Quantum mechanics and the Continuum Problem

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Abstract

A new approach to quantum mechanics based on independence of the Continuum Hypothesis is proposed. In one-dimensional case, it is shown that the basic principles of quantum mechanics are properties of the set of intermediate cardinality and of the simplest map from the intermediate set to the set of real numbers.

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The concept of discrete space is always regarded as the unique alternative of the continuous space. Nevertheless, it is important to stress that there is the intermediate possibility connected with the Continuum Problem (discrete space is a countable set). According to the independence of the Continuum Hypothesis (CH) we can neither prove nor disprove existence of a set of cardinality between cardinalities of countably infinite set and continuum. If Zermelo-Fraenkel set theory is consistent and complete, then the uncertain status of the intermediate set is unavoidable: no one can state that the set does not exist but at the same time we in principle cannot get it anyhow. In other words, uncertainty seems to be a property of the intermediate set. On the other hand, this set standing midway between countable set and continuum may combine properties of continuity (wave) and countability (particle). Hence, Wave-Particle duality may be considered as the direct pointing to the Continuum Problem and the intermediate set. Let us show that the basic principles of quantum mechanics reduce to properties of the set of intermediate cardinality.

Suppose there exists a set $S$ such that

$$\text{card}(N) < \text{card}(S) < \text{card}(R),$$

where $N$ is the set of natural numbers, $R$ is the set of all real numbers (let $S$ be called an interset). Then $R$ contains a subset $M$ equivalent to $S$, i.e.,

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there exists a bijection $f : S \rightarrow M = f(S) \subset R$ separating $M$ from the set of real numbers. But it follows from independence of CH that the interset cannot be separated from continuum: any procedure of separation is a proof of the negation of the Continuum Hypothesis. Hence, for any real number $r \in R$ we do not know, in principle, if the sentence $r \in M$ is true or false. Therefore any map cannot take a point $s \in S$ to a unique real number as well as to a definite subset of $R$ (the uncertain status of the intermediate set causes uncertainty of its members with respect to real numbers).

Then the point $s$ corresponds to entire continuum $R$ and we can assign to the point a function $\psi(r)$ defined on the same domain $R$. If a map takes $s$ to a random real number, then the function $\psi(r)$ has to be connected with probability $P(r)$ of finding the point at $r$: $P(r) = P(\psi(r))$. For two intervals of real numbers $a$ and $b$ probability $P_{a+b} \neq P_a + P_b$ because the point $s$ corresponds to both intervals at the same time (the events are not mutually exclusive). Then it is convenient to choose the function $\psi(r)$ such that

$$\psi_{a+b} = \psi_a + \psi_b.$$  \hspace{1cm} (1)

But

$$P_{a+b} = P(\psi_{a+b}) = P(\psi_a + \psi_b) \neq P(\psi_a) + P(\psi_b),$$

i.e., the dependence $P(\psi(r))$ is nonlinear. The simplest nonlinear dependence is a square dependence:

$$P(r) = |\psi(r)|^2.$$  \hspace{1cm} (2)

So we have superposition principle (1) and Born postulate (2).

CH is under discussion for more than one hundred years but no one can find in the set theory literature any description of probable properties of the interset. It is due to the following reason. According to the separation axiom schema for any set $X$ and for any property expressed by formula $\varphi$ there exists a subset of the set $X$, which contains only members of $X$ having $\varphi$:

$$\forall X \exists Y \forall u (u \in Y \leftrightarrow u \in X \land \varphi(u)).$$

If the interset cannot be separated from $R$, then its members have no exceptional properties with respect to continuum (uncertainty is not a set theory property). Hence, the properties of the interset are

1. uncertainty with respect to continuum,
2. separate properties of countable sets and continuum,
3. undecidable equivalents of the negation of CH.
Only these properties do not contradict the independence of CH because they do not allow separation of the interset from continuum in accordance with the separation axiom schema. Item (2) may be called a complementarity principle for set theory. Moreover, we claim that any description of the intermediate set must be based on combining separate properties of the countable set and continuum.

Now we know why the set theorists do not like to discuss probable properties of the interset: in some sense, it has no its own properties. It may be stated that this is the reason of independence of CH. If the interset had any unique property, then it would be possible to prove its existence.

The sets of natural and real numbers have certain structures, which may be called natural. In consequence of the complementarity principle, the structure of the interset must be a combination of the natural structures of the sets of natural and real numbers. Let us form such a combination.

Any interval of continuum is equivalent to any other interval and to entire continuum:

\[ \text{card}(R) = \text{card}((0, 10^{10^{\ldots}})) = \ldots = \text{card}((0, 10^{-10^{\ldots}})) = \ldots \]

In other words, any arbitrarily small interval contains the same number of points as the set of all real numbers. We will consider this property as a unique property of continuum. And for the interset we shall substitute the complementary property of the set of natural numbers for the property of continuum: there exists a unit (minimal) set and, consequently, different intervals of the interset are not equivalent (the above intervals of real numbers contain subsets of different intermediate cardinalities). Cardinality of the unit set is an infinite fundamental constant.

Coordinate of a point in the interset in units of the minimal set is a natural number \( n \). The function \( \psi \), necessarily, depends on \( n \): \( \psi(r) = \psi(n, r) \). Since \( n \) is accurate up to a constant (shift) and the function \( \psi \) is defined up to the factor \( e^{i\text{const}} \), we have

\[ \psi(n + \text{const}) = e^{i\text{const}} \psi(n). \]

Hence, the function \( \psi \) is of the following form:

\[ \psi(r, n) = A(r)e^{in} = A(r)e^{i\nu t}, \]

where natural number \( \nu \) is the (constant) time rate of change of cardinality (number of the unit sets per second).

In accordance with the complementary description of the interset, cardinality of an interval of the interset depends on its size; then a macroscopic
interval is approaching continuum and may be satisfactorily described by real numbers and differential equations (actually, this is the definition of a macroscopic interval). Thus the function $\psi$ describes the real relationship between the levels of the intermediate set.

Consider the shortest path in the interset. In other words, consider the motion of the point $s$ in such a way that the number $m$ (cardinality of the path in units of the minimal set) is a minimum with reference to all other paths. If we apply the condition to the macroscopic motion we shall get one-dimensional Lagrangian mechanics because this statement may be considered as the general formulation of the principle of least action. Indeed, since for the macroscopic path $r(t)$ ($t_0 \leq t \leq t_1$) the number $m(t)$ is a functional $m = F(r(t))$, we get

$$m = \int_{t_0}^{t_1} \frac{dm}{dt} dt = \int_{t_0}^{t_1} \frac{dF(r(t))}{dr} dr dt.$$  

The impression under the integral is some function of $r(t)$, $\dot{r}$, and $t$. This is sufficient to identify cardinality with action

$$I = \int_{t_0}^{t_1} L(r(t), \dot{r}, t) dt.$$  

But cardinality is the dimensionless natural number, while the value of action depends on units of measurement. We have lost the natural unit because macroscopic cardinality $m$ is regarded as a continuous variable (note that this step is the inverse of quantization). Hence, we need a parameter $h$ depending on units only such that

$$mh = \int_{t_0}^{t_1} L(r(t), \dot{r}, t) dt.$$  

Subtracting $mh = I_m$ from $(m + 1)h = I_{m+1}$, we get

$$h = I_{m+1} - I_m,$$  

i.e., $h$ is the least change in action.

Recall that action for any classical path in units of quantum of action is really a dimensionless natural number. Its time rate of change has units of $s^{-1}$. In quantum mechanics the rate is called a frequency because it appears in the wave function.

Finally, the function $\psi$ has the form

$$\psi(r) = A(r)e^{im} = A(r)e^{i(I/h)} = A(r)e^{i(pr/h)}.$$  

4
Thus De Broglie wave represents the simplest map from the interset to continuum. This means that we can replace Wave-Particle duality by the mathematical concept.

We see that the intermediate set is absolutely new set of stepwise changing infinite cardinality (and consequently changing structure). In fact, this is a spectrum of sets. The countable set and continuum are the ends of the spectrum. Existence of infinite quantum of cardinality is a unique property of the set.

The description of the interset falls into two complementary parts: continuous and countable (quantum). It is interesting to note that, even in the case of pure geometry, we need some parameter $t$ for correlation between continuous and countable description of the intermediate set.