Inhomogeneous Multidimensional Cosmologies

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Abstract

Einstein’s equations for a 4+n-dimensional inhomogeneous space-time are presented, and an special family of solutions is exhibited for an arbitrary $n$. The solutions depend on two arbitrary functions of time. The time development of a particular member of this family is studied. This solution exhibits a singularity at $t = 0$ and dynamical compactification of the $n$ dimensions.

PACS number(s): 04.50.+h 98.80.-k 11.25.Mj

I. INTRODUCTION

Over the last two decades, increasing attention has been paid to theories that unify the fundamental interactions in more than three spatial dimensions. The story of this kind of theories started in the 20’s, when Kaluza [1] and Klein [2] augmented the dimensionality of space to describe both gravity and electromagnetism as manifestations of geometry, using the degrees of freedom available from the 5-dimensional metric tensor [3]. The idea was restored

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in the 60’s by deWit [4] who tried to incorporate non-abelian interactions into the scheme. The original idea of Kaluza and Klein turned out to be incomplete for several reasons, but it still pervades in one way or another many of the unifying schemes currently thought to be viable (most notably in the case of string theory. See for instance [5]). However, if we are willing to accept any of these theories in which space has more than three dimensions, we are faced with several questions, particularly on the cosmological side. Perhaps the most obvious one is related to the fact that we live in a 4-dimensional space-time, so every theory formulated in more than 4 dimensions must say something about the fate of the extra dimensions. A convenient working hypothesis would be to assume that they have been compactified up to some small size. From a theoretical point of view, the most satisfactory way to achieve the compactification of the extra dimensions would be the dynamical one. This means that the theory has solutions in which the size of the extra dimensions diminishes as the universe evolves. Solutions of this type have been found for the more symmetric cases both in 4+1 and 4+n dimensions [3], but only a few with some degree of inhomogeneity can be found in the literature, and always for the 4+1 case (see [6] and references therein). We will be concerned here with a 4+n dimensional model, with arbitrary n. This case may have a paramount importance, as shown by the recent work of Arkani-Hamed et al [7], in which the existence of n sub-millimeter dimensions (with n ≥ 2) yields a new framework for solving the hierarchy problem, which does not rely on supersymmetry or technicolor. The central idea of this scheme is that the existence of this extra dimensions brings quantum gravity to the TeV scale through the relationship between the Planck scales of the 4+n dimensional theory and the long-distance 4-dimensional theory. It must be remarked that the extra dimensions are supposed to have a characteristic length of less than a millimeter, in accordance with the lower bound at which gravity has been tested up to date [8]. In this framework, the fields of the Standard Model are localised on a 3-brane in the higher dimensional space. Many of the important consequences of these ideas in phenomenology, astrophysics, and cosmology can be found in [9]. Many papers related to these matters have appeared lately; we mention here only a few. Argyres et al [10] have studied the properties of black holes
with Schwarzschild radius smaller than the size of the extra dimensions, and concluded that
the spectrum of primordial black holes in a 4+n dimensional spacetime differs from the
usual one. Moreover, these primordial black holes would provide dark matter candidates
and seeds for early galaxy and QSO formation. Mirabelli et al [11] have recently analyzed
the missing-energy signatures that should be present in high-energy particle collisions due
to the radiation of gravitons if gravity is important at TeV scale. They argue that collision
experiments provide the strongest present constraint on the size of the extra dimensions.
Nath and Yamaguchi [12] have explored the effect of the excitations associated with extra
dimensions on the Fermi constant. They give stringent constraints on the compactification
radius from current precision determinations of the Fermi constant, of the fine structure
constant, an of the mass of the W and the Z bosons.

A salient feature of the model we propose here is its inhomogeneity. This type of models
might describe an early phase of the universe, or may be of use on a super-horizon scale,
as suggested by chaotic inflation [13]. Besides, the work of Mustapha et al [14] indicates
that there is no unquestionable observational evidence for spatial homogeneity. This makes
worthwhile analysing models that are isotropic but exhibit some degree of inhomogeneity
[15].

The aim of this paper is then to show the existence of analytical solutions in inhomoge-
neous cosmological models in 4 + n dimensions. Although some exact solutions for the 4+1
dimensional inhomogeneous case have been worked out [6] [16], the case dealt with here has
not been studied previously. Due to the complexity of the equations of motion (which have
not been displayed before in the literature in the case of an arbitrary n), the inhomogeneity
has been restricted to the n internal dimensions. We shall show that there exist solutions for
which the system becomes homogeneous for large t, while the extra dimensions compactify
due to the evolution of the system.

The starting point is the 4 + n dimensional metric, given by

\[ ds^2 = -dt^2 + e^{2\lambda(t,r)}(dr^2 + r^2 d\Omega^2) + e^{2\mu(t,r)}dy^2, \]  \tag{1}
where $d\Omega^2$ is the surface element on the 2-sphere, and $dy^2 = \sum_{i=4}^n dy_i^2$. For simplicity we will work with a plane 3-space, and we assume a single scale factor for the internal dimensions.

We adopt the following stress-energy tensor for the matter content of the model:

$$T_{\mu\nu} = \text{diag}(\rho, -p, -p, -p', ..., -p'),$$

with $p'$ the internal pressure. The nonvanishing field equations in this 4 + $n$ dimensional space-time are then

$$-2\dot{\lambda} - n\dot{\mu}' + n\dot{\lambda} + \dot{\mu} = 0 \quad (3a)$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda} + n\dot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[\frac{2\lambda'}{r} + \frac{2n}{r}\dot{\mu}' + 2n\dot{\lambda}' + \lambda' - \frac{n(1-n)}{2}\mu^2\right] = -8\pi p \quad (3b)$$

$$2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda} + n\dot{\mu} + \frac{n(n+1)}{2}\dot{\mu}^2 - e^{-2\lambda}\left[\lambda'' + \frac{\lambda'}{r} + n\mu'' + \frac{n}{r}\mu' + \frac{n(n+1)}{2}\mu^2\right] = -8\pi p \quad (3c)$$

$$(n-1)\ddot{\mu} + \frac{n(n-1)}{2}\dot{\mu}^2 + 3(n-1)\dot{\lambda} + 3\ddot{\lambda} + 6\dot{\lambda}^2 - e^{-2\lambda}\left[(n-1)\lambda' + \left(\frac{n(n-1)}{2}\mu^2 + \frac{2(n-1)}{r}\mu' + 2\lambda' + \frac{4}{r}\lambda + \lambda'\right)\right] = -8\pi p' \quad (3d)$$

$$3\ddot{\lambda} + \frac{n(n-1)}{2}\dot{\mu}^2 + 3n\dot{\lambda} + e^{-2\lambda}\left[2\lambda'' + n\mu'' + \frac{4}{r}\lambda' + n\lambda'\mu' + \left(\frac{n(n+1)}{2}\mu^2 + \frac{2n}{r}\mu' + \lambda^2\right)\right] = 8\pi \rho \quad (3e)$$

(as usual, a dot denotes derivative with respect to time, and a prime, with respect to the radial coordinate).

As was said above, we shall restrict here to the case $\lambda = \lambda(t)$, so the equations are the following:

$$-\dot{\mu}' + \dot{\lambda}\mu' - \dot{\mu}\mu' = 0 \quad (4a)$$
\[2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda} \dot{\mu} + n\ddot{\mu} + \frac{n(n + 1)}{2} \mu^2 - e^{-2\lambda} \left[ 2n \frac{\mu'}{r} - \frac{n(1 - n)}{2} \mu'^2 \right] = -8\pi p \]  
(4b)

\[2\ddot{\lambda} + 3\dot{\lambda}^2 + 2n\dot{\lambda} \dot{\mu} + n\ddot{\mu} + \frac{n(n + 1)}{2} \mu^2 - e^{-2\lambda} \left[ n\mu'' + \frac{n}{r} \mu' + \frac{n(n + 1)}{2} \mu'^2 \right] = -8\pi p \]  
(4c)

\[(n - 1)\ddot{\mu} + \frac{n(n - 1)}{2} \mu^2 + 3(n - 1)\dot{\lambda} \dot{\mu} + 3\ddot{\lambda} + 6\dot{\lambda}^2 - e^{-2\lambda} [(n - 1)\mu'' + \frac{n(n - 1)}{2} \mu'^2 + 2\frac{n}{r} \mu'] = -8\pi p' \]  
(4d)

\[3\dot{\lambda}^2 + \frac{n(n - 1)}{2} \mu^2 + 3n\dot{\lambda} \dot{\mu} - e^{-2\lambda} \left[ n\mu'' + \frac{n(n + 1)}{2} \mu'^2 + \frac{2n}{r} \mu' \right] = 8\pi \rho \]  
(4e)

In the case \( n = 1 \), these equations reduce to the ones given in Chaterjee et al [6].

It is easy to show that Eq.(4a) can be rewritten as

\[
\mu'' + \mu'^2 + \frac{1}{2} \mu' \Phi(r) = 0,
\]
(5)

where \( \Phi(r) \) is an arbitrary function. Besides, if we substract Eq.(4b) to Eq.(4c) we get

\[
\mu'' + \mu'^2 - \frac{1}{r} \mu' = 0.
\]
(6)

So for the last two equations to be compatible we must choose \( \Phi(r) = -\frac{2}{r} \). Eq.(6) is integrable and the result is

\[
e^{\mu(t, r)} = \beta(t)r^2 + \gamma(t),
\]
(7)

where \( \beta(t) \) and \( \gamma(t) \) are arbitrary functions of time. Now from Eqs.(4a) and (7) we get \( \dot{\lambda} = \frac{\dot{\beta}}{\beta} \), which yields

\[e^{\lambda(t)} = A \, \beta(t),\]
(8)

where \( A \) is an arbitrary constant. The remaining Eqs. (4b), (4d), and (4e) shall be taken as definitions of \( p(t, r) \), \( p'(t, r) \), and \( \rho(t, r) \), respectively.

The model may display several different features according to the explicit form of the functions \( \beta \) and \( \gamma \). In the following we will restrict to a particular choice of these functions,
but first we list certain quantities that will be of interest in the subsequent analysis: the scalar curvature of the 3 + \( n \) space, the Kretschmann scalar, the expansion scalar, and the shear scalar.

\[
R^{(3+n)} = -9\ddot{\lambda} - 21\dot{\lambda}^2 - 3n\ddot{\mu} - n(2n + 1)\dot{\mu}^2 - 12n\dot{\lambda}\dot{\mu} + ne^{-2\lambda} \left[ 2\dddot{\mu} + (1 + n)\dot{\mu}^2 + \frac{4}{r}\dot{\mu} \right]
\]  
\[\text{(9)}\]

\[
K = 24\dot{\lambda}^4 + 24\ddot{\lambda}\dot{\lambda}^2 + 12n\dot{\lambda}^2\dot{\mu}^2 + 2n(n + 1)\dot{\mu}^4 + 12\dddot{\lambda}^2 + 4n\dddot{\mu}^2 + 8n\dot{\mu}\dddot{\mu}^2 + e^{-2\lambda} \left[ -8n\dot{\lambda}^2\dot{\mu}^2 + 16n\dot{\mu}'\dot{\lambda}\dot{\mu}' - 16n\dot{\mu}'\dot{\mu}\dot{\mu}' - \frac{16n}{r}\dot{\lambda}\dot{\mu}' - 8n\dot{\mu}'^2 - 4n(n + 1)\dot{\mu}^2\dot{\mu}^2 + 8n\dot{\mu}\ddot{\mu}'\dot{\lambda}' - 8n\dot{\mu}\dot{\mu}\dot{\mu}' \right] + e^{-4\lambda} \left[ \frac{8n}{r^2}\dot{\mu}^2 + 2n(n + 1)\dddot{\mu}^4 + 4n\dddot{\mu}'^2 + 8n\dddot{\mu}'\dddot{\mu}^2 \right]
\]  
\[\text{(10)}\]

\[
\theta = 3\dot{\lambda} + n\dot{\mu}
\]  
\[\text{(11)}\]

\[
\sigma^2 = \sigma_{\mu\nu}\sigma^{\mu\nu} = (n + 4)\dot{\lambda}^2 + n\frac{n(n + 1) + 9}{9}\dddot{\mu}^2 + 2\frac{n(n + 1)}{3}\dddot{\mu}\dot{\lambda}.
\]  
\[\text{(12)}\]

The scalar curvature of the 3 + \( n \) space was calculated by means of the expression [17]

\[
R_{\mu}^{\mu} = R^{(3+n)} + \dot{\theta} + \theta^2 - 2\omega^2 - \dot{\omega}^{\mu}_{\mu}
\]  
\[\text{(13)}\]

(It can be seen that in this model the last two terms of this expression are null).

Now, if we want this model to describe some features of the universe we observe today, we must impose the conditions \( \dot{\lambda} > 0 \), and \( \dot{\mu} < 0 \) (the latter must be valid at least for the some part of the evolution of the universe). This can be easily achieved by taking advantage of the freedom we have in the arbitrary functions \( \beta \) and \( \gamma \). Let us take as an example

\[
\beta(t) = -b\ln(1 + t^\alpha) \quad \gamma(t) = a\ln(1 + t^\alpha) - kt^\delta.
\]  
\[\text{(14)}\]

The positively defined and arbitrary constants \( a, b, k, \alpha, \) and \( \delta \) are to be chosen in a convenient way. Note that in order that \( \dot{\mu} \) be negative from some point of the evolution onwards, we have to demand that \( a - br^2 > 0 \). The scale factor of the three space is monotonically increasing, while the scale factor of the internal dimensions grows until a time \( t_{\text{max}} \), given for each \( r \) by
\[ r^2 = \frac{a}{b} - \frac{k\delta}{b\alpha} t^{\delta - \alpha}_{\text{max}} (1 + t^\alpha_{\text{max}}), \] (15)

and compactifies dynamically to zero size at different times \( t_0 \) for each \( r \), given by

\[ r^2 = \frac{a \ln(1 + t_0^\alpha) - kt_0^\delta}{b \ln(1 + t_0^\alpha)}. \] (16)

We move now to the analysis of the asymptotic behaviour of the model. From the explicit expression of \( K \) and \( R \) it is seen that both of them diverge as \( t \to 0 \), the first one as \( t^{-4} \), and the second one as \( t^{-2} \). The existence of the initial singularity (expected from the absence of repulsive terms in Raychaudhuri’s equation [18] for this case) is confirmed by the divergence at \( t = 0 \) of \( \rho, p \) and \( p' \). At \( t = t_0 \), all the matter functions, the curvature scalars and the shear scalar diverge. It has been argued however [19] that there might exist some sort of stabilization mechanism (probably due to quantum gravity effects) which could prevent the formation of the final singularity [20]. This would allow the evolution of the ordinary 3-space independently of the internal (and microscopical) space, in such a way that \( e^\mu \) attains a constant value during all the subsequent evolution. In such a case, it can be seen from Eqs. (9)-(12) that the post-compactification phase of the model describes an ever-expanding effective 4-dimensional universe, with plane 3-sections, and asymptotically vanishing shear.

We have shown then that there exists a family of solutions, parameterized by the functions \( \beta \) and \( \gamma \), for the very complex system of equations corresponding to the case of a \( 4 + n \) inhomogeneous model. A general feature of this solutions is that the time at which dynamical compactification of the extra dimensions begins is different for each value of the \( r \) coordinate. The particular example that was analyzed here evolves from an inhomogeneous multidimensional phase to a 4-dimensional phase with asymptotically vanishing shear, in which the extra dimensions may attain a microscopic size. The important abovementioned consequences of the existence of these internal dimensions is enough motivation to search for more general solutions, which we hope to report in some future communication.

Acknowledgements: The author would like to thank CLAF-CNPq for financial support, and J. Salim and M. Novello for helpful comments.
REFERENCES


