Collective motion of organisms in three dimensions

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We study a model of flocking in order to describe the transitions during the collective motion of organisms in three dimensions (e.g., birds). In this model the particles representing the organisms are self-propelled, i.e., they move with the same absolute velocity $v_0$. In addition, the particles locally interact by choosing at each time step the average direction of motion of their neighbors and the effects of fluctuations are taken into account as well. We present the first results for large scale flocking in the presence of noise in three dimensions. We show that depending on the control parameters both disordered and long-range ordered phases can be observed. The corresponding phase diagram has a number of features which are qualitatively different from those typical for the analogous equilibrium models.

I. INTRODUCTION

The collective motion of organisms (birds, for example), is a fascinating phenomenon many times capturing our eyes when we observe our natural environment. In addition to the aesthetic aspects of collective motion, it has some applied aspects as well: a better understanding of the swimming patterns of large schools of fish can be useful in the context of large scale fishing strategies. In this paper we address the question whether there are some global, perhaps universal transitions in this type of motion when many organisms are involved and such parameters as the level of perturbations or the mean distance of the organisms is changed.

Our interest is also motivated by the recent developments in areas related to statistical physics. During the last 15 years or so there has been an increasing interest in the studies of far-from-equilibrium systems typical in our natural and social environment. Concepts originated from the physics of phase transitions in equilibrium systems [1] such as collective behaviour, scale invariance and renormalization have been shown to be useful in the understanding of various non-equilibrium systems as well. Simple algorithmic models have been helpful in the extraction of the basic properties of various far-from-equilibrium phenomena, like diffusion limited growth [2], self-organized criticality [3] or surface roughening [4]. Motion and related transport phenomena represent a further characteristic aspect of non-equilibrium processes, including traffic models [5], thermal ratchets [6] or driven granular materials [7].

Self-propulsion is an essential feature of most living systems. In addition, the motion of the organisms is usually controlled by interactions with other organisms in their neighbourhood and randomness plays an important role as well. In Ref. [8] a simple model of self propelled particles (SPP) was introduced capturing these features with a view toward modelling the collective motion of large groups of organisms [9] such as schools of fish, herds of quadrupeds, flocks of birds, or groups of migrating bacteria [10–12], correlated motion of ants [13] or pedestrians [14]. Our original SPP model represents a statistical physics-like approach to collective biological motion complementing models which take into account much more details of the actual behaviour of the organism, but, as a consequence, treat only a moderate number of organisms and concentrate less on the large scale transitions [9,16].

In this paper the large scale transitions during collective motion in three dimensions is considered for the first time. Interestingly, biological motion is typical in both two and three dimensions, because many organisms move on surfaces (ants, mammals, etc), but can fly (insects, birds) or swim (fish). In our previous publications we demonstrated that, in spite of its analogies with the ferromagnetic models, the transitions in our SSP systems are quite different from those observed in equilibrium models. In particular, in the case of equilibrium systems possessing continuous rotational symmetry the ordered phase is destroyed at finite temperatures in two dimensions [15]. However, in the 2d version of the non-equilibrium SSP model phase transitions can exist at finite noise levels (temperatures) as was demonstrated by simulations [8] and by a theory based on a continuum equation developed by Toner and Tu [17]. Thus, the question of how the ordered phase emerges due to the non-equilibrium nature of the model is of considerable theoretical interest as well.

In section 2 we describe our model. The results are presented in section 3 and the conclusions are given in section 4.
II. MODEL

The model consists of particles moving in three dimensions with periodic boundary conditions. The particles are characterised by their (off-lattice) location \( \vec{x}_i \) and velocity \( \vec{v}_i \) pointing in the direction \( \vartheta_i \). To account for the self-propelled nature of the particles the magnitude of the velocity is fixed to \( v_0 \). A simple local interaction is defined in the model: at each time step a given particle assumes the average direction of motion of the particles in its local neighbourhood \( S(i) \) with some uncertainty, as described by

\[
\vec{v}_i(t + \Delta t) = N(\vec{v}(\vec{u}(t))_{S(i)} + \vec{\xi}),
\]

where \( N(\vec{u}) = \vec{u}/|\vec{u}| \) and the noise \( \vec{\xi} \) is uniformly distributed in a sphere of radius \( \eta \).

The positions of the particles are updated according to

\[
\vec{x}_i(t + \Delta t) = \vec{x}_i(t) + v_0 \vec{v}_i(t) \Delta t,
\]

The model defined by Eqs. (2) and (3) is a transport related, non-equilibrium analogue of the ferromagnetic models. The analogy is as follows: the Hamiltonian tending to align the spins in the same direction in the case of equilibrium ferromagnets is replaced by the rule of aligning the direction of motion of particles, and the amplitude of the random perturbations can be considered proportional to the temperature.

We studied this model by performing Monte-Carlo simulations. Due to the simplicity of the model, only two control parameters should be distinguished: the (average) density of particles \( \rho \) and the amplitude of the noise \( \eta \). In the simulations random initial conditions and periodic boundary conditions were applied.

III. RESULTS

For the statistical characterisation of the configurations, a well-suited order parameter is the magnitude of the average momentum of the system: \( \phi \equiv \sum_j \vec{v}_j / N \). This measure of the net flow is non-zero in the ordered phase, and vanishes (for an infinite system) in the disordered phase.

The simulations were started from a disordered configuration, thus \( \phi(t = 0) \approx 0 \). After some relaxation time a steady state emerges indicated, e.g., by the convergence of the cumulative average \( (1/\tau) \int_0^\tau \phi(t) dt \).

The stationary values of \( \phi \) are plotted in Fig. 1. vs \( \eta \) for \( \rho = 2 \) and various system sizes \( L \) (indicated in the plot by the number of particles). For weak noise the model displays long-range ordered motion (up to the actual system size \( L \)) disappearing in a continuous manner by increasing \( \eta \).

These numerical results suggest the existence of a kinetic phase transition as \( L \rightarrow \infty \) described by

\[
\phi(\eta) \sim \begin{cases} \left( \frac{\eta_+(\rho)-\eta}{\eta_-(\rho)} \right)^\beta & \text{for } \eta < \eta_+(\rho) \\ 0 & \text{for } \eta > \eta_+(\rho) \end{cases}
\]

where \( \eta_+(\rho) \) is the critical noise amplitude that separates the ordered and disordered phases. Due to the nature of our non-equilibrium model it is difficult to carry out simulations on a scale large enough to allow the precise determination of the critical exponent \( \beta \). We find that the exponent 1/2 (corresponding to the mean field result for equilibrium magnetic systems) fits our results within the errors. This fit is shown as a solid line.

Next we discuss the role of density. In Fig. 2a, \( \phi(\eta) \) is plotted for various values of \( \rho \) (by keeping \( N = \text{Const.} \) and changing \( L \)). One can observe that the long-range ordered phase is present for any \( \rho \), but for a fixed value of \( \eta \), \( \phi \) vanishes with decreasing \( \rho \). To demonstrate how much this behaviour is different from that of the diluted ferromagnets we have also determined \( \phi(\eta) \) for \( v_0 = 0 \). In this limit our model reduces to an equilibrium system of randomly distributed "spins" with a ferromagnetic-like interaction. This system is analogous to the three dimensional diluted Heisenberg model. In Fig. 2b we display the results of the corresponding simulations. There is a major difference between the self-propelled and the static models: in the static case the system does not order for densities below a critical value close to 1 which in the units we are using corresponds to the percolation threshold of randomly distributed spheres in 3d.

This situation is demonstrated in the phase diagram shown in Fig 3. Here the diamonds show our estimates for the critical noise for a given density for the SPP model and the crosses show the same for the static case. The SPP system becomes ordered in the whole region below the curved line connecting the diamonds, while in the static case the ordered region extends only down to \( \rho \approx 1 \).
IV. DISCUSSION

A model (such as SSP) based on particles whose motion is biased by fluctuations is likely to have a behaviour strongly dependent on dimensionality around 2 dimensions since the critical dimension for random walks is 2. An other facet of this aspect of the problem is that a diffusing particle returns to the vicinity of any point of its trajectory with probability 1, while the probability of the same to occur in 3d is less than 1. In other words, the diffusing particles and clusters of particles are likely to frequently interact in 2d, but in a three dimensional simulation they may not interact frequently enough to ensure ordering.

Our calculations, however, show that for any finite density for small enough noise there is an ordering in the SSP model.

On the other hand, in the 3d case it is very difficult to estimate the precise value of the exponent describing the ordering as a function of the noise. The value we get within the errors agrees with the exponent which is obtained for the equilibrium systems in the mean filed limit. It is possible that the correlations in the direction of motion of the particles spread so efficiently due to their motion that the SSP model behaves already in 3d similarly to an infinite-dimensional static system. Indeed, the motion leads to an effective long-range interaction, since particles moving in opposite direction will soon get close enough to interact.

Finally, these findings indicate that the three dimensional SPP system can be described using the framework of classical critical phenomena, but shows surprising new features when compared to the analogous equilibrium systems. The velocity $v_0$ provides a control parameter which switches between the SPP behavior ($v_0 > 0$) and equilibrium type models ($v_0 = 0$).

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FIG. 1. The order parameter $\phi$ vs the noise amplitude $\eta$ for the 3D SPP model for various system sizes. In these simulations the density was fixed and the system size (number of particles $N$) was increased to demonstrate that for any system size the ordered phase disappears in a continuous manner beyond a size dependent critical noise.

FIG. 2. (a) The order parameter $\phi$ vs the noise amplitude $\eta$ ($N=1000$). (b) As a comparison, when $v_0 = 0$ the behavior of the model is similar to diluted ferromagnets: $\phi$ vanishes below the percolation threshold ($\rho^* \simeq 1$).

FIG. 3. The diamonds show our estimates for the critical noise for a given density of the particles in the SPP model and the crosses show the critical noise for the static case as a function of density. The SPP system becomes ordered in the whole region below the curved line connecting the diamonds, while in the static case the ordered region extends only down to $\rho \simeq 1$ corresponding to the percolation transition in the units we are using.