Testing Abelian Flavor Symmetries with Neutrino Parameters

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The experimental data on atmospheric and solar neutrinos are used to test the framework of Abelian horizontal gauge symmetries with only three light active neutrinos. We assume that the hierarchy in mass-squared splittings is not accidental and that the small breaking parameters are not considerably larger than 0.2. We find that the small angle MSW solution of the solar neutrino problem cannot be accommodated in this framework. In a large class of models, the large mixing angle MSW solution cannot be accommodated either. This leaves the vacuum oscillation solution as the favored one. These conclusions apply to any continuous Abelian symmetry, whether it is broken by a single parameter, as would be the case for an anomalous $U(1)$, or by two parameters of opposite charges and equal magnitudes, as with a non-anomalous gauge symmetry. They do not apply to discrete symmetries or to continuous ones that are broken into a discrete subgroup.
1. Introduction and Results

Approximate Abelian horizontal symmetries can explain the smallness and the hierarchy in the flavor parameters — fermion masses and mixing angles — in a natural and simple way [1]. One can think of three types of evidence for such symmetries: First, the full theory involves fields that are related to the spontaneous symmetry breaking and to the communication of the breaking to the observable sector. Direct discovery of such particles is, however, very unlikely because constraints from flavor changing neutral current (FCNC) processes and from Landau poles imply that they should be very heavy [2]. Second, the supersymmetric flavor parameters are also determined by the selection rules of the horizontal symmetry [3,4]. (This is likely to be the case if supersymmetry breaking is mediated to the observable sector by Planck-scale interactions [5]; in contrast, gauge mediation would erase the effects of the horizontal symmetry from the sfermion flavor parameters.) The spectrum of supersymmetric particles and, in particular, supersymmetric effects on FCNC and on CP violation could then provide evidence for the horizontal symmetry. Third, it could be that the Yukawa parameters themselves obey simple order of magnitude relations that follow from the horizontal symmetry [6]. In this context, Abelian horizontal symmetries have much more predictive power in the lepton sector than in the quark sector [7]. Neutrino parameters provide then an important input for testing and refining this framework.

As concerns neutrino parameters, recent measurements of the flux of atmospheric neutrinos (AN) suggest the following mass-squared difference and mixing between $\nu_\mu$ and $\nu_\tau$ [8]:

\[
\Delta m_{23}^2 \sim 2 \times 10^{-3} \, eV^2 , \quad \sin^2 2\theta_{23} \sim 1 .
\] (1.1)

On the other hand, measurements of the solar neutrino (SN) flux can be explained by one of the following three options for the parameters of $\nu_e - \nu_x$ ($x = \mu$ or $\tau$) oscillations (for a recent analysis, see [9]):

\[
\begin{array}{ccc}
\text{MSW(SMA)} & \Delta m_{1x}^2 [eV^2] & \sin^2 2\theta_{1x} \\
\text{MSW(LMA)} & 5 \times 10^{-6} & 6 \times 10^{-3} \\
\text{VO} & 2 \times 10^{-5} & 0.8 \\
\end{array}
\] (1.2)

\[8 \times 10^{-11} \quad 0.8 \]
Here MSW refers to matter-enhanced oscillations, VO refers to vacuum oscillations, and SMA (LMA) stand for small (large) mixing angle. Only central values are quoted for the various parameters.

Our basic assumption will be that eqs. (1.1) and (1.2) imply that the ratio between the mass splittings is suppressed by the small breaking parameter of an Abelian horizontal symmetry, $\lambda \sim 0.2$, while the $\nu_\mu - \nu_\tau$ mixing angle is not:

$$\sin \theta_{23} \sim 1,$$

$$\frac{\Delta m^2_{12}}{\Delta m^2_{23}} \sim \begin{cases} \lambda^2 - \lambda^4 & \text{MSW}, \\ \lambda^{10} - \lambda^{12} & \text{VO}. \end{cases} \tag{1.3}$$

We note, however, that it is not impossible that, if the solar neutrino problem is a result of the MSW mechanism, the ratio between $\Delta m^2_{23}^{SN}$ and $\Delta m^2_{23}^{AN}$ is accidentally, rather than parametrically, suppressed [10-13]. Then, the analysis of this work and of ref. [14] is irrelevant, and the framework of Abelian horizontal symmetries can accommodate the neutrino parameters in a simple way.

In a previous work [14] we investigated the implications of (1.3) for models where an Abelian horizontal symmetry $H$ is broken by a single small parameter. (For recent related work, see [15-31].) This assumption is best-motivated in models where the horizontal symmetry is an anomalous $U(1)$ gauge symmetry [33-36]. The anomaly is cancelled by the Green-Schwarz mechanism [37]. The contribution of the Fayet-Iliopoulos term to the D-term cancels against the contribution from a VEV of a Standard Model (SM) singlet field $S$ with $H$-charge that is opposite in sign to $\text{tr}H$. (Without loss of generality, we choose the $H$-charge of $S$ to be $-1$). The information about the breaking is communicated to the observable sector (MSSM) at the string scale. The ratio $\lambda = \langle S \rangle / m_{Pl} \sim \frac{\text{tr}H}{192\pi^2}$ [38-40] provides the small breaking parameter of $H$. The single VEV assumption is also plausible if the horizontal symmetry is discrete. In the single VEV framework, as explained in [14], it is non-trivial to get large mixing together with a large hierarchy as implied by (1.3). To obtain that one needs to invoke either holomorphic zeros or discrete symmetries, often with a symmetry group that is a direct product of two factors.

The situation is different if the Abelian horizontal symmetry is a non-anomalous gauge symmetry. If Supersymmetry is not to be broken at the scale of spontaneous $H$-breaking,
then $H$ should be broken along a D-flat direction. The simplest possibility then is that two scalars, $S$ and $\bar{S}$, of opposite $H$-charges (say, $\pm 1$) assume equal VEVs, $\langle S \rangle = \langle \bar{S} \rangle$ [41-47,13]. In this work we investigate the implications of (1.3) in the two-VEVs framework.

It is straightforward to see that our previous mechanisms are irrelevant in the new framework. First, with discrete symmetries there is no motivation for the two-VEVs scenario. There is also no sense in talking about negative charges. Second, with VEVs of opposite charges there can be no holomorphic zeros. On the other hand, this framework is in some sense less predictive and, consequently, allows new mechanisms to accommodate simultaneous large mixing and mass hierarchy between neutrinos. In particular, this situation can be obtained “naturally”, with a single $U(1)$ horizontal symmetry.

As we shall see, when the horizontal symmetry is continuous, whether it is an anomalous $U(1)$ with a single breaking parameter or whether it is non-anomalous, with two breaking parameters, it is impossible to accommodate the small-angle MSW solution. The reason is that if we arrange for large mixing as well as mass hierarchy of the two heavier neutrinos, the mass hierarchy obtained is greater than that required for the MSW solution. The only way to overcome this problem is to have the lighter of these two states combine with the remaining neutrino to form a pseudo-Dirac neutrino. The solar problem is then explained by oscillations between two states of a pseudo-Dirac type whose mixing is large. Thus, only the VO solution or the large angle MSW solution may be obtained.

Furthermore, in the two VEV framework, it is often hard to obtain even the large angle MSW solution. In a large class of models, even when allowing the first and second generation neutrinos to form a pseudo-Dirac fermion, the mass hierarchy obtained is at least $\lambda^5$, whereas the MSW solution requires $\lambda^2 - \lambda^4$. Here the small breaking parameter, $\lambda$, is taken to be $\lesssim 0.2$.

Another constraint associated with the large angle MSW solution is that $\sin 2\theta_{12} < 0.9$. However, as mentioned above, all models with continuous horizontal symmetries employ pseudo-Dirac first and second generation neutrinos whose mixing is close to maximal [17]. In fact, the deviation from maximal mixing is at most $O(\lambda^2)$. These models are therefore marginally viable. They are only consistent if the $O(\lambda^2)$ correction is enhanced by about three.
The plan of this paper is as follows. In section 2 we specify our theoretical framework. In Section 3 we argue that, in the framework of Abelian horizontal symmetries, $\nu_\mu$ and $\nu_\tau$ cannot pair to a pseudo-Dirac neutrino. In section 4 we review previous results in the framework of an anomalous $U(1)$ symmetry. In section 5 we analyze in detail the lepton parameters in the framework of non-anomalous Abelian gauge symmetry. (Proofs of some of the statements of this section are given in the appendix.) We conclude in section 6, emphasizing the difficulties that the data on solar and atmospheric neutrinos pose to the framework of Abelian horizontal gauge symmetries.

2. The Theoretical Framework

Our theoretical framework is defined as follows. We consider a low energy effective theory with particle content that is the same as in the Supersymmetric Standard Model. In addition to supersymmetry and to the Standard Model gauge symmetry, there is an approximate $U(1)_H$ symmetry that is broken by two small parameters $\lambda$ and $\bar{\lambda}$ [48,41-47,13]. The two parameters are assumed to be equal in magnitude:

$$\lambda = \bar{\lambda} \sim 0.2. \quad (2.1)$$

(The choice of numerical value comes from the quark sector, where the largest small parameter is $\sin \theta_C = 0.22$.) To derive selection rules, we attribute to the breaking parameters $U(1)_H$ charges:

$$H(\lambda) = +1, \quad H(\bar{\lambda}) = -1. \quad (2.2)$$

Then, the following selection rule applies: Terms in the superpotential or in the Kahler potential that carry (integer) $H$-charge $n$ are suppressed by $\lambda^{|n|}$.\(^1\) Whenever we do not specify the charges of singlet neutrinos, we implicitly assume that the singlet neutrinos are in vector representations of the horizontal symmetry [14].

We will only consider models where all fields carry integer $H$-charges (in units of the charge of the breaking parameters). It is possible that some or all of the lepton fields carry

\(^1\) The same selection rule would apply in a theory with a single breaking parameter and no supersymmetry.
half-integer charges [17]. In such a case there is a residual, unbroken discrete symmetry. Such models can be phenomenologically viable and lead to interesting predictions. We leave the investigation of this class of models to a future publication [49].

Note that if the horizontal symmetry is a continuous symmetry with a single breaking parameter, as would be the case for an anomalous $U(1)$, the selection rule stated above is modified. Superpotential terms that carry negative $H$-charge cannot appear, as they would require powers of $\lambda^\dagger$, which is forbidden by holomorphy [2]. We refer to these absent terms as "holomorphic zeros".

The theory is limited in the sense that it cannot predict the exact coefficients of $O(1)$ for the various terms. Wherever we use the symbol "~" below we mean to say that the unknown coefficients of $O(1)$ are omitted.

RGE effects could enhance the neutrino mixing angle [50-54,13]. The enhancement can take place if $\tan \beta$ is large and if the mass ratio between the corresponding neutrinos is not small. These enhancement effects are not important in our framework and we will not take them into account.

### 3. On Pseudo-Dirac Neutrinos

As a first step in our discussion, we would like to make a general comment about pseudo-Dirac neutrinos in the framework of Abelian horizontal symmetries. In our examples (here and in [14]), two of the three active neutrinos pair to form a pseudo-Dirac neutrino. In all of these examples, the parameters of the pseudo-Dirac neutrino (maximal mixing and very small mass splitting) are fitted to solve the SN problem. Since AN observations seem to favor maximal, and not just generic $O(1)$, mixing, one may wonder whether we could find a model where, indeed, the mass splitting between the components of the pseudo-Dirac neutrino corresponds to $\Delta m^2_{\text{AN}}$. We argue now that this is impossible.

The argument goes as follows. Let us define $m_{\text{pD}}$ to be the mass of the pseudo-Dirac neutrino and $\delta_{\text{pD}} \ll m_{\text{pD}}$ to be the mass splitting between its components. An Abelian symmetry cannot give an exact relation between three entries in the mass matrix. (The symmetric structure of $M_\nu$ relates pairs of entries, which enables us to find models with a
pseudo-Dirac neutrino.) Therefore, the mass-squared splitting between the pseudo-Dirac neutrino and the other mass eigenstate is at least $O(m_{pD}^2)$.

On the other hand, the mass-squared splitting between the components of the pseudo-Dirac neutrino is $O(\delta_{pD} m_{pD})$. Since $\delta_{pD} \ll m_{pD}$, the mass-squared splitting between the components of the pseudo-Dirac neutrino is much smaller than the mass-squared splitting between the pseudo-Dirac neutrino and the third mass eigenstate. Therefore, the former corresponds to $\Delta m^2_{SN}$ and the latter to $\Delta m^2_{\text{AN}}$.

It is worth emphasizing that the discussion of this section applies to any Abelian symmetry, continuous or discrete.

4. Continuous Symmetry with a Single Breaking Parameter

Before moving on to our discussion of continuous symmetries with two breaking parameters, let us recall some results of [14] in the single VEV framework.

The main obstacle in obtaining, within the single-VEV framework, large mixing between hierarchically separated neutrinos can be explained as follows. Consider a single $U(1)_H$ symmetry. Large mixing between, say, $\nu_2$ and $\nu_3$ can only be obtained in two cases: either the $H$-charges of the lepton doublets are equal [7], $H(L_2) = H(L_3)$, or they are opposite [15], $H(L_2) = -H(L_3)$. In the first case the mixing is $O(1)$ but the masses are of the same order of magnitude, $m(\nu_2) \sim m(\nu_3)$. In the second case, to a good approximation, the mixing is maximal, $\sin^2 2\theta_{23} = 1$ and the masses are equal, $m(\nu_2) = m(\nu_3)$. (This is the case of a pseudo-Dirac neutrino.) In either case, there is no mass hierarchy. If the symmetry is continuous but more complicated, say $U(1)_1 \times U(1)_2$ with the respective breaking parameters of order $\lambda^m$ and $\lambda^n$, then one can still define an effective $H$-charge,

$$H_{\text{eff}} = mH_1 + nH_2. \quad (4.1)$$

Large mixing can only be obtained for $H_{\text{eff}}(L_2) = \pm H_{\text{eff}}(L_3)$, so that, again, there is no mass hierarchy. This conclusion can only be evaded if holomorphic zeros appear in the

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2 It is of course possible to fine tune the $O(1)$ coefficients to get a stronger degeneracy [44].

3 If one considers only the AN problem, then it is of course possible that $\nu_\mu$ and $\nu_\tau$ form a pseudo-Dirac neutrino [46].
neutrino mass matrix such that one of the two mass eigenvalues vanishes. To summarize, in the single VEV framework, when considering $\nu_2$ and $\nu_3$ only, the only way to get large mixing and a large hierarchy between them is to make $\nu_2$ massless. Clearly though, we need to generate a mass for $\nu_2$ as well in order to account for (1.3). This can be arranged by having $\nu_2$ combine with $\nu_1$ to form a pseudo-Dirac neutrino. But then, $\sin 2\theta_{12}$ is large, and we cannot obtain the small angle MSW solution.

As we shall see in the next section, the situation in the two VEV scenario is qualitatively similar.

5. Continuous Symmetry with Two Breaking Parameters

5.1. Generalities: Problems with three hierarchical masses

Let us now consider an Abelian symmetry with two equal breaking parameters of opposite charges. For most of this section we will restrict our attention to two neutrinos only, $\nu_\mu$ and $\nu_\tau$. That is, we will assume that the mixing and the hierarchy between $\nu_\mu$ and $\nu_\tau$ can be described effectively by a $2 \times 2$ matrix (and, in particular, that $m(\nu_\mu) \gg m(\nu_\tau)$).

Also, for reasons explained in section 3, we assume that $\nu_\mu$ and $\nu_\tau$ do not pair to a pseudo-Dirac neutrino.

Finally, we assume that the light neutrino masses are a result of a see-saw mechanism. The heavy sterile neutrinos could be either in vector representations of $H$ (in which case we can apply the selection rules directly on the $2 \times 2$ Majorana mass matrix for the active neutrinos) or chiral under $H$ (in which case we will consider a full $4 \times 4$ matrix for two active and two sterile neutrinos).

In the appendix we will show that a large mixing between $\nu_2$ and $\nu_3$ requires that the horizontal charges of $L_2$ and $L_3$ obey

$$H(L_2) - H(L_3) = 0(\text{mod } 2), \quad (5.1)$$

and that the mass hierarchy $m_2/m_3$ is given by

$$\frac{m(\nu_2)}{m(\nu_3)} = \lambda^{2a[H(L_2)+bH(L_3)]+4c_iH_i}, \quad (5.2)$$
where \( a, b = \pm 1 \), \( c_i \) are integers, and \( i \) stands for any of the neutrino fields. We learn from eqs. (5.1) and (5.2) that \( \frac{m(\nu_2)}{m(\nu_3)} \sim \lambda^{4n} \) where \( n \) is an integer. If in addition, the mass parameters involving the first generation are negligible compared to \( m_2 \), we have

\[
\frac{\Delta m_{SN}^2}{\Delta m_{AN}^2} \sim \frac{m^2(\nu_2)}{m^2(\nu_3)} \sim \lambda^{8n}.
\]  

(5.3)

This creates a phenomenological problem. If the hierarchy is \( \lambda^0 \), \( \Delta m_{SN}^2 \) is too large. The next weakest possibilities are \( \lambda^8 \) or \( \lambda^{16} \). But for the MSW solutions we need \( \lambda^{2-4} \) and for the VO solution we need \( \lambda^{10-12} \). We can achieve neither in this framework. The MSW solutions are particularly disfavored; the VO solution may still correspond to the \( \lambda^8 \) hierarchy if \( \lambda \) is actually close to 0.1 (rather than the value of 0.2 that we usually use). \(^4\)

So far, we assumed that the second highest mass scale is determined by \( m_2 \). However, if we consider three generations of neutrinos, there is another possibility. Namely, if the \((12)\) entry of the neutrino mass matrix is large, \( \nu_\mu \) and \( \nu_e \) form a pseudo-Dirac neutrino with close to maximal mixing. Therefore the small angle MSW solution can not be obtained, and only the large angle MSW solution and the VO solution are allowed. In the next three subsections we will give some examples involving these two solutions.

Before doing that, however, note that since \( \nu_\mu \) and \( \nu_e \) form a pseudo Dirac neutrino, their mixing is close to maximal. In fact, the deviation of \( \sin 2\theta_{12} \) from 1 is suppressed by at least \( \lambda^2 \) in all models involving continuous Abelian symmetries. (Note that even if \( s_{12} = \sqrt{2}/2 + O(\lambda) \), we obtain \( \sin 2\theta_{12} = 1 - O(\lambda^2) \).) But the large angle MSW solution requires \( \sin 2\theta_{12} < 0.9 \). Thus we can only obtain this solution if the \( O(\lambda^2) \) corrections to \( \sin 2\theta_{12} \) are accidentally enhanced by a factor of about 3 or more.

5.2. Large mixing from unequal charges: Mass hierarchy

A large 2–3 mixing for \( H(L_2) \neq H(L_3) \) can be obtained from the charged lepton mass matrix by choosing

\[
H(L_2) + H(\bar{L}_3) = -[H(L_3) + H(\bar{L}_3)] \implies (M_\ell)_{23} \sim (M_\ell)_{33}.
\]  

(5.4)

\(^4\) The MSW solution can still be accommodated if the small breaking parameters are large, \( \lambda \sim 0.5 \) [42].
Consider then the following choice of charges:

\[ L_1(w), \ L_2(x), \ L_3(y), \ \bar{\ell}_3(z), \]  

(5.5)

with \( z = -(x + y)/2 \). We obtain:

\[ M_\nu \sim \frac{\langle \phi_u \rangle^2}{M} \begin{pmatrix} \lambda^2 |w| & \lambda |x+w| & \lambda |y+w| \\ \lambda |x+w| & \lambda^2 |x| & \lambda |y+x| \\ \lambda |y+w| & \lambda |y+x| & \lambda^2 |y| \end{pmatrix}. \]

(5.6)

Our assumptions above mean that \( 2|y| < |x+w| < 2|x|, 2|w|, |y+x|, |y+w| \). The hierarchy between the mass-squared differences is given by

\[ \frac{\Delta m_{SN}^2}{\Delta m_{\nu}^2} \sim \frac{\lambda^2 |x|+|x+w|}{\lambda^4 |y|} = \lambda^2 (|x| - |y|) \times \lambda |x+w|-2|y| \lesssim \lambda^5. \]

(5.7)

In deriving the inequality we used the fact that \( |x| - |y| \) is positive and even and that \( |x+w| > 2|y| \). We conclude then that the SN problem cannot be solved by the MSW mechanism when the mechanism for large 2–3 mixing is the one employed here.

To demonstrate that the VO solution can be implemented in our framework, consider the following set of charges:

\[ L_1(-7), \ L_2(+4), \ L_3(0), \]

\[ \ell_1(+14), \ \ell_2(-8), \ \bar{\ell}_3(-2). \]

(5.8)

The neutrino mass matrix is

\[ M_\nu \sim \frac{\langle \phi_u \rangle^2}{M} \begin{pmatrix} \lambda^{14} & \lambda^3 & \lambda^7 \\ \lambda^3 & \lambda^8 & \lambda^4 \\ \lambda^7 & \lambda^4 & 1 \end{pmatrix}, \]

(5.9)

which yields

\[ \frac{\Delta m_{SN}^2}{\Delta m_{\nu}^2} \sim \lambda^{11}. \]

(5.10)

For the charged lepton mass matrix we find

\[ M_\ell \sim \langle \phi_d \rangle \begin{pmatrix} \lambda^7 & \lambda^{15} & \lambda^9 \\ \lambda^{18} & \lambda^4 & \lambda^2 \\ \lambda^{14} & \lambda^8 & \lambda^2 \end{pmatrix}, \]

(5.11)

which gives \( \sin \theta_{23} \sim 1 \) and, for \( \tan \beta \sim \lambda^{-1} \), the required charged lepton mass hierarchy.

Simple modifications of the above set of charges lead to \( \frac{\Delta m_{SN}^2}{\Delta m_{\nu}^2} \sim \lambda^{10} \) or \( \lambda^{12} \), and to viable charged lepton masses with \( \tan \beta \sim 1 \).
5.3. Large mixing from unequal charges: Hierarchy of mass splittings without hierarchy of masses

The mechanism for large mixings that we presented above, that is $\mathcal{O}(1) \nu_\mu - \nu_\tau$ mixing from unequal charges, and maximal $\nu_e - \nu_\mu$ mixing from their pairing to a pseudo-Dirac combination, opens up the interesting possibility that there is actually no mass hierarchy: All three neutrino masses may be of the same order of magnitude, which is the scale set by AN, with the mass-squared splittings hierarchically separated.

It is simple to see that all three neutrino masses are of the same order of magnitude if we take in (5.5) $|x + w| = 2|y|$. The inequality in (5.7) is now weaker, $\frac{\Delta m^2_{\text{SN}}}{\Delta m^2_{\text{AN}}} \sim \lambda^2|x| - |y| \ll \lambda^4$. Yet, the MSW(LMA) solution cannot be achieved because the deviation from maximal mixing is suppressed by at least $\mathcal{O}(\lambda^4)$. (A deviation of $\mathcal{O}(\lambda^2)$ can be achieved only if $w - x$ is odd.) Therefore, also in this class of models only the VO solution can be accommodated. We now demonstrate this by an explicit example.

Consider the following set of $H$-charges for the lepton fields:

\begin{align*}
L_1 (+7), & \quad L_2 (-5), \quad L_3 (+1), \\
\bar{\ell}_1 (-15), & \quad \bar{\ell}_2 (+10), \quad \bar{\ell}_3 (+2).
\end{align*} \hspace{1cm} (5.12)

The neutrino mass matrix is of the form

\[ M_\nu \sim \frac{\langle \phi_u \rangle^2}{M} \begin{pmatrix}
\lambda^{14} & A\lambda^2 & \lambda^8 \\
A\lambda^2 & \lambda^{10} & \lambda^4 \\
\lambda^8 & \lambda^4 & B\lambda^2
\end{pmatrix}. \hspace{1cm} (5.13)\]

For later purposes we explicitly wrote down the $\mathcal{O}(1)$ coefficients, $A$ and $B$, of the dominant entries. We see that all three neutrinos have masses of the same order of magnitude, that is

\[ m(\nu_i) \sim \frac{\langle \phi_u \rangle^2}{M} \lambda^2 \quad \text{for} \quad i = 1, 2, 3. \hspace{1cm} (5.14)\]

The mass splittings are, however, hierarchical:

\[ \frac{\Delta m^2_{12}}{\Delta m^2_{23}} \sim \lambda^8, \hspace{1cm} (5.15)\]

which fits the VO solution for $\lambda \sim 0.1$. A large $2 - 3$ mixing is obtained from the charged lepton sector:

\[ M_\ell \sim \langle \phi_d \rangle \begin{pmatrix}
\lambda^8 & \lambda^{17} & \lambda^9 \\
\lambda^{20} & \lambda^5 & \lambda^3 \\
\lambda^{14} & \lambda^{11} & \lambda^3
\end{pmatrix}. \hspace{1cm} (5.16)\]
We learn that
\[ \sin^2 2\theta_{12} \simeq 1, \quad \sin \theta_{13} \sim \lambda^6, \quad \sin \theta_{23} \sim 1. \] (5.17)

Note that this scenario of hierarchical mass-squared splittings is not included in ref. [17]. The form of the neutrino mass matrix (5.13) in the charged lepton mass basis is given to a good approximation by
\[ M_\nu \sim \frac{\langle \phi_u \rangle^2}{M} \lambda^2 \begin{pmatrix} 0 & Ac & As \\ Ac &Bs^2 & Bcs \\ As & Bcs & Bc^2 \end{pmatrix}, \] (5.18)
where \( c \equiv \cos \theta_{23} \) and \( s \equiv \sin \theta_{23} \). Indeed, this corresponds to neither of the two forms advocated in [17]. It is amusing to note, however, that it can be presented as the sum of these two forms.

5.4. Large mixing from equal effective charges

A different way of obtaining large mixing together with a large hierarchy is the analog of the ‘holomorphic zeros’ mechanism of ref. [14]. In both frameworks we take the horizontal symmetry to be \( U(1)_1 \times U(1)_2 \), with equal effective charges (see eq. (4.1)) for \( L_2 \) and \( L_3 \), so that the 2–3 mixing is \( \mathcal{O}(1) \). The separate charges can, however, be chosen so as to induce a holomorphic zero in the single VEV framework and to suppress one of the masses in the two VEV framework.

As an example consistent with the MSW(LMA) solution, consider the following set of charges for the lepton fields [14]:
\[ \begin{align*}
L_1(1, 0), & \quad L_2(-1, 1), \quad L_3(0, 0), \\
\bar{\ell}_1(3, 4), & \quad \bar{\ell}_2(3, 2), \quad \bar{\ell}_3(3, 0).
\end{align*} \] (5.19)

The lepton mass matrices are of the form
\[ M_\nu \sim \frac{\langle \phi_u \rangle^2}{M} \lambda^2 \begin{pmatrix} \lambda^2 & \lambda & \lambda \\ \lambda & \lambda^4 & \lambda^2 \\ \lambda & \lambda^2 & 1 \end{pmatrix}, \quad M_{\ell} \sim \langle \phi_d \rangle \begin{pmatrix} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^7 & \lambda^5 & \lambda^3 \\ \lambda^7 & \lambda^5 & \lambda^3 \end{pmatrix}. \] (5.20)

Without the positively charged \( \bar{\lambda}_1 \), the (22), (23) and (32) entries in \( M_\nu \) would vanish because of holomorphy [14]. Here, the holomorphic zeros are lifted, but the new entries
are small and affect neither the mass hierarchy nor the mixing. Thus, the analysis of [14] is still valid, yielding
\[
\frac{\Delta m_{12}^2}{\Delta m_{23}^2} \sim \lambda^3, \quad \sin 2\theta_{12} = 1 - \mathcal{O}(\lambda^2), \quad s_{23} \sim 1, \quad s_{13} \sim \lambda.
\] (5.21)

The VO solution is similarly obtained with
\[
L_1(1, -4), \quad L_2(-2, 2), \quad L_3(0, 0),
\]
\[
\bar{\ell}_1(6, 5), \quad \bar{\ell}_2(3, 2), \quad \bar{\ell}_3(3, 0).
\] (5.22)
This gives
\[
M_\nu = \begin{pmatrix}
\lambda^{10} & \lambda^3 & \lambda^5 \\
\lambda^3 & \lambda^8 & \lambda^4 \\
\lambda^5 & \lambda^4 & 1
\end{pmatrix},
\] (5.23)
yielding \(\frac{\Delta m_{12}^2}{\Delta m_{23}^2} \sim \lambda^{11}\).

6. Conclusions

In models with a non-anomalous Abelian horizontal gauge symmetry, one expects that the symmetry is spontaneously broken by fields of opposite horizontal charges that acquire equal VEVs. We find two general mechanisms by which such models can accommodate the neutrino parameters that explain both the atmospheric neutrino and the solar neutrino problems. In particular, these mechanisms allow for
\[
\sin \theta_{23} \sim 1, \quad \frac{\Delta m_{12}^2}{\Delta m_{23}^2} \ll 1.
\] (6.1)

The possible scenarios divide into two general classes:
(i) The three neutrino masses are hierarchical. Then the hierarchy between the mass-squared splittings of eq. (6.1) is an integer power of \(\lambda^8\). The MSW solutions are disfavored. The VO solution is accommodated if the small breaking parameter is somewhat smaller than the ‘canonical’ value of 0.2 related to the Cabibbo mixing.
(ii) The two lighter neutrinos form a pseudo-Dirac neutrino. The mixing related to the solar neutrino solution is then close to maximal, so that obviously only large angle solutions to the SN problem are possible. In many of our models, though not in all,
also the large angle MSW solution cannot be accommodated, and the VO solution is, again, favored. A special case in this class is that of three same order-of-magnitude neutrino masses, with hierarchical splittings. This corresponds to the neutrino mass matrix of eq. (5.18), which is different from the two textures discussed in ref. [17].

We emphasize that our arguments are not valid for *discrete* Abelian symmetries [14]. Moreover, they can be circumvented even if the symmetry is continuous but the spontaneous symmetry breaking is not complete, leaving a residual exact discrete symmetry [17,49]. In particular, it has been demonstrated that the MSW(SMA) solution can be naturally generated in these cases [14,17].

While the conclusion of both this work and the one of ref. [14] is that large mixing and large hierarchy can be accommodated in the framework of Abelian horizontal symmetries, we would still like to emphasize the following points:

a. The most predictive class of models of Abelian horizontal symmetries is that of an anomalous $U(1)$ with holomorphic zeros having no effect on the physical parameters [7]. The various solutions suggested here and in [14] require that either the symmetry is non-anomalous with two breaking parameters, or the symmetry is discrete, or that holomorphic zeros do play a role. In all these cases there is a loss of predictive power. If, indeed, (6.1) holds in nature, it would mean that neutrino parameters by themselves will not make a convincing case for the Abelian horizontal symmetry idea, even if they cannot rule it out.

b. We argued here that, if (6.1) holds, the neutrino parameters that correspond to the atmospheric neutrino oscillations are not related to a pseudo-Dirac neutrino. Consequently, while Abelian horizontal symmetries allow for $O(1)$ $\nu_\mu - \nu_\tau$ mixing, they cannot explain *maximal* mixing (except as an accidental result). If the case for $\sin^2 2\theta_{23} = 1$ is experimentally made with high accuracy, and the solar neutrino problem is indeed solved by neutrino oscillations, the framework of Abelian horizontal symmetries would become less attractive.

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Appendix A. Mass Hierarchy with Large Mixing

We will now prove eqs. (5.1) and (5.2). We consider a framework with two active neutrinos in lepton doublets $L_2$ and $L_3$ and two singlet neutrinos $N_2$ and $N_3$.

The first step in our proof is to show that the large mixing between $\nu_2$ and $\nu_3$ requires that the horizontal charges of $L_2$ and $L_3$ obey

$$H(L_2) - H(L_3) = 0 \text{ (mod 2)}. \quad (A.1)$$

There are three possible sources (in the interaction basis) for a large mixing: (i) the charged lepton mass matrix $M_\ell$, (ii) the neutrino Dirac mass matrix $M_{\nu}^{\text{Dir}}$, and (iii) the sterile neutrino Majorana mass matrix $M_{\nu_s}^{\text{Maj}}$. We now examine the three mechanisms in turn.

(i) If the large mixing arises from $M_\ell$, then

$$(M_\ell)_{23} \sim (M_\ell)_{33} \implies |H(L_2) + H((\bar{\ell}_3))| = |H(L_3) + H((\bar{\ell}_3))|. \quad (A.2)$$

Therefore, either $H(L_2) = H(L_3)$, or $H(L_2) + H(L_3) = -2H((\bar{\ell}_3))$. In either case, (A.1) holds.

(ii) If the large mixing arises from $M_{\nu}^{\text{Dir}}$, then a similar argument holds with $H((\bar{\ell}_3))$ replaced by $H(N_3)$.

(iii) Let us examine the conditions for large mixing induced by $M_{\nu_s}^{\text{Maj}}$. To do so, let us define the matrix elements by

$$(M_{\nu_s}^{\text{Maj}})^{-1} = \begin{pmatrix} r_{11} & r_{12} \\ r_{12} & r_{22} \end{pmatrix}. \quad (A.3)$$

For simplicity, we take $M_{\nu}^{\text{Dir}}$ to be diagonal (consistent with our assumption that the large mixing does not come from this matrix):

$$M_{\nu}^{\text{Dir}} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}. \quad (A.4)$$
Then
\[ M_{\nu_3}^{\text{Maj}} = (M_{\nu}^{\text{Dir}})(M_{\nu_3}^{\text{Maj}})^{-1}(M_{\nu}^{\text{Dir}})^T = \begin{pmatrix} d_1^2 r_{11} & d_1 d_2 r_{12} \\ d_1 d_2 r_{12} & d_2^2 r_{22} \end{pmatrix}. \] (A.5)

Large mixing can be induced in two cases: first, \( d_1 d_2 r_{12} \gg d_2^2 r_{22} \) which leads to a pesudo-Dirac neutrino in contrast to our assumptions; second, \( d_1 d_2 r_{12} \sim d_2^2 r_{22} \), which can be achieved with
\[
\frac{d_1 r_{12}}{d_2 r_{22}} \sim \lambda_{H(L_2)+H(N_2)\lessgtr|H(L_3)+H(N_3)|+|H(N_2)+H(N_3)|-2|H(N_3)| \sim 1. \] (A.6)

The condition on the exponent is then of the form
\[
a_2 H(L_2) + a_3 H(L_3) + 2b_2 H(N_2) + 2b_3 H(N_3) = 0, \tag{A.7}
\]
where \( a_2, a_3 = \pm{1} \), \( b_2 = 0, \pm{1} \) and \( b_3 = 0, \pm{1}, \pm{2} \). Clearly, it leads to (A.1).

The second step is to find the hierarchy between the masses in terms of the lepton charges. Consider the neutrino mass matrices:
\[
M_{\nu}^{\text{Dir}} \sim \langle \phi_d \rangle \begin{pmatrix} \lambda [H(L_2)+H(N_2)] & \lambda [H(L_2)+H(N_3)] \\ \lambda [H(L_3)+H(N_2)] & \lambda [H(L_3)+H(N_3)] \end{pmatrix}, \tag{A.8}
\]
\[
M_{\nu_3}^{\text{Maj}} \sim M \begin{pmatrix} \lambda^2 [H(N_2)] & \lambda [H(N_2)+H(N_3)] \\ \lambda [H(N_3)+H(N_2)] & \lambda^2 [H(N_3)] \end{pmatrix}. \tag{A.9}
\]

One can easily see that
\[
\det M_{\nu} = m(\nu_2)m(\nu_3)m(N_2)m(N_3) \\
\sim \langle \phi_d \rangle^4 \max \left\{ \lambda^2 |H(L_2)+H(N_2)| + |H(L_3)+H(N_3)|, \lambda^2 |H(L_2)+H(N_3)| + |H(L_3)+H(N_2)| \right\}, \tag{A.10}
\]
\[
\det M_{\nu}^{(3)} = m(\nu_3)m(N_2)m(N_3) \\
\sim M \langle \phi_d \rangle^2 \max \left\{ \lambda^2 |H(L_2)+H(N_2)| + |H(N_3)|, \lambda^2 |H(L_2)+H(N_3)| + |H(N_2)|, \ldots \right\}, \tag{A.11}
\]
\[
\det M_{\nu}^{(2)} = m(N_2)m(N_3) \\
\sim M^2 \lambda^2 |H(N_2)+H(N_3)|. \tag{A.12}
\]

In (A.11) we did not write the full list of possible dominant terms. We can use these equations to estimate the mass hierarchy:
\[
\frac{m(\nu_2)}{m(\nu_3)} \sim \frac{\det M_{\nu} \det M_{\nu}^{(2)}}{|\det M_{\nu}^{(3)}|^2}. \tag{A.13}
\]
We find the following dependence of the mass ratio on the various charges:

\[
\frac{m(\nu_2)}{m(\nu_3)} = \lambda^{2a[H(L_2)+bH(L_3)]+4c_i} H_i
\]  

where \(a, b = \pm 1\), \(c_i\) are integers, and \(i\) stands for any of the four neutrino fields. Since all \(H_i\) are integers and, furthermore, \(H(L_2) \pm H(L_3)\) is even, we find that indeed \(\frac{m(\nu_2)}{m(\nu_3)} \sim \lambda^{4n}\), leading to (5.3).

Our proof generalizes trivially to the case of a horizontal symmetry that is a product of \(U(1)s\).

References