ON THE UNSTABLE SPECTRUM OF
THE EULER EQUATION

Susan FRIEDLANDER

9902063

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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Introduction

In this short paper we survey some recent results concerning the unstable spectrum of the Euler equations linearised about a steady flow of an ideal fluid. We discuss the concept of a “fluid Lyapunov exponent” which determines the essential spectral radius of the linearised Euler operator. A positive lower bound for this exponent is computable for many classes of explicit flows and hence gives an effective method for detecting instability in the essential spectrum. The problem of existence of unstable eigenvalues is more difficult and, at least at present, must be treated on a case by case basis. In the case of certain specific flows containing regions of spatial oscillation it is possible to investigate the discrete unstable spectrum using averaging techniques and to demonstrate the existence of unstable eigenvalues. In the case of plane parallel shear flow with a sinusoidal profile we construct the complete unstable spectrum. We discuss a result which proves under certain rather general conditions that linear instability implies nonlinear instability. We show how this abstract theorem can be used to conclude that some specific examples of oscillating flows with discrete unstable eigenvalues are nonlinearly unstable.

1. The Euler Equations for Fluid Motion

The Euler equations for the motion of an inviscid incompressible fluid are

$$q_t + (q \cdot \nabla) q = -\nabla P \quad (1.1)$$

$$\nabla \cdot q = 0. \quad (1.2)$$

The evolution of the velocity field $q(x, t)$ is determined by its current state, as well as by the internal pressure forces $-\nabla p(x, t)$.

Let $U_0(x)$ be a steady Euler flow. From (1.1), (1.2) the equilibrium equation for U_0 can be written in the form

$$U_0 \times \text{curl} U_0 = -\nabla H, \quad \nabla \cdot U_0 = 0 \quad (1.3)$$

where H is a scalar field known as the reduced pressure. We consider the linearised Euler equations for a small perturbation $v(x, t)$ about a steady flow U_0 :

$$\left. \begin{aligned} v_t &= -(U_0 \cdot \nabla) v - (v \cdot \nabla) U_0 - \nabla p \equiv Lv \\ \nabla \cdot v &= 0 \end{aligned} \right\} \quad (1.4)$$

with initial condition

$$v(x, 0) = v_0(x). \quad (1.5)$$

The results that we will present are proved for

$$U_0(x) \in (C^\infty(T^n))^n, \quad \nabla \cdot U_0 = 0 \quad (1.6)$$

$$v_0(x) \in (L^2(T^n))^n, \quad \nabla \cdot v_0 = 0 \quad (1.7)$$

where $T^n = \mathbb{R}^n/2\pi\mathbb{Z}^n$ is an n -dimensional torus. The applications we will discuss are examples in 2 and 3 dimensions.

In this paper we discuss the spectral problem associated with equation (1.4), in particular the nature of the stability spectrum σ . The stability spectrum of a viscous flow in a bounded domain is simple: it consists of a sequence of eigenvalues. However the more difficult mathematical problem of the spectrum of *inviscid* flows still has open questions. Unlike the viscous problem where the relevant Navier-Stokes operator is elliptic, the linearised Euler operator L is degenerate non-elliptic and a continuous spectrum is usually present. In fluid dynamics the inviscid limit is well known to be a subtle problem that is mathematically and physically challenging [1]. We will give some partial answers concerning the structure of the stability spectrum for the inviscid equations (1.4).

2. The Essential Spectral Radius

Let e^{Lt} be the evolution operator for the linearised Euler equation (1.4). We use the following classification of the elements of the spectrum σ . A point $z \in \sigma$ is called a point of the *discrete* spectrum σ_{disc} if it satisfies the following three conditions.

1. z is an isolated point in σ .
2. z has finite multiplicity.
3. The range of $z - e^{Lt}$ is closed.

If z does not satisfy 1 – 3, it is called an element of the essential spectrum σ_{ess} . Thus

$$\sigma = \sigma_{\text{disc}} \cup \sigma_{\text{ess}}$$

and the union is disjoint. We denote by r_{ess} the essential spectral radius in L^2 , where, following Nussbaum [2] we define

$$r_{\text{ess}}(e^{Lt}) = \sup \{|z| \mid z \in \sigma_{\text{ess}}(e^{Lt})\}.$$

The following theorem due to Vishik [3] gives an expression for $r_{\text{ess}}(e^{Lt})$ in terms of a geometric quantity that can be considered as a Lyapunov exponent for fluid flow.

Theorem 2.1.

$$r_{\text{ess}}(e^{Lt}) = e^{\mu t}$$

where

$$\mu = \lim_{t \rightarrow \infty} \frac{1}{t} \log \sup_{\substack{x_0, \xi_0, b_0 \\ b_0 \cdot \xi_0 = 0 \\ |b_0| = 1, |\xi_0| = 1}} |b(x_0, \xi_0, b_0; t)|. \quad (2.1)$$

Here the vector b is determined by the following system of ODE's which we call the bicharacteristic amplitude equations.

$$\left. \begin{aligned} \dot{x} &= U_0(x) \\ \dot{\xi} &= - \left(\frac{\partial U_0}{\partial x} \right)^T \xi \\ \dot{b} &= - \left(\frac{\partial U_0}{\partial x} \right) b + 2 \left[\left(\frac{\partial U_0}{\partial x} \right) b \cdot \xi \right] \xi / |\xi|^2 \end{aligned} \right\} \quad \begin{aligned} (2.2.1) \\ (2.2.2) \\ (2.2.3) \end{aligned}$$

with $x(0) = x_0$, $\xi(0) = \xi_0$, $b(0) = b_0$.

This theorem is proved by writing the evolution operator e^{Lt} as a product of a pseudodifferential operator and a shift operator along the trajectories of the equilibrium flow U_0 . This allows the growth of the evolution operator to be studied to precise exponential asymptotics. This approach refines the result of Vishik and Friedlander [4] which proves that $e^{\mu t}$ gives a lower bound for $r_{\text{ess}}(e^{Lt})$. A heuristic derivation of this result is obtained by applying a “geometric optics” treatment based on high frequency perturbations to equations (1.4). In the language of geometric optics, equation (2.2.2) is the Eikonal equation and equation (2.2.3) is the transport equation. Equation (2.2.3) is the evolution equation for the amplitude of a high frequency wavelet initially localised at x_0 , with initial wave number vector ξ_0 . The quantity on the RHS of (2.1) is the Lyapunov exponent corresponding to the maximal exponential growth rate of such an amplitude vector $b(t)$.

The result of Theorem 2.1 gives one piece of information concerning the stability spectrum for inviscid flows, namely the maximum growth rate of instability in the essential spectrum. Moreover it implies that any point z in the spectrum $\sigma(e^{Lt})$ such that $|z| > e^{\mu t}$ is necessarily an eigenvalue of finite multiplicity. As we will discuss later, a positive lower bound for the value of the Lyapunov exponent μ can be explicitly computed in several examples [5], [6]. Furthermore Theorem 2.1 provides an effective sufficient condition for instability of large classes of inviscid fluid flows. Since expression (2.1) involves the supremum over initial conditions (x_0, ξ_0, b_0) , it is only necessary to show there exists at least one initial condition for which the solution to the system (2.2) of ODE gives

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log |b| > 0$$

to conclude that $\mu > 0$, and hence the unstable essential spectrum is non-empty.

3. Examples of Instability in the Essential Spectrum

The idea that exponential stretching of fluid particles could imply instability for the Euler equations is originally due to Arnold [7]. Friedlander and Vishik [6] use the result discussed in Section 2 to prove every flow with exponential stretching somewhere in the flow is linearly unstable.

Theorem 3.1. (*3-dimensions*).

Let the flow U_0 have a positive classical Lyapunov exponent at some point x_0 . Then $\mu > 0$, and hence the flow U_0 is unstable.

The proof of this theorem follows from a result obtained from the system (2.2) of ODE, namely:

$$\frac{d}{dt} (b_1 \times b_2 \cdot \xi) = 0 \quad (3.1)$$

where b_1 and b_2 are two linearly independent solutions of (2.2.3) corresponding to a co-tangent vector ξ that satisfies (2.2.2). Since the flow is volume preserving the existence of a positive classical Lyapunov exponent (i.e. an exponentially growing tangent vector) implies the existence of an exponentially decaying co-tangent vector ξ . From (3.1) we then conclude the existence of at least one exponentially growing amplitude vector b , which implies $\mu > 0$. Therefore we note that in 3-dimensions the classical Lyapunov exponent provides a lower bound on μ .

In 2-dimensions, the system (2.2) of ODE provides an even stronger constraint on the relation between $b(t)$ and $\xi(t)$, namely

$$\frac{d}{dt} (|b(t)| |\xi(t)|) = 0. \quad (3.2)$$

Hence in 2-dimensions the fluid Lyapunov exponent μ and the maximal classical Lyapunov exponent for the dynamical system $\dot{x} = U_0(x)$ are the same.

It follows from Theorem 3.1 that any flow U_0 (in 2 or 3 dimensions) with a hyperbolic stagnation point x_0 is unstable [i.e. $U(x_0) = 0$ and there exists an eigenvalue of $(\frac{\partial U}{\partial x})_{x_0}$ with positive real part]. There are large classes of fluid flows U_0 with such stagnation points. For all such flows $r_{\text{ess}} > 1$.

A class of flows with presumably chaotic stream lines was identified by Arnold [8]. An example is the so-called *ABC* flow $U_0 = (\dot{x}, \dot{y}, \dot{z})$ where

$$\begin{aligned} \dot{x} &= A \sin z + C \cos y \\ \dot{y} &= B \sin x + A \cos z \\ \dot{z} &= C \sin y + B \cos x. \end{aligned} \quad (3.3)$$

For general values of the constants A , B and C numerical investigations [9], [10] indicate that *ABC* flows exhibit the phenomenon of Lagrangian chaos which suggests strong exponential stretching. Analytic treatment of *ABC* flows [11], [12], proves that for certain ranges of A , B and C there is exponential stretching either at hyperbolic points or associated with hyperbolic closed trajectories. The result of Theorem 3.1 then proves that these *ABC* flows are hydrodynamically unstable.

In 3-dimensions (unlike 2-dimensions), it is possible to have flows U_0 for which the classical Lyapunov exponents are all zero yet the fluid exponent μ is positive. Such an example is constructed in [6]. It is proved that the integrable flow $U_0 \times \text{curl } U_0 = -\nabla H$

with $\nabla H \neq 0$ has μ positive provided that a certain geometric condition is satisfied by the stream lines. The following (non sharp) condition ensures $\mu > 0$:

$$\int_0^T \{ \kappa \hat{n} \cdot \nabla H - \tau_g U_0 \cdot \text{curl } U_0 / |\nabla H|^2 \} dt \geq 0 \quad (3.4)$$

where, for any stream line of the flow as it wraps around the toroidal surface $H = H_0$, T is the period, κ the curvature, τ_g the geodesic torsion and $\hat{n}$ the principal unit normal to the stream line.

We have described many fluid flows where it can be shown that $\mu > 0$. In a few cases μ can be computed explicitly. For example, the 2-dimensional cellular flow

$$U_0 = (-\sin x \cos y, \cos x \sin y).$$

In this case μ is given by the positive real eigenvalue of the matrix $(\frac{\partial U_0}{\partial x})$ at a hyperbolic stagnation point. Thus $\mu = 1$ for this simple cellular flow. There are certain classes of 2-dimensional flows for which it follows from (3.2) that $\mu = 0$ [13]. In particular $\mu = 0$ (i.e. there is no unstable essential spectrum) for

- (1) 2-dimensional flows with no stagnation point,
- (2) 2-dimensional plane parallel shear flows.

4. Examples of Instability in the Discrete Spectrum

We now turn to the question of existence and distribution of unstable eigenvalues in the discrete spectrum of equation (1.4). As we remarked, the linearised Euler operator is degenerate, non-elliptic and there are no general theorems that may be applied to prove the existence of unstable discrete eigenvalues. However in certain rather special examples it is possible to construct unstable eigenvalues.

The spectral problem for the linearised Euler operator is considerably simpler in 2-dimensions rather than 3-dimensions. In particular, in 2-dimensions we can define a scalar stream function to replace the divergence free velocity field. We write

$$U_0 = \hat{k} \times \nabla \Psi(x, y), \quad v = \hat{k} \times \nabla \phi(x, y, t). \quad (4.1)$$

Hence

$$\nabla \times U_0 = \hat{k} \nabla^2 \Psi(x, y), \quad \nabla \times v = \hat{k} \nabla^2 \phi(x, y, t). \quad (4.2)$$

Here $\hat{k}$ is the unit vector perpendicular to the 2-dimensional plane with cartesian coordinates (x, y) . The equilibrium equation (1.3) will be satisfied when Ψ satisfies an elliptic equation of the form

$$\nabla^2 \Psi = -F(\Psi). \quad (4.3)$$

Taking the curl of equation (1.4) gives the equation for the evolution of the perturbation vorticity $\omega \equiv \nabla \times v$:

$$\frac{\partial \omega}{\partial t} = \{U_0, \omega\} + \{v, \nabla \times U_0\} \quad (4.4)$$

where $\{ , \}$ denotes the Poisson bracket of two vector fields, i.e.

$$\{A, B\} = (B \cdot \nabla)A - (A \cdot \nabla)B. \quad (4.5)$$

In general the second Poisson bracket on the RHS of (4.4) is very difficult to analyze. However in 2-dimensions the problem greatly simplifies because $\widehat{k} \cdot \nabla(\cdot) \equiv 0$. We note that for 2-dimensional flows it is proved in [14] that there is a solution of the linearised Euler equation (1.4) of the form $e^{\sigma t} v(x)$ with $v \in L^2$, $\nabla \cdot v = 0$, if and only if there is a non-trivial solution $\omega \in H^{-1}$ of the spectral problem for the 2-dimensional reduction of equation (4.4) given by

$$\sigma \omega + (U_0 \cdot \nabla) \omega + (v \cdot \nabla)(\nabla \times U_0) = 0. \quad (4.6)$$

We therefore consider the eigenvalue PDE given by substituting (4.1) – (4.3) into (4.6), namely

$$\sigma \nabla^2 \phi = \left(\Psi_y \frac{\partial}{\partial x} - \Psi_x \frac{\partial}{\partial y} \right) (\nabla^2 \phi + F'(\Psi) \phi). \quad (4.7)$$

We take the boundary conditions to be 2π periodicity in (x, y) .

A simple and very classical example that has received much attention in the literature of the past 100 years is plane parallel shear flow (see, for example, [15], [16], [17]). In this case $U_0 = (U(y), 0)$ and (4.7) becomes the so-called Rayleigh equation:

$$(\sigma/ik + U(y)) \left(\frac{d^2}{dy^2} - k^2 \right) \Phi(y) - U''(y) \Phi(y) = 0 \quad (4.8)$$

where we have written

$$\phi(x, y, t) = \Phi(y) e^{ikx} e^{\sigma t}. \quad (4.9)$$

The celebrated Rayleigh stability criterion [15] says that a necessary condition for instability is the presence of an inflection point in the profile $U(y)$. As we remarked in Section 3, the concept of the “fluid Lyapunov exponent” μ given by expression (2.1), can be used to prove that equation (4.8) with periodic boundary conditions has no unstable essential spectrum for any profile $U(y)$. It remains to discuss the possibility of discrete unstable eigenvalues (i.e. σ such that $\text{Re } \sigma > 0$) associated with equation (4.8) for profiles $U(y)$ that contain at least one inflection point.

Meshalkin and Sinai [18], followed by Yudovich [19] investigated the instability of a *viscous* shear flow $U(y) = \sin my$ using techniques of continued fractions. More recently Friedlander et al [13], [20], [21] showed that these techniques could be used for the inviscid equation (4.8) with $U(y) = \sin my$. Eigenfunctions are constructed in terms of Fourier series that converge to C^∞ -smooth functions for eigenvalues σ that satisfy the characteristic equation. We write

$$\Phi(y) = \sum_{n=-\infty}^{\infty} a_n e^{iny}. \quad (4.10)$$

The recurrence relation equivalent to (4.8) yields the following tridiagonal infinite algebraic system:

$$d_{n+m} = \beta_n(\sigma) d_n + d_{n-m}, \quad n \in \mathbb{Z} \quad (4.11)$$

where

$$\beta_n(\sigma) = \frac{2\sigma}{k} \cdot \frac{(k^2 + n^2)}{k^2 + n^2 - m^2} \quad (4.12)$$

and

$$d_n = a_n(k^2 + n^2 - m^2). \quad (4.13)$$

The system (4.11) is treated using continued fractions to yield the characteristic equation for σ namely

$$\left(\beta_j + \frac{1}{[\beta_{j+m}, \beta_{j+2m}, \dots]} \right) \left(\beta_{j-m} + \frac{1}{[\beta_{j-2m}, \beta_{j-3m}, \dots]} \right) + 1 = 0 \quad (4.14)$$

for each integer $j = 0, 1, \dots, [m/2]$. Here $[\dots]$ denotes an infinite continued fraction, i.e.

$$[\beta_{j+m}, \beta_{j+2m}, \beta_{j+3m}, \dots] = \frac{1}{\beta_{j+m} + \frac{1}{\beta_{j+2m} + \frac{1}{\beta_{j+3m} + \dots}}} \quad (4.15)$$

In [20], [21] an analysis is given of all the roots σ with $\text{Re } \sigma > 0$, of equation (4.14) for each fixed integer m and $0 \leq j \leq m/2$. It is shown that for $k^2 > m^2 - j^2$ there are no eigenvalues. For $k_c^2 < k^2 < m^2 - j^2$, for a critical wave number k_c , there are purely real roots and for $0 < k^2 < k_c^2$ there is a complex conjugate pair of roots. This describes the complete unstable spectrum associated with the flows $U = \sin my$.

The existence of unstable eigenvalues for equation (4.8) with a general rapidly oscillating profile $U(my)$, $m \gg 1$, was demonstrated in [21] using homogeneization techniques to compute the spectral asymptotics. It is proved that one root of the asymptotic characteristic equation is given by

$$\sigma/k = \pm \sqrt{\langle U^2 \rangle} \quad \text{as } m \rightarrow \infty \quad (4.16)$$

where $\langle \cdot \rangle$ denotes the 2π -average with respect to the fast variable my . Numerical analysis is used to describe the qualitative behaviour of the distribution of roots of the asymptotic characteristic equation in the parameter space of the wave numbers n and k .

The following is an example of a stream function Ψ satisfying an equation of the form (4.3) which exhibits both the features of exponential stretching at a hyperbolic stagnation point and oscillatory shear flow behaviour. We consider

$$\Psi = \frac{1}{m} (\cos(x + my) + a \cos(x - my)) \quad (4.17)$$

where a is a constant such that $0 < a < 1$. This flow has hyperbolic points at

$$x + my = 2n\pi, \quad x - my = (2j - 1)\pi.$$

Hence by the results of Section 3 there is a non empty unstable essential spectrum associated with this flow. The fluid Lyapunov exponent μ can be calculated explicitly in this example to give

$$\mu = 2a^{1/2}. \quad (4.17a)$$

Again homogenisation techniques can be used to demonstrate the existence of unstable eigenvalues for equation (4.7) with Ψ given by (4.17) with $m \gg 1$ [22]. We introduce a change of variables $\eta = my$ and write (4.7) in the form

$$\begin{aligned} & \sigma \left(\frac{\partial^2}{\partial x^2} + m^2 \frac{\partial^2}{\partial \eta^2} \right) \phi \\ & + \left[(\sin(x + \eta) - a \sin(x - \eta)) \frac{\partial}{\partial x} - (\sin(x + \eta) + a \sin(x - \eta)) \frac{\partial}{\partial \eta} \right] \\ & \cdot \left[\frac{\partial^2}{\partial x^2} + m^2 \frac{\partial^2}{\partial \eta^2} + (1 + m^2) \right] \phi = 0. \end{aligned} \quad (4.18)$$

We seek Bloch eigenfunctions of the form

$$\phi = e^{ip\eta/m} G(x, \eta) \quad (4.19)$$

where p is an integer such that $p \ll m$, and we seek smooth functions $G(x, \eta)$ given by a regular expansion in powers of $1/m$:

$$\begin{aligned} G(x, \eta) &= \Phi_0(x) + \tilde{\phi}_0(x, \eta) + \frac{1}{m} (\Phi_1(x) + \tilde{\phi}_1(x, \eta)) + \dots \\ \text{with } \langle \tilde{\phi}_j \rangle &= \int_0^{2\pi} \tilde{\phi}_j d\eta = 0 \\ \text{and } \sigma &= \sigma_0 + \frac{1}{m} \sigma_1 + \dots \end{aligned} \quad (4.20)$$

The solvability conditions for the hierarchy of equations for $\tilde{\phi}_j$ gives the ODE that define the averaged part of the eigenfunctions Φ_j . The equation for Φ_0 becomes

$$\begin{aligned} & \sigma_0^2 \left(\frac{d^2}{dx^2} - p^2 \right) \Phi_0 + \left(\frac{1+a^2}{2} + a \cos 2x \right) \left(\frac{d^2}{dx^2} - p^2 \right) \frac{d^2 \Phi_0}{dx^2} \\ & - 4a \sin 2x \left(\frac{d^2}{dx^2} - p^2 \right) \frac{d\Phi_0}{dx} - 4a \cos 2x \frac{d^2}{dx^2} \Phi_0 = 0. \end{aligned} \quad (4.21)$$

We consider the behaviour of (4.21) with the parameter a . In the limit $a = 0$ equation (4.21) reduces to the equation governing the spectral asymptotics for oscillatory shear flows. When $a = 1$ the equation is degenerate at the hyperbolic stagnation points where $\cos 2x = -1$. In this case the flow has no shear flow type regions and the degeneracy of (4.21) implies there are no discrete unstable eigenvalues with smooth eigenfunctions. For $0 < a < 1$, equation (4.21) has discrete real eigenvalues σ_0 which are the leading order part

of an expansion in powers of $(1/m)$ of the eigenvalues of (4.18). For large wave number k , Weyl's formula gives the asymptotic behaviour of σ_0 as

$$\sigma_0^2/k^2 \rightarrow (1 - a^2)/2 \quad (4.22)$$

which as $a \rightarrow 0$ is in agreement with expression (4.16) for the purely sheared flow.

As we remarked, the fully 3-dimensional vorticity equation (4.4) is very difficult to study. However some "2 1/2" dimensional examples are tractable. For example, the stability problem of a three component velocity of the form

$$U(x, y) = \hat{k} \times \nabla \Psi + \hat{k} g(\Psi) \quad (4.23)$$

is governed by an equation for a z -independent perturbation of the form of equation (4.7). A specific example is the "ABC" type flow with

$$\dot{x} = \cos my, \quad \dot{y} = \frac{1}{m} \sin mx, \quad \dot{z} = \sin my + \frac{1}{m} \cos mx. \quad (4.24)$$

For large m , averaging techniques again lead to an equation for the average with respect to my of the leading order eigenfunction $\Phi_0(x)$:

$$\sigma_0^2 \left(\frac{d^2}{dx^2} - p^2 \right) \Phi_0 + \frac{1}{2} \left(\frac{d^2}{dx^2} - p^2 \right) \frac{d^2 \Phi_0}{dx^2} = 0. \quad (4.25)$$

Hence

$$\sigma_0^2/k^2 = 1/2$$

and we conclude there exist discrete unstable eigenvalues.

The 3-dimensional general *ABC* flow given by (3.3) is not integrable. It has been investigated numerically by a number of authors including Henon [9] and Dombre et al [10]. It appears that the Lagrangian trajectories of the flow (3.3) densely fill some open domains of T^3 . Stagnation points may occur and when they do there is numerical evidence that they are connected by a web of heteroclinic streamlines. Poincaré sections computed for more general *ABC* flows include patterns that are qualitatively similar to the streamlines of the special flow given by (4.24). This suggests that the unstable spectrum associated with general *ABC* flows contains both an essential part and discrete eigenvalues.

5. Nonlinear Instability

In the previous sections we have discussed the unstable spectrum of the Euler equation linearised about a given steady flow U_0 . The following general theorem of Friedlander et al [13] can be applied to prove that some of the examples we considered are also nonlinearly unstable.

Consider the evolution equation

$$w_t = Lw + N(w), \quad w(0) = w_0 \quad (5.1)$$

and fix a pair of Banach spaces $X \hookrightarrow Z$ with a dense embedding. Here L is the generator of a C_0 -group of operators in $\mathcal{L}(Z)$, e^{Lt} leaves X invariant for $t \in \mathbb{R}$, $X \subset D(L)$ and N is a nonlinear operator $N : X \rightarrow Z$. We assume there is *local* existence theorem for (5.1) for $w_0 \in X$.

Definition of nonlinear stability:

The trivial solution $w = 0$ of (5.1) is called nonlinearly stable in X if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|w(0)\|_X < \delta$ implies

(i) there exists a unique solution $w(t) \in L^\infty((0, \infty); X) \cap C([0, \infty), Z)$
and

(ii) $\|w(t)\|_X < \varepsilon$ for a.e. $t \in [0, \infty)$.

The trivial solution $w = 0$ is called nonlinearly unstable if it is not stable.

Note: by this definition the “blowing up” of a solution is a particular case of instability.

Theorem 5.1. *Let (5.1) admit a local existence theorem in X . Let N and L satisfy the following conditions.*

$$(1) \quad \|N(w)\|_Z \leq C_0 \|w\|_X \|w\|_Z \text{ for } w \in X \text{ with } \|w\|_X < \rho \text{ for some } \rho > 0. \quad (5.2)$$

(2) A spectral “gap” condition, i.e.

$$\sigma(e^{Lt}) = \sigma_+ \cup \sigma_- \text{ with } \sigma_+ \neq \emptyset$$

where

$$\begin{aligned} \sigma_+ &\subset \{z \in \mathbb{C} \mid e^{Mt} < |z| < e^{\Lambda t}\} \\ \sigma_- &\subset \{z \in \mathbb{C} \mid e^{\lambda t} < |z| < e^{\mu t}\} \end{aligned} \quad (5.3)$$

with

$$-\infty < \lambda < \mu < M < \Lambda < \infty \text{ and } M > 0. \quad (5.4)$$

Then the trivial solution $w = 0$ to equation (5.1) is nonlinearly unstable.

The proof of this theorem is by contradiction and it uses the gap condition to permit a projection onto the subspace of growing modes that is guaranteed to exist by virtue of the spectral gap. We note that a stronger result, i.e. replacing condition (2) by the condition that the unstable spectrum $\sigma(e^{Lt})$ is non-empty, may in fact be true but would require a different method of proof.

We apply Theorem 5.1 to the Euler equations in the following way. We rewrite (1.1) – (1.2) with the notation

$$q = U_0 + w \quad (5.5)$$

where $U_0 \in (C^\infty(T^n))^n$ is a steady solution of (1.1), i.e.

$$w_t = -(U_0 \cdot \nabla) w - (w \cdot \nabla) U_0 - (w \cdot \nabla) w - \nabla P \quad (5.6)$$

$$\nabla \cdot w = 0. \quad (5.7)$$

Hence in this notation

$$Lw \equiv -(U_0 \cdot \nabla)w - (w \cdot \nabla)U_0 - \nabla p \quad (5.8)$$

$$N(w) \equiv -(w \cdot \nabla)w - \nabla q \quad (5.9)$$

and instability of the trivial solution $w = 0$ of the equation $w_t = Lw + N(w)$ corresponds to instability of the steady Euler flow U_0 . We make the “natural” choices of the spaces X and Z :

$$X = X_s = \{w \in (H^s(T^n))^n \mid \nabla \cdot w = 0\}, \quad s > \frac{n}{2} + 1 \quad (5.10)$$

$$Z = \{w \in (L^2(T^n))^n \mid \nabla \cdot w = 0\}. \quad (5.11)$$

For this choice of X local existence is a classical result [23] and condition (5.2) follows from the Sobolev embedding theorem.

The condition that is difficult to check is the spectral gap condition (5.3) – (5.4). We note that in the case of the Navier-Stokes equations in a finite domain the additional viscous term $\nu \nabla^2 q$ ensures that the equations are elliptic. Hence the spectrum is purely discrete and the existence of any unstable eigenvalue ensures the gap condition is satisfied. For the Euler equations, we return to the examples of Sections 3 and 4 where the structure of the unstable spectrum was analysed for certain choices of U_0 .

In Section 4 we discussed how any plane parallel shear flow with a sufficiently oscillating profile has a non-empty discrete unstable spectrum. The results of Section 3 can be used to show that the unstable essential spectrum associated with this flow is empty. Hence the spectral gap condition required by the proof of Theorem 5.1 is satisfied and therefore all shear flows with rapidly oscillating profiles are nonlinearly unstable. We note this result is proved with periodic boundary conditions. The cellular 2-dimensional flow given by (4.17) is also shown to have discrete unstable eigenvalues. In this case the essential unstable spectrum is non-empty. However for $a \neq 1$ the computable expressions for the essential spectral radius (see (4.17a)) and the spectral asymptotics (see (4.22)) show that there exist unstable eigenvalues outside the essential spectral radius. Hence again we can apply Theorem 5.1 to prove nonlinear instability.

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References

- [1] Constantine, P., A few results and open problems regarding incompressible fluids, Notices of AMS, 42, n° 6 (1995), 658-663.
- [2] Nussbaum, R., The radius of the essential spectrum, Duke Math. J., 37 (1970), 473-478.

- [3] Vishik, M.M., The spectrum of small oscillations of an ideal fluid and Lyapunov exponents, *J. Math. Pures et Appl.*, 75 (1996), 531-557.
- [4] Vishik, M.M. and Friedlander, S., Dynamo theory methods for hydrodynamic stability, *J. Math. Pures et Appl.*, 72 (1993), 145-180.
- [5] Friedlander, S. and Vishik, M.M., Instability criteria for the flow of an inviscid incompressible fluid, *Phys. Rev. Lett.*, 66, n° 17 (1991), 2204-2206.
- [6] Friedlander, S. and Vishik, M.M., Instability criteria for steady flows of a perfect fluid, *Chaos*, 2, n° 3 (1992), 455-460.
- [7] Arnold, V.I., Notes on the 3-dimensional flow pattern of a perfect fluid in the presence of small perturbation of the initial velocity field, *Appl. Math. Mech.*, 36, n° 2 (1972), 236-242.
- [8] Arnold, V.I., Sur la topologie des écoulements stationnaires des fluides parfaits, *C.R. Acad. Sci. Paris*, 261 (1965), 17-20.
- [9] Henon, M., Sur la topologie des lignes de courant dans un cas particulier, *C.R. Acad. Sci. Paris*, 262 (1966), 312.
- [10] Dombre, T., Frisch, U., Greene, J.M., Henon, M., Mehr, A. and Soward, A., Chaotic streamlines in *ABC* flows, *J. Fluid. Mech.*, 167 (1986), 353-391.
- [11] Friedlander, S., Gilbert, A. and Vishik, M.M., Hydrodynamic instability for certain *ABC* flows, *Geophys. Astrophys. Fluid Dyn.*, 73 (1993), 97-107.
- [12] Chicone, C., A geometric approach to regular perturbation theory with application to hydrodynamics, *Trans. AMS*, 347, n° 12 (1995), 4559-4598.
- [13] Friedlander, S., Strauss, W. and Vishik, M.M., Nonlinear instability in an ideal fluid, *Ann. Inst. Henri Poincaré, Analyse non Linéaire* 14, n° 2 (1997), 187-209.
- [14] Friedlander, S., Strauss, W. and Vishik, M.M., Robustness of instability for the two dimensional Euler equations, submitted for publication.
- [15] Rayleigh (Lord), On the stability or instability of certain fluid motions, *Proc. Lond. Math. Soc.*, 9 (1880), 57-70.
- [16] Tollmein, W., Ein allgenienes kriterium der instabilitat laminarer geschwindigkeitsverteilungen, *Nachr. Ges. Wiss. Gottingen Math. Phys.*, 50 (1935), 79-114.
- [17] Lin, C.C., *The Theory of Hydrodynamic Stability*, Cambridge University Press (1955).
- [18] Meshalkin, L. and Sinai, Y., Investigation of the stability of a stationary solution of a system of equations for the plane movement of an incompressible viscous liquid, *App. Math. and Mech.*, 25 (1961), 1140-1143.
- [19] Yudovich, V.I., An example of secondary stationary flow or periodic flow appearing while a laminar flow of a viscous incompressible fluid loses its stability, *App. Math. and Mech.*, 29 (1965), 453-467.
- [20] Friedlander, S. and Howard, L., Instability in parallel flows revisited, *Studies in App. Math.*, to appear 1998.
- [21] Belenkaya, L., Friedlander, S. and Yudovich, V., The unstable spectrum of oscillating shear flows, *SIAM J. App. Math.*, to appear 1998.

- [22] Friedlander, S., Vishik, M.M. and Yudovich, V., The unstable spectrum of a 2-dimensional periodic flow of an ideal fluid, in preparation.
- [23] Temam, R., Local existence of C^∞ solutions of the Euler equations of incompressible perfect fluids, Lecture Notes in Math., 565 (1976), 184-194.

Susan FRIEDLANDER

Department of Mathematics (M/C 249)

University of Illinois-Chicago

Chicago, Illinois 60607

e-mail : susan@math.nwu.edu