alpha e + x_\alpha r^2 d\theta + y_\alpha \sin \theta d\phi,$$

where $e = (2h)^{-1} du - e^{-2f} h dv$ and $\alpha = 1, 2, 3$. Then the general volume element along $\gamma$ has the form

$$V(\tau) = z_1 \wedge z_2 \wedge z_3$$
$$= 6a_{[1} x_{2} y_{3]} r^4 \sin \theta e \wedge d\theta \wedge d\phi.$$

The norm $\|W\|$ of a $p$–form $W = W_{[i_1...i_p]} dx^{i_1} \wedge ... \wedge dx^{i_p}$ is given by

$$\|W\|^2 = W_{[i_1...i_p]} W^{i_1...i_p},$$

where the vertical bars indicate summation only over $i_1 < i_2 < ... < i_p$. This gives

$$\|V(\tau)\| = 6|a_{[1} x_{2} y_{3]}| r^2. \tag{2.14}$$

The existence of six independent solutions of the geodesic deviation equation indicates that in a general space-time, the different $V(\tau)$ form a six parameter family. This surfeit of possibilities would produce problems if one wanted to make statements about singularity strengths based on the behaviour of all such $V(\tau)$. However the definition of $J_{\tau_1}$ reduces the number significantly. Note that $z_\alpha$ vanishes if and only if each of $a_\alpha, x_\alpha$ and $y_\alpha$ vanish. The general solution for $a$ has the form

$$a = c_+ a_+ + c_- a_-$$

where $c_\pm$ are arbitrary constants and $a_\pm$ are any two independent solutions of (2.13). A similar result holds for $x$ and $y$. The initial condition $a(\tau_1) = 0$ fixes the ratio $c_+/c_-$, so that for the problem in hand, there is only one choice, up to a constant multiple, for each of $a, x$ and $y$. Therefore the norm of every relevant volume element has the simple form

$$\|V(\tau)\| = |a x y| r^2. \tag{2.15}$$

where $a, x$ and $y$ here represent the general solutions of the appropriate equations (2.12) and (2.13), into which constants may be absorbed.

We thus have a simple and direct way of assessing the strength of a singularity. We determine the limiting behaviour of solutions of (2.12) and (2.13) as the singularity is approached and then use (2.15) to calculate $\lim_{r \to 0^-} \|V(\tau)\|$.
III. POSSIBLE SOLUTIONS OF THE MAIN EQUATIONS

We now consider the various possible limiting behaviours which can occur for solutions of the main equations, i.e. (2.12) and (2.13).

From (2.15), the forms $rx$ and $ry$ arise naturally and occur in the following proposition which applies to both $x$ and $y$.

**Proposition One** (i) For a non-central singularity, $\lim_{\tau \to 0^-} rx$ is finite and non-zero.

(ii) For a central singularity, if

$$\lim_{\tau \to 0^-} \int_{\tau_1}^{\tau} r^{-2}(\tau') d\tau' < \infty,$$

then $\lim_{\tau \to 0^-} rx = 0$. Otherwise

$$\lim_{\tau \to 0^-} rx = -\lim_{\tau \to 0^-} \frac{1}{r'(\tau)}.$$

Here and below, the limit refers to the limit as $\tau \to 0^-$ along a geodesic which approaches the singularity at $\tau = 0$. The proofs of part (i) and the first part of part (ii) follow immediately from (2.12a), and that of the second part from (2.12a) and l’Hôpital’s rule. A consequence of this is that the strength of a non-central singularity is completely determined by the limiting behaviour of $a$ at the singularity.

Next we summarise the possible behaviour of $a$ in the appropriate limit. Define $F(\tau) = 2e^{2f}f_{uv}(\tau)$ and $G(\tau) = \tau^2 F(\tau)$. Then (2.13) is equivalent to

$$\tau^2 a + G(\tau) a = 0,$$

(3.1)

and we wish to determine the behaviour of $a$ at $\tau = 0$ which is a singular point of (3.1). We quote the following results from Bender and Orszag [13], which should also be found in any text on linear differential equations. All the asymptotic relations below hold as $\tau \to 0^-$ and $c, c_\pm$ are arbitrary constants.

The equation has a regular singular point at $\tau = 0$ if $G(\tau) = O(1)$ as $\tau \to 0^-$. Define $G_0 = \lim_{\tau \to 0^-} G(\tau)$. The method of Frobenius applies. The roots of the indicial equation are

$$v_{1,2} = \frac{1}{2} \pm \frac{1}{2} (1 - 4G_0)^{1/2}.$$

The following possibilities arise.

**RSP1** $v_1 - v_2 \not\in Z$. Then $a(\tau) \sim c_+ |\tau|^{v_1} + c_- |\tau|^{v_2}$. Three subcases arise depending on the value of $G_0$.

**RSP1a** $1/4 < G_0$. Then $a(\tau) \sim c |\tau|^{1/2}$, so that $\lim_{\tau \to 0^-} a(\tau) = 0$.

**RSP1b** $0 < G_0 < 1/4$. Then $v_{1,2}$ are both positive so that $\lim_{\tau \to 0^-} a(\tau) = 0$.

**RSP1c** $G_0 < 0$. Then $v_2 < 0$ so that $\lim_{\tau \to 0^-} a(\tau) = \infty$.

**RSP2** $v_1 = v_2 \Leftrightarrow G_0 = 1/4$ Then $a(\tau) \sim c_+ |\tau|^{1/2} + c_- |\tau|^{1/2} \ln |\tau|$. Again, $\lim_{\tau \to 0^-} a(\tau) = 0$.

**RSP3** $v_1 - v_2 \in N - 0 \Leftrightarrow G_0 = (1 - k^2)/4, k \in \mathbb{N}$. Then $a(\tau) \sim c_+ |\tau|^{v_1} + c_- (|\tau|^{v_2} + d|\tau|^{v_1} \ln |\tau|)$ where $d$ is a fixed constant. We mention under this last heading one special case of particular importance, that for which $G_0 = 0$. This includes singularities whereat $F(\tau)$
is finite and, typically, space-times with weak non-central singularities.

(RSP3a) $G_0 = 0$. Then $a(\tau) \sim c_+ \tau + c_- |\tau| + c_{6} |\ln(\tau)|$, and so $\lim_{\tau \to -0} a(\tau)$ is finite and non-zero.

The second class of possibilities arises when (3.1) has an irregular singular point at $\tau = 0$, which occurs when $\lim_{\tau \to -0} G(\tau) = \infty$. The WKB approximation applies.

(ISP) $a(\tau) \sim e(F(\tau))^{-1/4} \exp\{\pm \int_{\tau}^{0} (F(\tau'))^{1/2} d\tau'\}$. There are two possibilities here.

(ISP1) $\lim_{\tau \to -0} F(\tau) = +\infty$. Then $\lim_{\tau \to -0} a(\tau) = +\infty$.

(ISP2) $\lim_{\tau \to -0} F(\tau) = -\infty$. Then $\lim_{\tau \to -0} a(\tau) = 0$.

Keep in mind that the behaviour described here is characteristic of a particular radial timelike geodesic which runs into the singularity, and not of the singularity itself. We will therefore refer to, for example, a type (RSP1a) geodesic, and to a type (RSP1a) singularity only if all the radial timelike geodesics terminating there are type (RSP1a).

In this language, the central singularity of Schwarzschild space-time is type (RSP1c), with $a(\tau) \sim c_+ |\tau|^{4/3} + c_- |\tau|^{-1/3}$. Also, $rx(\tau) \sim x_0 |\tau|^{1/3}$, $ry(\tau) \sim y_0 |\tau|^{1/3}$, so that overall, $||V(\tau)|| \sim d|\tau|^{1/3}$, giving a singularity which is strong along timelike approaches. Suppose instead the behaviour was $rx(\tau) \sim x_0 |\tau|^{1/6}$, $ry(\tau) \sim y_0 |\tau|^{1/6}$. Then $||V(\tau)|| \sim d$ (constant), so by the current definition, the singularity is weak along timelike approaches. It would be of very little comfort to an observer jumping into such a singularity to realise, as he watched his legs elongate and disintegrate, that such volume forms were preserved on his journey. This motivates the following addendum to the definition of a strong singularity.

We will say that a causal geodesic $\gamma : [\tau_0, 0) \to M$ approaching a singularity as $\tau \to 0^-$ terminates in a strong singularity if for all $\tau_1 \in [\tau_0, 0)$, except some suitably small (finite, countable, zero-measure) set, the general element of $J_{\tau_1}(\gamma)$ is degenerate or infinite in the limit $\tau \to 0^-$. We will say that $\gamma$ terminates in a weak singularity if the general element of $J_{\tau_1}(\gamma)$ is finite and non-degenerate in the limit. The terms will be applied to the singularity itself if all causal geodesics approaching the singularity behave in one of the two ways.

By degenerate, we mean that both of the independent Jacobi fields in some particular direction (or mutually orthogonal directions), orthogonal to $k^a$, the tangent to the geodesic, shrink to zero magnitude. A non-central type (RSP1b) singularity would be an example of such.

We now gather the results above into some general statements.

**Proposition Two** For a non-centralsingularity and for a central singularity for which $i^\ast$ has a finite and non-zero limit along every causal geodesic approaching the singularity, the strength of the singularity is determined by (2.13). If the singularity is of type (RSP3a), then it is weak. Singularities of the other types are strong. A central singularity for which $i^\ast$ has zero or infinite limit along every causal geodesic approaching the singularity is strong.

The proof follows from Proposition One and from the definitions above; essentially it amounts to some useful book-keeping. A great many singularities will have different behaviours along different geodesics approaching the singularity, and so will not be covered by this result. There remains the problem of determining the behaviour of $a(\tau)$ and $i(\tau)$ in the limit as the singularity is approached. However we have identified which elements of the geometry determine the strength of a singularity and listed the various possibilities.

We now give some applications of the theory laid out above.
IV. EXAMPLES

In this section, we study the strengths of some singularities in four different (classes) of space-times. The first three, two toy models and Roberts’ space-time [14], are used to illustrate the types of singularities which may arise and some of the points made above. The third is the marginally bound case of a Lemaître-Tolman-Bondi (LTB) collapsing dust sphere [15]. We use the theory above to demonstrate conclusively the weakness of shell-crossing singularities in this space-time.

A. A toy model

We consider the space-time with line-element

$$ds^2 = -du dv + \left(\frac{v-u}{2}\right)^{2\alpha} d\omega^2,$$

i.e. $2e^{-2f} = 1$ and $r = ((v-u)/2)^{\alpha}$. We take $\alpha \geq 1$. The case $\alpha = 1$ is flat space-time. The Ricci scalar is

$$R = \frac{2}{r^2} (1 - (3\alpha^2 - 2\alpha)r^{2-2/\alpha}),$$

and so there is a scalar curvature singularity at $r = 0$. Since $f$ is constant, the radial Jacobi field orthogonal to an arbitrary timelike geodesic will satisfy, according to (2.13), $\ddot{a} = 0$, with general solution $a = c_+ + c_- \tau$. Thus the strength of this central singularity will be determined by the behaviour of the tangential Jacobi fields.

Along a radial null geodesic, we have (without loss of generality) $v =$constant and $u = c\tau + d$, where $c, d$ are constants and $\tau$ is an affine parameter. Thus after a reparametrisation of $\tau$, we have

$$r = c|\tau|^{\alpha}.$$

The same result holds for all radial timelike geodesics. From (2.12) and (2.15), we find

$$\|V(\tau)\| \propto (c_+ + c_-|\tau|)|\tau|^{2-2\alpha}.$$

Thus the singularity is strong. Notice that $\|V(\tau)\| \to \infty$ as the singularity is approached along any radial causal geodesic. The deformation results from infinite tangential stretching.

The purpose of examining this model is to give an explicit example where the behaviour at the singularity is clearly pathological and destructive, but which would not previously have been described as a strong singularity. We note that

$$T_{ab} \left(\frac{\partial}{\partial u}\right)^a \left(\frac{\partial}{\partial u}\right)^b = -\frac{\alpha}{8\pi}(\alpha - 1)r^{-2/\alpha},$$

so that the weak energy condition is violated for the values of $\alpha$ of interest here.
B. Roberts’ solution

Roberts’ solution has been used in studies of cosmic censorship [14] and critical collapse [16]. The line-element is

$$ds^2 = -du dv + \frac{1}{4}(u^2 - 2uv + (1 - p^2)v^2)d\omega^2,$$

where $p$ is constant. $p = 0$ gives flat space-time. The Ricci scalar is

$$R = \frac{1}{2r^4}p^2 uv,$$

and so there is a central scalar curvature singularity. As above, the strength of the singularity is determined be the tangential Jacobi fields. In this case we find that $r = c|\tau|$ along any radial causal geodesic terminating at $r = 0$ at parameter value $\tau = 0$. We use (2.12) and (2.15) to obtain

$$\|V(\tau)\| \sim k(c_+ + c_-|\tau|)$$

with $rx, ry \sim$constant as $\tau \to 0^-$, and so this central singularity is weak. Thus the examples where this singularity is naked may not be genuine counterexamples to cosmic censorship. See also [3] for related comments.

C. Another toy model

A model with slightly more complicated dynamics and which illustrates well some of the points made above is that with the line element

$$ds^2 = -\left(\frac{v - u}{2}\right)^{2\alpha} du dv + \left(\frac{v - u}{2}\right)^2 d\omega^2.$$

We take $\alpha \geq 0$; $\alpha = 0$ is flat space-time. We find that

$$2e^{2f} f_{uv} = -4\alpha(2\alpha + 1)r^{-4\alpha - 2},$$

and so there is a scalar curvature singularity at $r = 0$ (recall that this term is an invariant). For an arbitrary radial null geodesic, we make take $u = u_0$=constant. Then we find

$$v - u = v - u_0 = (c\tau + d)^{1/(2\alpha + 1)},$$

so that

$$r = k|\tau|^{1/(2\alpha + 1)}$$

after an appropriate shift in the affine parameter $\tau$. Applying the second part of Proposition One, we see that all radial null geodesics approaching $r = 0$ terminate in a strong curvature singularity with the area element obeying $\|V(\tau)\| \to 0$ in every case.
To solve for the radial timelike geodesics, we make the change of variables 
\[ r = \frac{(v - u)}{2}, \quad t = \frac{(v + u)}{2}. \]
Then the line element takes the form
\[ ds^2 = r^{2\alpha}(-dt^2 + dr^2) + r^2d\omega^2. \]
The geodesic equations for radial infall yield
\[ \dot{r} = -r^{-2\alpha}(c^2 - r^{2\alpha})^{1/2}. \]
According to (2.13) and (3.1), we need to determine the behaviour of \( r \) as proper time \( \tau \to 0 \).
(As usual, we fix the origin of proper time so that \( r(0) = 0 \).) The previous equation may be solved asymptotically by expanding the right hand side and then inverting the resulting integral with the result
\[ r = c|\tau|^{1/(2\alpha+1)} + O(|\tau|^\beta) \quad (4.2) \]
where \( \beta > 1/(2\alpha + 1) \).

Then the tangential Jacobi fields have the asymptotic behaviour
\[
\begin{align*}
rx &\sim c|\tau|^{2\alpha/(2\alpha+1)} \quad (\alpha \neq 1/2), \\
rx &\sim c|\tau|^{1/2} \ln |\tau| \quad (\alpha = 1/2).
\end{align*}
\]

The behaviour of the radial Jacobi fields is dictated by (4.1) which from the above has the behaviour
\[ F(\tau) \sim c_1|\tau|^{-2} \]
where \( c_1 \) is a negative constant. Then in the notation used above, \( G_0 \) is a negative constant, and so this is a type \((\text{RSP}1c)\) singularity. The asymptotic behaviour of \( a \) is
\[ a(\tau) \sim c_+|\tau|^{v_1} + c_-|\tau|^{v_2} \]
where \( v_{1,2} = (1 \pm (1 - 4G_0)^{1/2})/2 \), and so
\[ \|V(\tau)\| \sim V_0|\tau|^{v_2+4\alpha/(2\alpha+1)}. \]
Therefore, for any value of \( \alpha \), there will exist radial timelike geodesics along which \( \|V(\tau)\| \) diverges, has zero limit and has finite limit as the singularity is approached. These different possibilities arise from the different choices available for \( c \) in (4.2) which give the value \( G_0 = -4\alpha(2\alpha + 1)c^{-4\alpha-2} \). Starting from some fixed value \( r = r_* \) at \( \tau = \tau_* < 0 \), we see that \( c \) is essentially a measure of the initial velocity of an observer falling radially inwards from \( r_* \). By tuning this velocity, an observer could in principle ensure that his \( \|V(\tau)\| \) is finite in the approach to the singularity. However, in practice this would be of little help to him since as pointed out above, the observer experiences infinite tangential crushing and radial stretching in the infall. Furthermore, his initial velocity would have to be tuned with infinite precision to obtain \( 0 \neq \lim_{\tau \to 0^-} \|V(\tau)\| < \infty \). According to the definition above, this is a strong singularity.
D. Marginally bound spherical dust

The marginally bound LTB space-time (spherically symmetric inhomogeneous dust) has line element

\[ ds^2 = -dt^2 + (r')^2 d\eta^2 + r^2 d\omega^2, \]

where the prime indicates differentiation with respect to the coordinate \( \eta \). For the collapsing case,

\[ r^3(\eta, t) = \frac{9}{2} m(\eta)(t_0(\eta) - t)^2, \]

where \( m, t_0 \) are arbitrary functions of \( \eta \). See [15] for details. The energy density \( \rho \) of the dust, which is proportional to the Ricci scalar, is given by

\[ 4\pi \rho = \frac{m'}{r^2 r'}. \]

Thus as well as the central singularity at \( r = 0 \) (occurring when \( t = t_0(\eta) \), there are so-called shell-crossing singularities occurring when \( r' = 0 \) [17]. These generally occur before the central singularity, at non-zero radius and so are non-central. It has long been believed that these scalar curvature singularities are weak. However, it seems that this has only been properly established for null geodesics approaching the singularity [6,11]. As we have seen above, this weakness is completely independent of the structure and nature of the singularity apart from the fact that it is non-central. We fill this gap by proving that all radial timelike geodesics terminate in a weak singularity.

According to Proposition Two, the strength of a shell-crossing singularity is governed by (2.13). Using (2.5), we find that

\[ F = 2e^{2f} f_{uv} = \frac{m'}{r^2 r'} - 2\frac{m}{r^3}. \]

The terms \( m'/r^2 \) and \( m/r^3 \) will both be finite in general in the approach to the singularity, the former being positive, assuming positive energy density. Thus the behaviour is governed by \( F = 1/r' \). We will show that

\[ \lim_{r \to 0} \frac{r^2}{r'} = 0 \quad (4.3) \]

along any radial timelike geodesic approaching the singularity. This shows that the singularity is type (RSP3a), and is therefore weak by Proposition Two.

The radial timelike geodesic equations are

\[ -\dot{t}^2 + (r')^2 \dot{\eta}^2 = -1 \quad (4.4a) \]

\[ r' \ddot{\eta} + 2r' \dot{t} \dot{\eta} + r'' \dot{\eta}^2 = 0 \quad (4.4b) \]

\[ \ddot{t} + r' r' \eta^2 = 0 \quad (4.4c) \]

where the overdot indicates differentiation with respect to proper time along the geodesic and the subscript is differentiation with respect to the global time coordinate \( t \). Along
each geodesic approaching the singularity, we choose the origin of proper time so that the singularity is at \( \tau = 0 \).

We find that
\[
r' = \frac{r}{3} \left( \frac{m'}{m} + \frac{2t'_0}{t_0 - t} \right),
\]
so that at a shell-crossing singularity,
\[
m' \frac{m}{m} = -\frac{2t'_0}{t_0 - t}.
\]
The following terms will enter into our analysis;
\[
r'' = \frac{2}{3} \frac{rt'_0}{(t_0 - t)^2}
\]
\[
r'' = \frac{(r')^2}{r} + \frac{r}{3} \left( \frac{m''}{m} - \left( \frac{m'}{m} \right)^2 + \frac{2t''_0}{t_0 - t} - \frac{2t'_0}{(t_0 - t)^2} \right).
\]
Then
\[
r''(0) = \frac{r}{3m^2t'_0} \left( mm''t'_0 - \frac{3}{2} (m')^2 t'_0 - mm't''_0 \right),
\]
where here and subsequently, evaluation at zero means in the limit \( \tau \to 0^- \) along a geodesic.

Generically, \( r'_t(0) \) and \( r''(0) \) will be non-zero. If this were not the case, there would be extra conditions imposed on \( m \) and \( t_0 \) for all values of \( \eta \), which would result in a loss of generality. For example, if \( r'_t(0) = 0 \), then \( t'_0(\eta) \equiv 0 \) for all \( \eta \). In this case, the space-time is homogeneous and isotropic. The condition \( r''(0) = 0 \) imposes less severe but nonetheless significant restrictions. So we assume henceforth that \( r'_t(0) \) and \( r''(0) \) are non-zero.

We also need to track the evolution of \( r \) and \( r' \) along the geodesics. We have
\[
r = r_t \dot{t} + r' \dot{\eta} = \sqrt{\frac{2m}{r}} t + r' \dot{\eta} \tag{4.8}
\]
\[
(r') = \frac{2}{3} \frac{rt'_0}{(t_0 - t)^2} + r'' \dot{\eta} \tag{4.9}
\]

We now prove (4.3), which demonstrates the weakness of the singularity.

**Case One:** \( \lim_{\tau \to 0^-} |\dot{\eta}| < \infty \).

By (4.4a), \( \dot{t}(0) = 1 \). The sign comes from the assumption that the geodesic is future directed and the fact that \( t \) is a global time coordinate. The past directed case proceeds in an identical manner. By (4.9), \( (r') \) will be finite in the limit \( \tau \to 0^- \). If this limit is non-zero, we can apply l'Hopital’s rule to \( \tau^2/r' \) to prove (4.3). The other possibility is that \( (r')(0) = 0 \) So now assume this to be the case.

Suppose further that \( |\dot{\eta}(0)| < \infty \). Then \( (r''\dot{\eta})(0) = 0 \), and so taking the limit of (4.4b), we have
\[
0 = \lim_{\tau \to 0^-} (2r'_t \dot{t} + r'' \dot{\eta})
\]
\[
= \lim_{\tau \to 0^-} (r'_t \dot{t} + (r')),
\]
which gives $r'(0) = 0$, in contradiction of one of our assumptions. So if $(\dot{r}')(0) = 0$, then we must have $|\dot{\eta}(0)| = \infty$.

Using l’Hopital’s rule twice, we have in this case
\[
\lim_{\tau \to 0^-} \frac{\tau^2}{r' \dot{r}} = \lim_{\tau \to 0^-} \frac{2}{(r')^2}.
\]

We calculate
\[
(\ddot{r}) = r'' \dot{t}^2 + r' \ddot{t} + 2r'' \dot{t} \dot{\eta} + r''' \dot{\eta} + r'''' \dot{\eta}^2.
\]

From (4.4c), $\ddot{t}(0) = 0$ and the terms $r'' \dot{t}$, $r'''$ and $r''''$ will be finite in the appropriate limit. Thus the dominant term is $r'' \dot{t}$, giving $\lim_{\tau \to 0^-} |\dot{r}'| = \infty$, proving (4.3).

**Case Two:** $\lim_{\tau \to 0^-} \dot{\eta} = \infty$.

Suppose that $|r' \dot{\eta}|(0) < \infty$. Then by (4.4a), $\ddot{t}(0)$ is finite and so (4.9) gives $|\dot{r}'|(0) = \infty$. We then use l’Hopital’s rule to prove (4.3).

Finally, suppose that $|r' \dot{\eta}|(0) = \infty$. Then by (4.4a),
\[
\lim_{\tau \to 0^-} \dot{t} = \lim_{\tau \to 0^-} \left(\frac{1}{|r' \dot{\eta}|} \left(1 + \frac{1}{|r' \dot{\eta}|}\right)^{1/2}\right) = \lim_{\tau \to 0^-} |r' \dot{\eta}|.
\]

Then by (4.9),
\[
\lim_{\tau \to 0^-} |\dot{r}'|(0) = |r'' \pm r' \dot{r}'(0)| \lim_{\tau \to 0^-} |\dot{\eta}|.
\]

This can be finite only if $(r'' \pm r' \dot{r}')(0) = 0$. But this limit is generically equal to $r''(0)$, which is non-zero, and so we have $|\dot{r}'|(0) = \infty$. Again, l’Hopital’s rule is used to prove (4.3).

This completes the proof of (4.3) for all radial timelike geodesics and thus demonstrates the weakness of the singularity.

**V. CONCLUSIONS**

The central results here are contained in equations (2.12), (2.13) and (2.15). These provide a set of covariant equations, the asymptotic solutions to which (which require information about causal geodesics) determine the strengths of singularities in spherically symmetric space-times. The notion of ‘strength’ is in a slightly modified form to Tipler’s original definition [5]; the modification is clearly motivated and is illustrated by the examples in Section 3.

Proposition One demonstrates the important point that the behaviour of null geodesics tells us nothing about the strength of a non-central singularity. Also, a null geodesic approaching a central singularity terminates in a strong singularity unless $\dot{r}$ has a finite, non-zero limit at the singularity. Proposition Two lists the possible ways in which strong or weak singularities may occur.

In addition to studying the toy models, we were able to demonstrate conclusively the weakness of the naked singularity in Roberts’ space-time and the shell-crossing singularities.
in collapsing spherical dust. This latter proof shows that while detailed qualitative information about causal geodesics is required, we do not need the full solution of the geodesic equations. Therefore there is good hope that the results above may be successfully applied to other situations.

One of the most important among these is the null Cauchy horizon singularity which is widely believed to occur generically in gravitational collapse (see [7] and references therein). It is also generally believed that this singularity is weak, the belief being based on the fact that the metric is $C^0$ at the singularity. See for example [18] and [19]. However, as we have shown elsewhere [20], a $C^0$ singularity is not necessarily weak, and so the issue of the gravitational strength of the Cauchy horizon singularity remains open. (This is not the case for the singularities studied by Brady et al [21], where, essentially, the behaviour of the curvature along geodesics striking the singularity was studied. These examples have been demonstrated to be weak.) It should be possible to address this issue using the results here applied to the several models of Cauchy horizon singularities that have appeared over the last ten or so years [7].

Clarke and Krolak [10] have given necessary and sufficient conditions, in arbitrary spacetimes, for a singularity to be strong, the conditions involving integrals of certain curvature terms along geodesics. An advantage of our work is that it deals with the full set of Jacobi fields $J_\tau$ rather than the volume element $V(\tau)$. As the toy model of Section 3.3 shows, this can be important. Also, the decisive term here $2e^{2f}f_{uv}$ is slightly simpler than the decisive terms in [10]. It may be possible to use the results here to investigate the connection between Tipler’s definition of strengths of singularities and Krolak’s limiting focusing conditions [22].
REFERENCES

7 See several of the articles in Internal structure of black holes and spacetime singularities, edited by L. M. Burko and A. Ori (IOP Publishing, Bristol, 1997), especially chapters 1 and 24.
16 P. R. Brady, Class. Quant. Grav. 11, 1255 (1994).