A TOPOLOGICAL INVARIANT OF RG FLOWS
IN 2D INTEGRABLE QUANTUM FIELD THEORIES

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We construct a topological invariant of the renormalization group trajectories of a large class of 2D quantum integrable models, described by the thermodynamic Bethe ansatz approach. A geometrical description of this invariant in terms of triangulations of three-dimensional manifolds is proposed and associated dilogarithm identities are proven.

1 Introduction

Many quantum integrable 2D models can be considered as perturbations of a Conformal Field Theory. The RG evolution of these models is described, in the thermodynamic Bethe ansatz (TBA) approach [1] by the ground state energy $E(R)$ of the system in an infinite cylinder of radius $R$, through an equation of the form $E(R) = \frac{1}{2\pi} \sum_{a=1}^{N} \int d\theta \nu_a(\theta) \log(1 + Y_a(\theta))$, where $\theta$ is the rapidity, $a$ labels the pseudoenergies (cf. [2]), $Y_a(\theta)$ are $R$-dependent functions, determined by a set of coupled integral equations known as TBA equations; the $\nu_a$’s are elementary functions of $R$ and $\theta$, related to the asymptotic behaviour of the $Y_a(\theta)$’s for $\theta \to \pm \infty$. An important observation of Al.B.Zamolodchikov[3] was that any solution $\{Y_a(\theta)\}$ of the TBA equations satisfies a set of simple functional equations, called the Y-system. Conversely, it can be shown that under appropriate analytic and asymptotic restrictions the Y-system is equivalent to the set TBA equations. For instance, the Y-system describing the flow from the tricritical Ising model to the critical Ising one is given by

$$Y_1(\theta + \frac{i\pi}{2})Y_1(\theta - \frac{i\pi}{2}) = 1 + Y_2(\theta) \quad , \quad Y_2(\theta + \frac{i\pi}{2})Y_2(\theta - \frac{i\pi}{2}) = 1 + Y_1(\theta) . \quad (1.1)$$
This system, like any other $Y$-system, can be considered as a recursion relation in the imaginary rapidity. To simplify the notations we can put $Y_n(R, \theta) = Y_{\frac{1}{2} + (-1)^n (\theta + n \frac{2\pi}{2})}$, then the eq. (1.1) becomes

$$Y_{n-1} Y_{n+1} = 1 + Y_n.$$  \hfill (1.2)

It is immediate to verify that this $Y$-system is periodic: $Y_{n+5} = Y_n$, i.e. $Y_1(\theta + \frac{5}{2}i\pi) = Y_2(\theta)$, $Y_2(\theta + \frac{5}{2}i\pi) = Y_1(\theta)$. In ref. [3] it was also conjectured that a similar periodicity $Y_a(\theta + P i\pi) = Y_{\alpha}(\theta)$ where the period $P$ is related to the dimension of the perturbing operator by $\Delta = 1 - 1/P$. In the UV limit $R \rightarrow 0$ the vacuum energy $E(R)$ is expected to behave as $E(R) \sim -\pi \tilde{c}/6R$, where $\tilde{c}$ is the effective central charge of the UV fixed point. In general one finds

$$\tilde{c} = \frac{6}{\pi^2} \sum_{a=1}^{N} \left[ L \left( \frac{Y_a(-\infty)}{1 + Y_a(-\infty)} \right) - L \left( \frac{Y_a(\infty)}{1 + Y_a(\infty)} \right) \right],$$  \hfill (1.3)

where $L(x)$ is the Rogers dilogarithm. It can be defined as the unique function three times differentiable which satisfies the following five term relationship, known also as the Abel functional equation (see for instance [4])

$$L(x) + L(y) + L \left( \frac{1-x}{1-xy} \right) + L(1-xy) = 3L(1),$$  \hfill (1.4)

with $0 \leq x, y \leq 1$ and the normalization $L(1) = \frac{\pi^2}{6}$. This equation has a $\mathbb{Z}_5$ symmetry [15, 7] which becomes manifest rewriting it in the form [7]

$$\sum_{n=1}^{5} L \left( \frac{Y_n}{1 + Y_n} \right) = 3L(1),$$  \hfill (1.5)

where the $Y_n$’s fulfill the recursion relation (1.2). Clearly eq. (1.5) defines an invariant of the RG flow of the tricritical Ising system, being the $L$ arguments non-trivial functions of $R$ and $\theta$, while the r.h.s. is constant along the RG trajectory. This behaviour suggests [7] the existence of large families of functional equations, which are generalizations of (1.5). The simplest examples are classified in terms of an ordered pair $G \times H$ of $ADE\, (T_n = A_{2n}/\mathbb{Z}_2)$ Dynkin diagrams [5]

$$Y^b_a(n - 1) Y^b_a(n + 1) = \prod_{c=1}^{r_G} (1 + Y^b_c(n))^{G_a} \prod_{d=1}^{r_H} \left( 1 + \frac{1}{Y^d_a(n)} \right)^{-H_{bd}}.$$  \hfill (1.6)

Here $G_{ac}$ and $H_{bd}$ are the adjacency matrices of the appropriate Dynkin diagrams whereas $r_G$ and $r_H$ are the ranks of the corresponding algebras. The solutions to this system are conjectured to be periodic functions: $Y^b_a(n + P) = Y^b_a(n)$, $P = h + g$, with
π and \(\overline{\pi}\) the conjugate nodes to \(a\) and \(b\), while \(h\) and \(g\) are the dual Coxeter numbers. Then, the \(Y\)'s solving eq. (1.6) are, according to our conjecture, arguments of functional equations for the Rogers dilogarithm, \(L(x)\), of the following form

\[
\sum_{a=1}^{r_G} \sum_{b=1}^{r_H} \sum_{n=0}^{h+g-1} L\left(\frac{Y_a^b(n)}{1 + Y_a^b(n)}\right) = r_G r_H g L(1) . \tag{1.7}
\]

Notice that since \(L(x)\) is a multi-valued function, the r.h.s of eq. (1.7) cannot be fixed unambiguously, unless a consistent choice of the sheet of the associated Riemann surface is implemented. The quoted value corresponds to solutions of the recursion relations with all the \(Y\)'s > 0. It is interesting that the \(A_N \times A_1\) \((N = 1, 2, 3)\) identities reduce respectively to the Euler, Abel and Newman functional equations (cf. [8]) when symmetric conditions on the conjugated nodes are chosen.

Other examples of \(Y\)-systems with their corresponding dilogarithm identities were found and discussed in ref. [9]. Furthermore, two new families of \(Y\)-systems and dilogarithm functional equations, classified in terms of a pair of co-prime integer \((p, q)\) associated to the sine-Gordon model at rational points and to its reductions, were proposed and partially studied [9, 10] (see [12] for even more exotic relations). Very soon after their discovery, the general solution of the \(A_N \times A_1\) systems was found and the corresponding dilogarithm identities independently proven in [10, 13]. We report here the solution proposed in ref. [10], which was related to the fact, already known to Lobachevskij, that dilogarithms define volumes of particular tetrahedra in hyperbolic space and other manifolds. The seed of the idea was introduced in 1992 in the field of integrable models by ref. [6]. Other ideas were then expound and elaborated in ref. [14], where a manifold on which Rogers dilogarithms define volumes of ideal tetrahedra was identified.

2 The geometrical picture

The starting point of ref. [10] was the assumption that the Abel equation mimics in some way a volume calculation of a compact manifold. Although the existence and the nature of such a manifold could not be further argued, this geometrical point of view was rather powerful and allowed to prove in a rigorous way, just using eq. (1.4) as the only ingredient, the \(A_N \times A_1\) identities and to solve the \(Y\) system of a general class of thermally perturbed minimal models. Let us mention that after the main results of ref.s [7, 10, 13] were already appeared, we became aware * of an undated manuscript by D.Zagier where this geometric approach is clarified and simplified. We can now rephrase

*We are grateful to the referee of an earlier version of [10] for a copy of Zagier’s notes.
our construction as follows. Consider an arbitrary triangulation of an arbitrary three-dimensional manifold, made by $M$ oriented tetrahedra $\{T_i\}$. The fact that this set forms a triangulation means that the boundary of the associated 3-chain is zero:

$$\partial \left( \sum_{i=1}^{M} T_i \right) = 0 ,$$

(2.8)

where the boundary of the tetrahedron $T \equiv [ABCD]$ is defined in the standard way:

$$\partial [ABCD] = [ABC] - [BCD] + [CDA] - [DAB] .$$

(2.9)

Now associate to each vertex $i$ $(i = 1, \ldots, N)$ a real number $\dagger x_i$. Through this map we can assign to each tetrahedron $T$, the cross-ratio $t = (abcd)$ of the values associated to its vertices

$$T \equiv [ABCD] \rightarrow t = (abcd) , \quad (abcd) = \frac{(a-c)(b-d)}{(a-d)(b-c)} .$$

(2.10)

In the following we shall use some elementary properties of the cross-ratios:

$$(abcd) = (cdab) = (badc) = 1 - (acbd) = (bacd)^{-1}$$

$$(aabc) = 1 , \quad (abbc) = \infty , \quad (abch) = 0 .$$

(2.11)

We shall prove that

$$\sum_{j=1}^{M} L(t_j) = n L(1) ,$$

(2.12)

where the integer $n$ is unambiguously fixed when the sequence of real numbers $\{x_i\}$ is ordered in such a way that all the $t_j$’s belong to the interval $[0, 1]$. Since the number $M$ of tetrahedra is less or equal to the number $N$ of vertices, there are simple algebraic relations among the $t_j$’s. For suitable triangulations they coincide with the Y-system of known integrable models. Before giving a general proof of eq. (2.12) let us see a couple of simple instances. Consider a polytope made of tetrahedra in four dimensional Euclidean space. Well known examples of this kind are the three regular polytopes 5-cell (or 4-simplex), 16-cell (dual to hypercube) and 600-cell. They are all triangulations of $S^3$ (see for example [11]). The 5-cell of fig. 1a is formed by the following five tetrahedra

$$T_1 = [BACD] , \quad T_2 = [CBDE] , \quad T_3 = [DCEA] , \quad T_4 = [EDAB] , \quad T_5 = [AEBC] .$$

The pentagonal relation (1.4) is then secured assigning five real numbers $\{a < b < c < \dagger$More generally, the $x_i$’s may be chosen as points of an arbitrary circle of the complex plane in order to have real cross-ratios in eq. (2.10).}
It is instructive to draw the adjacency graph of this set of tetrahedra; associating each tetrahedron to a vertex and connecting with a line two tetrahedra sharing the same face, we end up with the graph of the dual polytope, which in this case is again a 5-cell (see fig. 1b). For each loop of the adjacency graph one can write an algebraic relation among the cross-ratios of the tetrahedra touched by the loop. In particular, it is easy to verify that the $L$ arguments fulfil the recursion relation (1.2). Moreover the well known identity $L(x) + L(1 - x) = L(1)$, known as Euler equation, can be written as

\begin{equation}
L((abcd)) + L((cdbe)) + L((deca)) + L((eadb)) + L((abec)) = 3L(1) \ . \tag{2.13}
\end{equation}

In the following we shall use the short-hand notation $L\{abcde\} = 3L(1)$ for the identity (2.13) and $L\{abode\} = 2L(1)$ for the identity (2.15). As a more engaging example, let us consider the 16-cell. This polytope has 8 vertices, characterized by the property that each vertex is linked to all the others but one (see fig. 2a). Ordering the vertices in such
a way that the unlinked ones are given by the pair \( i, i + 4 \mod 8 \), the 16 tetrahedra split in two sets, associated to the following cross-ratios:

\[
t_j = (x_j x_{j-1} x_{j+1} x_{j+2}), \quad u_j = (x_j x_{j-1} x_{j+2} x_{j-3}), \quad (j = 1, 2 \ldots 8)
\]  

(2.16)

where all the indices are taken modulo 8 and \( x_1 < x_2 < \ldots x_8 \). Starting from the sum of the following five-term relations written as

\[
22L(1) = L\{x_1 x_2 x_3 x_4 x_5\} + L\{x_1 x_3 x_4 x_5 x_6\} + \tilde{L}\{x_1 x_2 x_4 x_5 x_7\} + L\{x_1 x_2 x_3 x_5 x_8\} \\
+ L\{x_5 x_6 x_7 x_8 x_1\} + L\{x_5 x_7 x_8 x_1 x_2\} + \tilde{L}\{x_5 x_6 x_8 x_1 x_3\} + L\{x_5 x_6 x_7 x_1 x_4\}
\]

one can easily verify that each five-term relation contains two tetrahedra of the triangulation (2.16); the remaining 24 tetrahedra combine in pairs according to eq. (2.14), so one is directly led to the sought-after 16-term relation

\[
\sum_{i=1}^{8} [L(t_i) + L(u_i)] = 10L(1) .
\]  

(2.17)

Drawing the adjacency graph of these tetrahedra we find the dual polytope, i.e. the 8-cell (or hypercube) represented in fig. 2b. There is an algebraic relation for any set of tetrahedra connected by a closed path of such a graph. In order to find a link with known Y-systems it is convenient to transform the argument \( t = (abcd) \) of \( L \), using the relation \( \tilde{t} = -(adcb) = t/(1 - t) \). Define now \( Y_1(n) = \tilde{t}_{3n+3} \) and \( Y_2(n) = \tilde{u}_{3n} \), where
all the indices are understood modulo 8. One can check that the following relations are fulfilled

\[
Y_1(n-1)Y_1(n+2) = (1 + Y_2(n))(1 + Y_2(n + 1))
\]
\[
Y_2(n-1)Y_2(n+1) = (1 + Y_1(n))/(1 + 1/Y_2(n)).
\] (2.18)

This is the Y-system corresponding to the $\phi_{13}$-thermal perturbation of the $M_{5,8}$ minimal model. While the discovery of the Y-system associated to the 600-cell is still an open challenge, two infinite families of $S_3$ triangulations associated to known integrable models may be found in ref. [10].

Dilogarithm functions obey a general property known as “beta-map” [13, 14, 15], that we will shortly introduce. The beta-map and various geometrical notions relating dilogarithms to three-dimensional compact manifolds will merge together giving a more unified point of view. Swapping between these two main tools we will then be able to prove many of the ADE-related dilogarithm identities. The beta-map condition adapted to the Rogers dilogarithm states that a sufficient condition for having the dilogarithm functional equations

\[
\sum_i L(K_i) \in \mathbb{Z}L(1),
\] (2.19)

is that the following condition is satisfied

\[
\sum_i K_i \land (1 - K_i) = 0,
\] (2.20)

where the wedge product is defined through the properties

\[
x \land z = -z \land x, (xy) \land z = x \land z + y \land z.
\] (2.21)

For a more detailed discussion of the relation (2.19-2.20) the reader is referred to [13, 14, 15]. If the $L$ argument is expressed as a cross-ratio $t$, we have $t \land (1 - t) = (abcd) \land (acbd)$ and

\[
(abcd) \land (acbd) = (abc) - (bcd) + (cda) - (dab),
\] (2.22)

where we have defined

\[
(abc) = (a - c) \land (a - b) + (a - c) \land (c - b) + (b - c) \land (a - b).
\]

Comparison between eq. (2.9) and eq. (2.22) shows that the beta map acts on the cross-ratios associated to the tetrahedra like the boundary operator $\partial$, hence we have the implication

\[
\partial \left( \sum_j T_j \right) = 0 \Rightarrow \sum_j t_j \land (1 - t_j) = 0,
\] (2.23)
which proves eq. (2.12). It is important to note that the implication (2.23) cannot
be inverted. Actually we found Y-systems which do not correspond strictly to a tri-
angulation. The simplest example is the solution of the $D_N \equiv D_N \times A_1$ systems of

$$\Upsilon_a(n+1) \Upsilon_a(n-1) = \prod_{b=1}^{N} (1 + \Upsilon_b(n)) D_{ab}^{(N)}.$$  (2.24)

These systems emerge from the TBA analysis of the sine-Gordon model at the reflection-
less points $\xi = 1/(N-1)$ [3] in the attractive regime, or equivalently (with the change
$\Upsilon \to 1/\Upsilon$) at the points $\xi = (N-1)$ [16] in the repulsive regime. The solution will
be written in terms of cross-ratios of a unique function $j \to h_j$ exactly
as in the models treated previously, but now $h_j$ is quasi-periodic, in the sense that
$h_j - N = q h_j + p$, where $p$ and $q$ are real numbers. For the

$$\Upsilon_a(n) = -(h_j h_i h_{i+1} h_{i+1}) , \quad i = \lfloor \frac{n-a-2}{2} \rfloor , \quad j = \lfloor \frac{n+a}{2} \rfloor , \quad a \leq N - 2 ,$$  (2.25)

with the square brackets indicating that the indices must be taken modulo $N$. For the
two nodes $N - 1$ and $N$ (the “fork” of the diagram) we can write

$$\Upsilon_N(n \pm 1) = Z_\pm(n \pm 1) , \quad \Upsilon_{N-1}(n \pm 1) = Z_\mp(n \pm 1) ,$$  (2.26)

with $Z_+(n) = -(h_j h_i h_{i+1} \infty)$, $Z_-(n) = -(h_{i+1} h_{i+1} h_N \infty)$, where $i = \lfloor \frac{n-N-1}{2} \rfloor$ and $j = \lfloor \frac{n+N-1}{2} \rfloor$. It is now easy to verify the conjectured periodicity $\Upsilon_a(n + 2N) = \Upsilon_{\pi}(n)$.

Introducing, besides the $N$ points associated to $h_j$, $N$ further vertices $j = 1, \ldots, N$
associated to the values $\overline{j} = q h_j + p$ and a vertex associated to $\infty$, it is possible to
represent this Y-system with a 3-chain $D_N$ of $N^2$ tetrahedra and $2N+1$ vertices. Now
the boundary of this chain does not vanish, however we have $\partial\{D_N + B_N\} = 0$, where
$B_N$ is the auxiliary 3-chain

$$B_N = \sum_{j=1}^{N-2} \{ [j, N, j + 1, \infty] + [\overline{j}, \overline{j+1}, N, \infty] \} .$$  (2.27)

We can then write a dilog identity associated to the triangulation $D_N + B_N$ (which
turns out to be again a triangulation of $S_3$). Since we have

$$(h_j h_N h_{j+1} \infty) + (h_{j+1} h_{j+1} h_N \infty) = 1 ,$$

the tetrahedra of $B_N$ can be paired off using the Euler identity $L(x) + L(1-x) = L(1)$,
so we end up with the result conjectured in eq. (1.7). Note that because of the pairing
just mentioned, the chain $B_N$ gives a vanishing contribution to the beta map (2.20),
3a) $D_N$ diagram.  
3b) Adjacency graph of $D_4$.

hence also the chain $D_N$, although it is not a triangulation, fulfills this condition. As an explicit example, the adjacency graph of the $D_4$ system is drawn in fig. 3b; the grey circles denote the tetrahedra of the $B_4$ chain and the dotted lines correspond to their faces. Other infinite families of such “quasi-triangulations” can be found in ref. [17].

3 General ADE × ADE identities

In this section we will generalize the proof of the functional dilogarithm identities to more general cases, where the existence of an associated triangulation is not known, using the only assumption that the associated $Y$ system is periodic. Let’s introduce an auxiliary set of equations again classified in terms of a pair of Dynkin diagrams $(G, H)$

$$T_b^h(n+1)T_a^h(n-1) = \prod_{d=1}^{r_H} T_d^h(n)^{H_{bd}} + \prod_{c=1}^{r_G} T_c^h(n)^{G_{ac}}.$$  

(3.28)

When specialized to $A_N \times ADE$ the three term relations (3.28) coincide with the well known (cf. [18]) fusion-relations appearing in integrable lattice models. They are often called $T-$systems. Analytically for the lower rank cases and numerically more generally one can again show that they fulfil the cyclic property

$$T_a^h(n+P) = T_a^h(n), \quad P = h + g.$$  

(3.29)
Furthermore defining

\[ Y^g_a(n) = \prod_{c=1}^{r_G} T^b_c(n)^{G_{ac}} \prod_{d=1}^{r_H} T^d_a(n)^{-H_{bd}}, \tag{3.30} \]

and

\[ 1 + Y^b_a(n) = T^b_a(n + 1) T^b_a(n - 1) \prod_{d=1}^{r_H} T^d_a(n)^{-H_{ba}}, \tag{3.31} \]

one can easily show that the functions \( Y \) defined in (3.30) satisfy eq.s (1.6). Now writing the \( Y \)'s in terms of the \( T \)'s using (3.30) and the periodicity (3.29) one can show that (2.20) is vanishing. We have

\[
\sum_{n=1}^{P} \sum_{a=1}^{r_H} \sum_{b=1}^{r_G} Y^b_a(n) \wedge (1 + Y^b_a(n)) = \sum_{\{n,a,b\}} (\prod_{c=1}^{r_G} T^b_c(n)^{G_{ac}}) (\prod_{d=1}^{r_H} T^d_a(n)^{-H_{bd}}) \wedge (\prod_{m=1}^{P} T^b_a(m)^{\hat{A}^{(P)}_{mn}}) (\prod_{e=1}^{r_H} T^e_a(n)^{-H_{be}}), \tag{3.32}
\]

where for convenience the incidence matrix \( \hat{A}^{(P)} \) of an Affine \( A_{P-1} \) Dynkin diagram has been introduced. Using \( ab \wedge cd = a \wedge c + b \wedge c + a \wedge d + b \wedge d \) and

\[
\sum_{a=1}^{\dim(M)} \sum_{b=1}^{\dim(K)} \left( \prod_{c=1}^{\dim(M)} X_{cb}^{M_{ac}} \right) \wedge \left( \prod_{e=1}^{\dim(K)} X_{ae}^{K_{ae}} \right) = \sum_{\{a,b,c,e\}} M_{ac}K_{eb}(X_{cb} \wedge X_{ae}) = \frac{1}{2} \sum_{\{a,b,c,e\}} M_{ac}K_{eb}(X_{cb} \wedge X_{ae} - X_{ae} \wedge X_{cb}) = 0, \tag{3.33}
\]

for \( M \) and \( K \) arbitrary symmetric matrices we conclude that eq. (3.32) vanishes and therefore we have dilogarithm identities.

### 4 Conclusions

We have described a general method to write functional dilogarithm identities associated to the TBA formulation of the renormalization group evolution of integrable quantum models. It is worth to observe that such identities define quantities of topological nature, because they do not vary under arbitrary deformations of the vertex function \( j \rightarrow x_j \), provided that the initial ordering of the set \( \{x_j\} \) is preserved. Our geometrical description of these identities in terms of triangulations (or quasi-triangulations) of three-dimensional manifolds provides us with the general solution of the recursion
relations associated to the $Y$ systems and hence a simple explanation of their periodicity. Another consequence of this approach is to show that the $Y$’s can be expressed in terms of a unique periodic (or quasi-periodic) function $x(\theta)$, related to $x_j$ through $x_j = x(\theta + i\pi \frac{j}{g})$, where $g$ depends on the model. In particular, it is easy to find the $x(\theta)$ corresponding to the stationary (i.e. $\theta$-independent) version of the $Y$ systems and their associated dilog identities. For instance one can verify, using eq.s (2.25,2.26) that the stationary solution of the $A_N$ system is given by $x(\theta) = e^{2\theta N + \frac{2}{N+1}}$ and that of the $D_N$ system is $h(\theta) = \theta$. From a physical point of view, these functional identities define a quantity which is conserved during the renormalization group evolution of the system. In particular in the UV limit the system is described by a conformal field theory, then the r.h.s. of such identities must be expressible only in terms of conformal quantities such as the effective central charge and the conformal dimension $\Delta$ of the perturbing operators. For instance, in the purely elastic ADE related models the integer $n$ of eq. (2.12) is given by the ratio $\frac{c_{eff} \Delta}{\Delta}$. The RG flow of any quantum field theory can be viewed as a typical dissipative process, because the integration of degrees of freedom responsible of the trajectory between the UV and the IR fixed points yields an information loss of the system. Then it is amazing that there are conserved quantities along these trajectories. It would be very interesting, also in connection with the $c$ theorem of A.B.Zamolodchikov, to find a field-theoretic description of such an invariant.

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References


