Defining the Force between Separated Sources on a Light Front

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Abstract

The Newtonian character of gauge theories on a light front requires that the longitudinal momentum $P^+$, which plays the role of Newtonian mass, be conserved. This requirement conflicts with the standard definition of the force between two sources in terms of the minimal energy of quantum gauge fields in the presence of a quark and anti-quark pinned to points separated by a distance $R$. We propose that, on a light front, the force be defined by minimizing the energy of gauge fields in the presence of a quark and an anti-quark pinned to lines (1-branes) oriented in the longitudinal direction singled out by the light front and separated by a transverse distance $R$. Such sources will have a limited 1+1 dimensional dynamics. We study this proposal for weak coupling gauge theories by showing how it leads to the Coulomb force law. For QCD we also show how asymptotic freedom emerges by evaluating the S-matrix through one loop for the scattering of a particle in the $N_c$ representation of color $SU(N_c)$ on a 1-brane by a particle in the $\bar{N}_c$ representation of color on a parallel 1-brane separated from the first by a distance $R \ll 1/\Lambda_{QCD}$. Potential applications to the problem of confinement on a light front are discussed.

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1 Introduction

The description of QCD using light-cone methods has little in common with the more traditional description using Euclidean path integrals. In particular, the latter approach admits Wilson’s elegant criterion for confinement, that the functional average of the gauge invariant Wilson loop
\[ \text{Tr} P \exp iq \oint_C dx^\mu A_\mu \]
fall off as \( e^{-T_0 A} \) as the area, \( A \), spanned by \( C \) gets large [1]. Further, for numerical work the path integral has a natural discretization by Wilson’s lattice gauge theory, which has been very effectively exploited for finer and finer lattices [2]. An appealing feature of lattice gauge theory is the manifest gauge invariance—there is no need for gauge fixing. Although the lattice spacing breaks \( O(4) \) (Euclidean Lorentz) invariance, a discrete subgroup of \( O(4) \) remains, so that violations of \( O(4) \) are irrelevant in the continuum limit.

In sharp contrast, light-cone quantization is only truly useful in a completely fixed gauge \( A^+ \equiv (A^0 + A^3)/\sqrt{2} = 0 \). Moreover, the principle advantage of quantization on a light front is the possibility of interpreting the resulting quantum system in the language of non-relativistic quantum mechanics [3–5]: the manifest space-time symmetry is a Galilei subgroup of the Poincaré group generated by \( P^- \equiv H, P^k, M^{+k}, M^{12} \). A seventh generator \( P^+ \) plays the role of total Newtonian mass and is also conserved. The time in this Newtonian analogue is \( t \equiv (x^0 + x^3)/\sqrt{2} \), and if one passes to an imaginary time formalism in which \( it \rightarrow \tau \), one is led to a path integral formalism which has no easily interpreted relation to that of the original Euclidean gauge theory. For example, it is not obvious how Wilson’s confinement criterion can be implemented in this approach.

The problem is that the Wilson line refers to curves in coordinate space, whereas the most effective way to exploit light-front dynamics is to replace \( x^- \) by its conjugate \( p^+ \). The closest one can get to coordinate space is the “mixed” space \( x^+, x, p^+ \). Recall that an \( R \times T \) rectangular Wilson loop oriented in the time direction has the interpretation \( e^{-TE(R)} \) as \( T \rightarrow \infty \) where \( E(R) \) is the lowest energy of a quark and anti-quark pinned to points separated by a distance \( R \). Here we propose instead that we consider a quark and anti-quark pinned to parallel lines separated by a distance \( R \). If there is confinement, the lowest energy of this system should still be \( \sim T_0 R \) for large \( R \), with \( T_0 \) the string tension. For light-cone dynamics, we can retain \( P^+ \) conservation by orienting the parallel lines in the \( x^3 \) direction.

As a first semi-classical illustration of how this setup works, imagine a pair of particles constrained to move on two such parallel lines and interacting (in 3-space) via a potential \( V(|\vec{r}|) \). Then in the center of mass system the Hamiltonian will be

\[ H = 2\sqrt{m^2 + p_3^2} + V(\sqrt{z^2 + R^2}). \]  

(1.1)

For very large \( R \) we can approximate

\[ V(\sqrt{z^2 + R^2}) \approx V(R) + \frac{1}{2} \frac{z^2}{R} V'(R) + \cdots \]  

(1.2)

which is a stable approximation only for an attractive force \( V' > 0 \). In that case the ground state energy will be, to a good approximation,

\[ 2m + V(R) + \frac{1}{2} \sqrt{\frac{2V''(R)}{mR}} \approx 2m + V(R), \]  

(1.3)

yielding the potential \( V(R) \), as desired.

In the remainder of this paper we will study this idea in the context of gauge theories. For convenience we shall take the particles living on the lines (1-branes) to be Dirac fermions. For

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brevity, we shall call these constrained particles *branions*. The gauge fields will of course live in the $3+1$ dimensional “bulk” space-time. In section 2, we address the problem of calculating the energy of a pair of 1-branes separated by a distance $R$. In the weak coupling limit some sort of ladder approximation is appropriate. We study three versions of this approximation: the Bethe-Salpeter equation in Feynman gauge and light-cone gauge, and a Tamm-Dancoff approximation [6] to the problem of minimizing the energy of such a system. For larger coupling, the ladder approximation fails, and the only simplification available is to the planar diagrams of large $N_c$ QCD [7]. In section 3, we study the “S-matrix” for a branion on one line scattering off an anti-branion on the other through one-loop in light-cone gauge. For simplicity we assume the (anti-) branions couple to large $N_c$ QCD in the bulk and are in the $(\bar{N}_c)N_c$ representation of $SU(N_c)$. We show in detail how asymptotic freedom emerges and exhibit all of the light-cone “$P^+ = 0$” divergences which are shown to disappear if virtual momenta are taken on-shell in a particular way. We conclude in section 4 with some comments on how our proposal can be put to use in studying quark confinement using light-front methods.

## 2 Calculating the Energy

Our purpose in studying separated 1-branes is to devise a light-front friendly method to extract the force between charged sources separated by a variable distance $R$. The proposal is to minimize the system energy $U(R)$ with the constraint that each 1-brane carries a non-zero charge or color, and to identify $-U'(R)$ with the force, at least in certain regimes.

Because the sources are constrained to lines, rather than to points, this minimization involves the 1+1 dimensional quantum dynamics of the sources, and this puts certain limitations on possible applications.

- First of all it gives no useful information about sources that repel one another. In this case the minimum energy configuration is that with the sources at opposite ends of their respective 1-branes yielding a minimum energy independent of $R$. This is no problem for settling the issue of confinement, since that requires the energy due to separated sources in an overall color singlet state. Such sources will attract one another whether or not confinement is realized in the theory.

- Even if the sources attract one another, the physical meaning of the minimal energy as a function of $R$ can be clouded by the brane dynamics. For example, if the branions are massless and the actual interaction energy falls off only as $1/r$ (i.e. confinement does not occur), the charge on each brane can spread to a size of order $R$ as $R \to \infty$ so, although the minimal energy might fall off as $1/R$, the coefficient is reduced compared to that of the actual potential. In that case the method would establish the absence of confinement but would not yield a direct measure of the interaction potential. Giving the branions a mass would provide a limitation to the growth of the charge size, so that the asymptotic behavior of the minimal energy would exactly track that of the actual potential.

- In the context of QCD, we expect confinement to show up as a linear growth $\sim T_0 R$ in the minimal energy at large $R$. However the brane dynamics would not automatically ensure (via asymptotic freedom) a valid weak coupling description of the opposite limit $R \to 0$. That would require that all relevant momentum scales be large compared to $\Lambda_{QCD}$: not only $1/R$ but also the momentum transfers involved in the binding dynamics. To track the force from a perturbative description at small $R$ to confinement at large $R$ we require
that the branions be very heavy: \( 1/a_b = N_c g^2 m \gg \Lambda_{QCD} \). Here \( a_b \) is the Bohr radius for the Coulombic bound state, and sets the size of charge distribution on each brane. If the inequality holds, the minimal energy will have a renormalization group improved Coulombic behavior \( N_c g^2(R)/4\pi R \) for \( a_b < R < 1/\Lambda_{QCD} \) and, if confinement occurs, a linear growth for \( R \gg 1/\Lambda_{QCD} \). The behavior of the minimal energy for \( R \lesssim a_b \) is reflective of the two branion composite system, and has no direct interpretation as an effective force between two point sources.

Weak coupling methods (valid in QCD for short distance phenomena) can only effectively describe bound states close to threshold, that is when the bound state mass \( M(R) \to 2m \) in the zero coupling limit. This is a nonrelativistic regime: in the center of mass system energy transfers and the squares of momentum transfers are of order \( \mathcal{O}(2m - M) \). Because energy transfers are much smaller than momentum transfers, the propagators of exchanged gluons describe effectively instantaneous interactions, and the nonrelativistic Schrödinger equation is applicable. This nonrelativistic dynamics can be identified in quantum field theory in a number of ways which we shall address below. For large distances, where \( N_c \alpha_s(R) \sim 1 \), a full-fledged nonperturbative treatment is required. We advocate an approach based on summing the planar graphs of \( 't \) Hooft’s large \( N_c \) limit. This limit leads to important simplifications, both because it suppresses pair production and because only planar graphs need be summed. Unfortunately, in the continuum theory no further simplification is possible. But we think it particularly interesting to consider a strong \( 't \) Hooft coupling limit in the context of light-cone quantization with discretization of both \( P^+ \) and \( x^+ \). Of course, this limit takes one far from the continuum theory, but it is likely to be described by string theory [8]. This possibility has been strengthened recently by developments surrounding the Maldacena conjecture [9–11].

In the remainder of this section we discuss the energy of two 1-brane sources in the weak coupling limit of threshold binding. We consider the configuration of a branion anti-branion pair each confined to 1 + 1-dimensional branes separated by a transverse distance \( R \). The gauge fields, which take values in the Lie algebra of \( U(N_c) \), live in the 4-dimensional bulk space-time. Our model is therefore described by the Lagrangian,

\[
\mathcal{L} = -\frac{1}{4} \text{Tr} F_{\mu\nu} F^{\mu\nu} + \delta(x) \bar{\psi}_1 [i\gamma^\alpha (\partial_\alpha - igA_\alpha) - m] \psi_1 \\
+ \delta(x - R) \bar{\psi}_2 [i\gamma^\alpha (\partial_\alpha - igA_\alpha) - m] \psi_2,
\]

where the index \( \alpha \) only runs over brane coordinates. The field strengths are given by \( F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \). Note that with this normalization \( \alpha_s = g^2/2\pi \).

It is useful to identify a subset of graphs which is responsible for binding in this regime as a starting point for understanding what happens as the coupling is increased. As \( R \) increases from small values, we can begin to see gradually how more complicated diagrams become important. Optimistically this inside-out approach may provide a framework for understanding the dynamics of confinement in the large \( N_c \) limit. Of course we expect weak coupling binding to be captured by the sum of ladder diagrams or by the associated Bethe-Salpeter equation. We first discuss that approach. Ladder diagrams are gauge dependent, but that dependence should disappear in the non-relativistic binding regime. We shall confirm this expectation by examining the ladder approximation in both Feynman and light-cone gauges. Later we shall show how Tamm-Dancoff truncation can capture the same physics.
2.1 Summing Ladder Feynman Diagrams

We write the integral equation describing the ladder sum for the channel where a branion (with 2-momentum $p$) moves on a 1-brane oriented parallel to the $z$-axis passing through $x = 0$ and an anti-branion (of 2-momentum $q$) moves on a parallel 1-brane passing through $x = R$. Call the Green function, describing the coupling of the branion and anti-branion to a bound system of 2-momentum $p + q$, $G(p, q)$. Then

$$G(p, q) = g^2 N_c \frac{m + \gamma \cdot q}{m^2 + q^2} \int \frac{dk}{(2\pi)^2} e^{ik \cdot R} \frac{d^2k}{(2\pi)^2} D_{\mu\nu}(k, k) \gamma^\mu G(p - k, q + k) \gamma^\nu \frac{m - \gamma \cdot p}{m^2 + p^2}. \quad (2.2)$$

In this equation Minkowski two-vectors are denoted $p, q, k$, etc. while vectors in the $xy$-plane are indicated by bold face type.

In Feynman gauge $D_{\mu\nu} = -i\eta_{\mu\nu}/(k^2 + k_3^2 - k_2^2)$. In the non-relativistic binding regime and in the center of mass system, $k_0 \sim (g^2 N_c)/R$ while $k_{1,2,3} \sim mg^2 N_c$. Thus, for $R \gg 1/m$, $k_0$ can be neglected in the denominator of the propagator, and only $\hat{G} \equiv \int G dk^0/2\pi$ appears on the r.h.s. Putting $p^0 \to p^0 - l^0$, $q^0 \to q^0 + l^0$, and integrating both sides with respect to $l^0$ yields by contour integration

$$\hat{G}(p^3, q^3) \approx g^2 N_c \frac{m + \gamma^3 q^3 - \gamma^0 \sqrt{m^2 + q_3^2}}{2\sqrt{m^2 + q_3^2}} \int \frac{dk}{(2\pi)^2} e^{ik \cdot R} \frac{dk}{(2\pi)^2} \gamma^\mu \frac{1}{k_2^2 + k_3^2} \gamma^0 (p^0 + q^0 - \sqrt{m^2 + q_3^2})\gamma^\mu m - \gamma^3 p^3 + \gamma^0 (p^0 + q^0 - \sqrt{m^2 + q_3^2}) m^2 + p_3^2 - (p^0 + q^0 - \sqrt{m^2 + q_3^2})^2 + g^2 N_c \frac{m + \gamma^3 q^3 - \gamma^0 (p^0 + q^0 + \sqrt{m^2 + p_3^2})}{m^2 + q_3^2 - (p^0 + q^0 + \sqrt{m^2 + p_3^2})^2} \int \frac{dk}{(2\pi)^2} e^{ik \cdot R} \frac{dk}{(2\pi)^2} \gamma^\mu \frac{1}{k_2^2 + k_3^2} \gamma^0 m - \gamma^3 p^3 + \gamma^0 \sqrt{m^2 + p_3^2} \quad (2.3)$$

In the center of mass system $q^3 = -p^3$ and $q^0 + p^0 \equiv M$, the mass of the bound system, is expected to be $2m - \mathcal{O}(g^2 N_c/R)$. Thus the first term on the r.h.s. is much larger than the second and, neglecting momenta small compared to $m$, the equation reduces to

$$\left(2m - M + \frac{q_3^2}{m}\right) \hat{G}(p^3) \approx g^2 N_c \int \frac{dk^3}{2\pi} \frac{dk}{(2\pi)^2} e^{ik \cdot R} \frac{1 - \gamma^0}{2} \gamma^\mu \hat{G}(p^3 - k^3) \gamma^\mu \frac{1 + \gamma^0}{2}. \quad (2.4)$$

This equation implies that $(1 + \gamma^0)\hat{G} = \hat{G}(1 - \gamma^0) = 0$, so that only $\mu = 0$ contributes in the contraction of the gamma matrices. The projection operators can be deleted and the equation thus reduces to

$$\left(2m - M + \frac{q_3^2}{m}\right) \hat{G}(p^3) \approx g^2 N_c \int \frac{dk^3}{2\pi} \frac{dk}{(2\pi)^2} e^{ik \cdot R} \hat{G}(p^3 - k^3) \approx g^2 N_c \int \frac{dk^3}{4\pi^2} K_0(|k|^3|R) \hat{G}(p^3 - k^3), \quad (2.5)$$
which is just the momentum space representation of the nonrelativistic Schrödinger equation

\[
\left(-\frac{1}{m} \frac{\partial^2}{\partial z^2} - \frac{N_c g^2}{4\pi\sqrt{z^2 + R^2}}\right) \psi = (M(R) - 2m)\psi. \tag{2.6}
\]

The reduction is a bit more efficient in light-cone gauge, wherein the only relevant component of the gluon propagator is \(D^{-} = -2ik^{-}/k^{+}(k^2 - 2k^{+}k^{-})\). In the nonrelativistic regime the particles on the branes have energies of order \(m + O(N_c \alpha_s/R)\) but momenta of order \(O(N_c \alpha_s m)\) so that the momentum transfer \(k^{-} = -k^{+} + O(N_c \alpha_s/R) = O(N_c \alpha_s m)\). Thus we can make the replacement \(D^{-} \to +2i/(k^2 + 2k^{+2})\) in the Bethe-Salpeter equation, which with the following definition \(\hat{G}(p^+, q^+) = \int dt^- \gamma^+ G(p - l, q + l)\gamma^+\), takes the form,

\[
\left[\frac{m^2}{2p^+} + \frac{m^2}{2q^+} - (p^- + q^-)\right] \hat{G}(p, q) \approx 2g^2 N_c \int \frac{dk^+}{(2\pi)^3} e^{ik R} \hat{G}(p - k, q + k) \frac{k^2 + 2k^{+2}}{k^2 + 2k^{+2}}
\]

\[
\approx \frac{g^2 N_c}{2\pi^2} \int dk^+ K_0(\sqrt{2}|k^+|R)\hat{G}(p - k, q + k). \tag{2.7}
\]

A bound state of mass \(M\) would have \(p^- + q^- = M^2/2(p^+ + q^+)\), so after a little rearrangement, we get

\[
\frac{(4m^2 - M^2)p^+q^+ + m^2(p^+ - q^+)^2}{2p^+q^+(p^+ + q^+)} \hat{G}(p, q) \approx \frac{g^2 N_c}{2\pi^2} \int dk^+ K_0(\sqrt{2}|k^+|R)\hat{G}(p - k, q + k). \tag{2.8}
\]

In the nonrelativistic regime, \(p^+ \approx q^+ \approx m/\sqrt{2}\). Since \((4m^2 - M^2)\approx (2m - M)4m\) and \((p^+ - q^+)^2\) are of the same order of smallness, it is permissible to make these substitutions in their coefficients:

\[
\left[\frac{2m - M + (p^+ - q^+)^2}{2m}\right] \hat{G}(p, q) \approx \frac{g^2 N_c}{2\sqrt{2}\pi^2} \int dk^+ K_0(\sqrt{2}|k^+|R)\hat{G}(p - k, q + k). \tag{2.9}
\]

At fixed \(p^+ + q^+\), \(\hat{G}\) can be regarded as a function of \(q^+ - p^+ \to p^\sqrt{2}\). Thus by changing variables, \(k^+ \to k^3/\sqrt{2}\), we regain Eq.(2.5).

### 2.2 Light Front Tamm-Dancoff Approach

In 2-dimensions the branion Dirac spinors have two components (see Eq. (2.1)), however, in light-cone quantization with \(A_{-} = 0\), they are not independent. Thus, in the operator Hamiltonian approach one must eliminate one of the components and express the dynamics in terms of single component spinors. This leads us to the light-cone Hamiltonian,

\[
P^{-} = \int dx^- \left[\frac{-im^2}{2} \psi_1^\dagger \frac{1}{\partial_-}\psi_1 - \frac{im^2}{2} \psi_2^\dagger \frac{1}{\partial_-}\psi_2 + \frac{g^2}{2} \delta(0) \text{Tr} \left( \left( \frac{1}{\partial_-} \psi_1^\dagger \psi_1 \right)^2 + \left( \frac{1}{\partial_-} \psi_2^\dagger \psi_2 \right)^2 \right) \right]
\]

\[
+ \int dxdx^- \text{Tr} \left[ \frac{1}{2} \nabla A^i \cdot \nabla A^i - g \nabla \cdot A \left( \delta(x) \frac{1}{\partial_-} \left( \psi_1^\dagger \psi_1 \right) + \delta(x - R) \frac{1}{\partial_-} \left( \psi_2^\dagger \psi_2 \right) \right) \right]
\]

\[
+ \text{gluon interaction terms } \right]. \tag{2.10}
\]

where \(\psi_1\) and \(\psi_2\) are one-component spinors and \(\psi^\dagger \psi\) is understood to be a color matrix \((\psi^\dagger \psi)_{\alpha \beta}\). Also \(\nabla\) is the derivative operator with respect to the transverse coordinates (transverse to the 1-branes) and the index \(i\) runs over transverse components of the gauge field. The terms in Eq. (2.10)
proportional to $\delta(0)$ are associated with branion dynamics on a single brane. The divergent coefficients are due to the zero thickness of the brane, not the UV singularities of the bulk field theory. To deal with them at higher order, they should be regulated by giving a small thickness to the branes. In this paper we shall only see the effect of these terms in the one-loop branion self energy, where it can be absorbed into mass and wave function renormalization. More generally when the four branion vertex from these terms is combined with the corresponding exchange graphs, the most severe quadratic divergence cancels leaving a net logarithmic divergence. However, for our problem of the force between separated branes we do not need to consider these terms to the order in which we are working.

The fundamental commutation relations of the quantum fields are implied by their relation to creation and annihilation operators:

$$A_{\alpha}^\beta(x,x^-) = \int_0^\infty \frac{dk^+}{\sqrt{4\pi k^+}} \left[ a_{\alpha}^\beta(x,k^+)e^{-ik^+x^-} + a_{\alpha}^\beta(x,k^+)e^{ik^+x^-} \right], \quad (2.11)$$

$$\psi_{\alpha}(x^-) = \int_0^\infty \frac{dp^+}{2\pi} \left[ b_{\alpha}(p^+)e^{-ip^+x^-} + d_{\alpha}(p^+)e^{ip^+x^-} \right], \quad (2.12)$$

where the index $a$ distinguishes between the branes and $\alpha, \beta$ are color indices. The creation and annihilation operators satisfy the usual commutation relations

$$[a_{k\alpha}^\beta(x,p^+), a_{l\beta}^\delta(y,q^+)] = \delta_{\alpha\delta} \delta_{\alpha\beta} \delta_{kl} \delta(x-y) \delta(p^+-q^+)$$

$$\{b_{r\alpha}(p^+), b_{s\beta}(q^+)\} = \delta_{\alpha\beta} \delta_{rs} \delta(p^+-q^+)$$

$$\{d_{r\alpha}(p^+), d_{s\beta}(q^+)\} = \delta_{\beta\alpha} \delta_{rs} \delta(p^+-q^+). \quad (2.13)$$

We look for an approximate $P^-$ eigenstate of the form

$$|\Psi\rangle = b_1^\dagger(p^+)d_2^\dagger(q^+)|0\rangle \phi(p^+, q^+) + b_1^\dagger(p^+)a_j^\dagger(x,k^+)d_2^\dagger(q^+)|0\rangle \chi_j^3(x,k^+, p^+, q^+), \quad (2.14)$$

where integration over all variables is implicitly understood. When we apply $P^-$ to $|\Psi\rangle$ we only keep terms proportional to the same states as in $|\Psi\rangle$ (Tamm-Dancoff truncation [6]). It is convenient to express $P^-=P_0^- + P_1^-$, where $P_0^-$ is the non-interacting Hamiltonian and $P_1^-$ are the terms of $O(g)$. Then

$$P_0^-|\Psi\rangle = b_1^\dagger(p^+)d_2^\dagger(q^+)|0\rangle \left( \frac{m^2}{2p^+} + \frac{m^2}{2q^+} \right) \phi(p^+, q^+)$$

$$+b_1^\dagger(p^+)a_j^\dagger(x,k^+)d_2^\dagger(q^+)|0\rangle \left( \frac{m^2}{2p^+} + \frac{m^2}{2q^+} - \frac{\nabla^2}{2k^+} \right) \chi_j^3(x,k^+, p^+, q^+). \quad (2.15)$$

The relevant part of the interaction Hamiltonian can be expressed as

$$P_1^- = -g \int dxdx^- \text{Tr} \nabla \cdot A : \left[ \delta(x) \frac{1}{\partial^-} \psi_1 \psi_1^\dagger + \delta(x-R) \frac{1}{\partial^-} \psi_2 \psi_2^\dagger \right] :$$

$$= -g \int_0^\infty \frac{dp^+ dq^+}{\sqrt{4\pi}} \text{Tr} \left[ b_1(q^+)b_1^\dagger(p^+) + d_1^\dagger(p^+)d_1(q^+) \right] \frac{\nabla \cdot a(0,p^+ - q^+) + \nabla \cdot a^\dagger(0,q^+ - p^+)}{i(p^+ - q^+)[p^+ - q^+]^{1/2}}$$

$$+d_1^\dagger(q^+)b_1^\dagger(p^+) \frac{\nabla \cdot a(0,p^+ + q^+) - b_1(q^+)d_1(p^+)}{i(p^+ + q^+)^{3/2}} - b_1(q^+)d_1(p^+) \frac{\nabla \cdot a^\dagger(0,p^+ + q^+)}{i(p^+ + q^+)^{3/2}} \right]. \quad (2.16)$$
where $a_j(x, k^+) = a_j^\dagger(x, k^+) = 0$ for $k^+ < 0$. With $P_i^-$ expressed in terms of creation and annihilation operators one can readily evaluate $P_i^-$ applied to $|\Psi\rangle$. Thus performing the advertised truncation, we find

\[
(P_i^- |\Psi\rangle)_{T-D} = b_1^\dagger(p'^+) a_j^\dagger(x, p^+ - p'^+) d_2^\dagger(q^+) |0\rangle \frac{i g \nabla j \delta(x)}{\sqrt{4\pi}} p^+/p'^+ - p^+ |3/2 \phi(p^+, q^+)
\]

\[-b_1^\dagger(p^+) a_j^\dagger(x, q^+ - p'^+) d_2^\dagger(p'^+) |0\rangle \frac{i g \nabla j \delta(x - R)}{\sqrt{4\pi}} p^+/p'^+ - q^+ |3/2 \phi(p^+, q^+)
\]

\[b_1^\dagger(p'^+) d_2^\dagger(q^+) |0\rangle \frac{i g N_c \delta(x)}{\sqrt{4\pi}} p^+/p'^+ - p^+ |3/2 \phi(p^+, q^+)
\]

\[+ b_1^\dagger(p^+) d_2^\dagger(p'^+) |0\rangle \frac{i g N_c \delta(x - R)}{\sqrt{4\pi}} p^+/p'^+ - q^+ |3/2 \phi(p^+, q^+),
\]

(2.17)

where the $N_c$ factor arises from a $\delta_{\alpha^a} = N_c$ when contracting annihilation and creation operators.

We would like to solve the following bound-state energy equation:

\[
P^−|\Psi\rangle = (P_0^− + P_i^−) |\Psi\rangle = E |\Psi\rangle.
\]

(2.18)

Substituting Eqns (2.15) and (2.17) into Eq. (2.18) yields two equations corresponding to each of the independent states in $|\Psi\rangle$. These are

\[
\left(\frac{m^2}{2p^+} + \frac{m^2}{2q^+} - E\right) \phi(p^+, q^+) = -\frac{ig N_c \delta(x)}{\sqrt{4\pi}} p^+/p'^+ |3/2 \nabla_j \chi_j(x, p^+ - p'^+, p'^+, q^+)
\]

\[+ \frac{ig N_c \delta(x - R)}{\sqrt{4\pi}} p^+/p'^+ - q^+ |3/2 \nabla_j \chi_j(x, p^+ - q^+, q^+, q^+),
\]

(2.19)

and

\[
\left(\frac{m^2}{2p^+} + \frac{m^2}{2q^+} - \frac{\nabla^2}{2k^+} - E\right) \chi_j(x, k^+, p^+, q^+) = -\frac{ig \nabla j \delta(x)}{\sqrt{4\pi}} p^+/k^+ |3/2 \phi(p^+ + k^+, k^+ + q^+)
\]

\[+ \frac{ig \nabla j \delta(x - R)}{\sqrt{4\pi}} k^+ |3/2 \phi(p^+, q^+ + k^+).
\]

(2.20)

We can use Eq. (2.20) to solve for $\nabla_j \chi_j$ in terms of $\phi$,

\[
\nabla_j \chi_j(x, k^+, p^+, q^+) = \left(x \begin{pmatrix} -\nabla^2 \\ -\nabla^2 + k^+(m^2/p^+ + m^2/q^+ - 2E) \end{pmatrix} 0 \right) \frac{i g}{\sqrt{\pi} |k^+| |3/2} \phi(p^+ + k^+, q^+)
\]

\[-\left(x \begin{pmatrix} -\nabla^2 \\ -\nabla^2 + k^+(m^2/p^+ + m^2/q^+ - 2E) \end{pmatrix} R \right) \frac{i g}{\sqrt{\pi} |k^+| |3/2} \phi(p^+, q^+ + k^+).
\]

(2.21)

Using Eq. (2.21) we can eliminate $\nabla_j \chi_j$ from Eq. (2.19), yielding

\[
\left(\frac{m^2}{2p^+} + \frac{m^2}{2q^+} - E\right) \phi(p^+, q^+) =
\int_0^{p^+} dp'^+ \frac{g^2 N_c}{2\pi |p^+ p'^+ - p^+ |^2} \left(\begin{pmatrix} 0 \\ -\nabla^2 \\ -\nabla^2 + M^2(p^+, q^+) \end{pmatrix} 0 \right) \phi(p^+, q^+)
\]

7
with two or more gluons in the eigenvalue equation \[6, 12, 13\]. Making the replacement gives

\[
\int_0^{q^+} dp^{+′} \frac{g^2 N_c}{2\pi p^{+′} - q^+} \left[ \left( \frac{p^{+′}}{q^+} \right) \phi(p^{+′} + q^+ - p^{+′}) - \left( \frac{p^{+′}}{q^+ - p^{+′}} \right) \phi(p^{+′}, q^+) \right], \tag{2.22}
\]

where

\[
M^2(p^+, q^+) = (p^+ - p^{+′}) \left( \frac{m^2}{p^{+′}} + \frac{m^2}{q^+} - 2E \right). \tag{2.23}
\]

Note in Eq. (2.22) we have restored the explicit integration over \(p^{+′}\).

The inverse kernel of the form \((x) \cdots (y)\) is readily evaluated as

\[
\left( \frac{p^{+′}}{q^+} \right) = \delta(R) - \left( \frac{M^2}{-\nabla^2 + M^2} \right) \delta(R) - \frac{M^2}{2\pi} K_0(MR), \tag{2.24}
\]

where for \(R \neq 0 \Rightarrow \delta(R) = 0\).

Substituting the result of Eq. (2.24) into Eq. (2.22) yields,

\[
\left( \frac{m_r^2}{2p^+} + \frac{m_r^2}{2q^+} - E + \frac{g^2 N_c}{4\pi^2} \int_0^{p^+} dp^{+′} \frac{M^2(p^+, q^+)}{|p^{+′} - p^+|^2} K_0(M(p^+, q^+) \cdot 0) \right)
\]

\[
\int_0^{q^+} dp^{+′} \frac{g^2 N_c}{2\pi p^{+′} - q^+} \left[ \left( \frac{p^{+′}}{q^+} \right) \phi(p^{+′} + q^+ - p^{+′}) - \left( \frac{p^{+′}}{q^+ - p^{+′}} \right) \phi(p^{+′}, q^+) \right] \phi(p^+, q^+)
\]

\[
= \frac{g^2 N_c}{4\pi^2} \int_0^{p^+} dp^{+′} \frac{M^2(p^+, q^+)}{|p^{+′} - p^+|^2} K_0(M(p^+, q^+) R) \phi(p^{+′}, p^+ + q^+ - p^{+′})
\]

\[
+ \frac{g^2 N_c}{4\pi^2} \int_0^{q^+} dp^{+′} \frac{g^2 N_c}{2\pi p^{+′} - q^+} \left( \frac{p^{+′}}{q^+} \right) K_0(M(q^+, p^+) R) \phi(p^+ + q^+ - p^{+′}, p^{+′}). \tag{2.25}
\]

In Eq. (2.25) we have absorbed the divergent \(\delta(0)\) factors into the renormalized branion mass, \(m_r\).

This equation is still ill-defined both because of the \(K_0(M \cdot 0)\) terms and because the \(p^{+′}\) integrations are divergent at \(p^{+′} = p^+, q^+\). These divergences can be associated with the truncation of terms with two or more gluons in the eigenvalue equation \[6, 12, 13\].

One approach to this problem is the \textit{ad hoc} replacement \(2E \rightarrow m_r^2/p^+ + m_r^2/q^+\) in the expression for \(M^2\), which causes the non-integrable singularities to disappear. The only justification is that this replacement is formally valid to order \(g^2 N_c\), and so might arguably be provided if higher terms are properly included \[6, 12, 13\]. Making the replacement gives \(M^2(p^+, q^+) \rightarrow M^2(p^+) = m_r^2|p^{+′} - p^+|^2/p^+ p^{+′}\). With this simplification the factors proportional to \(K_0(M \cdot 0)\) have the right momentum dependence to be absorbed into a further renormalization of the branion mass, \(m_r\).

Then the integral equation becomes

\[
\left( \frac{m_r^2}{2p^+} + \frac{m_r^2}{2q^+} - E \right) \phi(p^+, q^+) = \frac{g^2 N_c}{4\pi^2} \int_0^{p^+} dp^{+′} \frac{M^2(p^+, q^+)}{p^{+′} R} K_0(M(p^+) R) \phi(p^{+′}, p^+ + q^+ - p^{+′})
\]

\[
+ \frac{g^2 N_c}{4\pi^2} \int_0^{q^+} dp^{+′} \frac{g^2 N_c}{2\pi p^{+′} - q^+} \left( \frac{p^{+′}}{q^+} \right) K_0(M(q^+, p^+) R) \phi(p^+ + q^+ - p^{+′}, p^{+′}). \tag{2.26}
\]
If we perform a shift in the integration variable in each integral this becomes

\[
\left( \frac{m_e^2}{2p^+} + \frac{m_r^2}{2q^+} - E \right) \phi(p^+, q^+ ) =
\]

\[
g^2 N_c \frac{m_e^2}{4\pi^2} p^+ \int_0^{p^+} \frac{dk^+}{p^+ - k^+} K_0 \left( \frac{m_r |k^+| R}{\sqrt{p^+(p^+ - k^+)}} \right) \phi(p^+ - k^+, q^+ + k^+ )
\]

\[
+ g^2 N_c \frac{m_r^2}{4\pi^2} q^+ \int_{-q^+}^{0} \frac{dk^+}{q^+ + k^+} K_0 \left( \frac{m_r |k^+| R}{\sqrt{q^+(q^+ + k^+)}} \right) \phi(p^+ - k^+, q^+ + k^+ ).
\] (2.27)

We can show that this reduces to the nonrelativistic Schrödinger equation, Eq. (2.5). First we identify \( E = p^- + q^- \). In the nonrelativistic limit \( p^+ \approx q^+ \approx m/\sqrt{2} \). We also recognize that the integrals on the r.h.s. of Eq. (2.27) are dominated by the Bessel function, \( K_0 \), near \( k^+ \approx 0 \). Thus in the nonrelativistic limit Eq. (2.27) reduces to

\[
\left[ \frac{m_e^2}{2p^+} + \frac{m_r^2}{2q^+} - (p^- + q^-) \right] \phi(p^+, q^+ ) \approx \frac{g^2 N_c}{2\pi^2} \int dk^+ K_0(\sqrt{2}k^+ R)\phi(p^+ - k^+, q^+ + k^+ ),
\] (2.28)

which is identical to Eq. (2.7).

### 3 Scattering to One Loop

In this section we show asymptotic freedom for the configuration of a branion anti-branion pair confined to 1-branes separated by a distance \( R \). We shall show this by calculating the four-point scattering amplitude. For simplicity we shall only calculate the planar diagrams of ’t Hooft’s large \( N_c \) limit [7].

We shall work in light-cone gauge \((A_-=0)\), and use the following conventions:

\[
\gamma^\pm = (\gamma^0 \pm \gamma^3)/\sqrt{2}, \quad \{\gamma^\mu, \gamma^\nu\} = -2\eta^{\mu\nu}.
\] (3.1)

Also the space-time metric is given by \( \eta^{\mu\nu} = \text{diag}[-1,1,1,1] \). We shall also adopt the following notation; lower case momenta are 2-vectors on the brane while upper case momenta are bulk 4-vectors. So for a 4-vector \( Q \) we distinguish its components longitudinal to the brane via \( Q_{\parallel} \) (a 2-vector with light-cone components \( Q^+, Q^- \)) from its transverse components \( Q \) (in **bold-face** - a Euclidean 2-vector).

When we compute the individual Feynman diagrams for the one-loop four-point “S-matrix” it will be convenient to express them in terms of a Fourier integral

\[
\tilde{\Gamma}(p, q, Q_{\parallel}, R) = \int \frac{dQ}{(2\pi)^2} e^{iQ\cdot R} \Gamma(p, q, Q_{\parallel}, Q),
\] (3.2)

where we shall be calculating \( \Gamma(p, q, Q_{\parallel}, Q) \), which has the explicit \( Q \) dependence.

With these conventions the light-cone Feynman rules (using the “double line” notation) are presented in Fig.2. In the following we shall regulate the integration by an ultraviolet cutoff on the transverse momenta, \( K^2 < A^2 \), and an infrared cutoff on \( K^+ > \epsilon \).
The Feynman graph for the self-energy is depicted in Fig. 3 below. Using the light-cone Feynman rules of Fig. 2 the self-energy is given by

$$\frac{\partial \Sigma}{\partial p^-} = N_c \frac{\partial}{\partial p^-} \int \frac{d^4 K}{(2\pi)^4} (ig\gamma^+) \frac{-i}{\gamma^0(p + K_\parallel)_{\alpha} + m} (ig\gamma^+)D^{--}(K).$$

(3.3)

Thus with some manipulation

$$\frac{\partial \Sigma}{\partial p^-} = ig^2 N_c \gamma^+ \left[ \ln \frac{p^+}{\epsilon} \left( \ln \frac{\Lambda^2}{m^2 + p^2} - 1 \right) + \frac{1}{2} \ln^2 \frac{p^+}{\epsilon} + \int_0^1 \frac{dx}{x} \ln \left( \frac{(1-x)(m^2 + p^2)}{(m^2 + (1-x)p^2)} \right) \right].$$

(3.4)

The branion propagator sandwiched between two vertices proportional to $\gamma^+$ can be replaced by:

$$-\frac{i}{m^2 + p^2} \gamma^0(m - \gamma \cdot p) \gamma^+ \rightarrow -\frac{ip^+ \gamma^+ \gamma^-}{m^2 - 2p^+ p^-} = \frac{i\gamma^+}{p^- - m^2/2p^+},$$

(3.5)

from which we can infer that the fermion wave function renormalization is

$$Z_2 = 1 - g^2 N_c \frac{4\pi^2}{4\pi^2} \left[ \ln \frac{p^+}{\epsilon} \ln \frac{\Lambda^2}{m^2 + p^2} + \frac{1}{2} \ln^2 \frac{p^+}{\epsilon} + \text{Finite} \right].$$

(3.6)
to one loop. The renormalized four point function will include a factor of $\sqrt{Z_2}$ for each leg, yielding an overall factor

$$1 - \frac{g^2 N_c}{8\pi^2} \left[ \ln \frac{p^+}{\epsilon} \ln \frac{\Lambda^2}{m^2 + p^2} + \frac{1}{2} \ln^2 \frac{p^+}{\epsilon} + \ln \frac{g^+}{\epsilon} \ln \frac{\Lambda^2}{m^2 + q^2} + \frac{1}{2} \ln^2 \frac{q^+}{\epsilon} + \ln \frac{(p^+ + Q^+)}{\epsilon} \ln \frac{\Lambda^2}{m^2 + (p + Q^+)^2} + \frac{1}{2} \ln^2 \frac{(p^+ + Q^+)}{\epsilon} + \right]

\ln \left( \frac{(q^+ - Q^+)}{\epsilon} \ln \frac{\Lambda^2}{m^2 + (q - Q^+)^2} + \frac{1}{2} \ln^2 \frac{(q^+ - Q^+)}{\epsilon} + \right) \text{finite}, \quad (3.7)

multiplying the tree amplitude, $A^{\text{tree}}_4 = 2ig^2\gamma_1^+\gamma_2^+ Q^-/Q^+Q^2$.

### 3.2 Gluon Propagator to one Loop

The one loop corrections to the gluon propagator, shown in Fig. 4, in light-cone gauge have been evaluated in [14]. After subtraction of quadratic divergences, the propagator to one loop is given by:

$$D^{--}(Q, Q^+, Q^-) = -\frac{2iQ^-}{Q^+Q^2} \left[ 1 - \frac{g^2 N_c}{16\pi^2} \left( 8 \ln \frac{Q^+}{\epsilon} - \frac{22}{3} \right) \ln \frac{\Lambda^2}{Q^2} + 4 \ln^2 \frac{Q^+}{\epsilon} + \frac{4\pi^2}{3} - \frac{134}{9} \right]

+ \frac{i}{Q^+Q^2} \left[ 8 \ln \frac{Q^+}{\epsilon} - 1 \right] \ln \frac{\Lambda^2}{Q^2} + 4 \ln^2 \frac{Q^+}{\epsilon} + \frac{4\pi^2}{3} - \frac{206}{9} \right]. \quad (3.8)

### 3.3 Triangle Graph

We now calculate the triangle graphs contributing to the four-point amplitude. The Feynman diagrams for these contributions are portrayed in Fig. 5. Using the light-cone Feynman rules of

$$\Gamma^{\text{left}}_{\text{Triangle}} = N_c \int \frac{d^4 K}{(2\pi)^4} (ig\gamma_1^+) \frac{-i}{\gamma_1^+(p - K^\parallel)_{\alpha} + m} (ig\gamma_1^+) D^{-\mu_1}(K) D^{-\mu_2}(K + Q) D^{-\mu_3}(Q)$$

Fig. 2 we can readily write the Feynman integral corresponding to the diagram on the l.h.s. of Fig. 5.
Thus the Feynman integral corresponding to this is given by

\[ V_{\L,\L_2}(K, -K - Q, Q)(ig\gamma_5^+) \]

where

\[ F = K^-(Q^2(2Q^+ + K^+) + K \cdot Q(2K^+ + Q^+)) - Q^-[K^2(2K^+ + Q^+) + K \cdot Q(2Q^+ + K^+)]. \]

The box graph contributing to the four-point amplitude which is portrayed in Fig. 6.

### 3.4 Box Graph

The subscripts on the \( \gamma \)'s distinguish between the different branes. All branion-branion-gluon vertices only include the \( + \) component of \( \gamma^\mu \), since the gluon propagator, \( D^{\mu\nu} \), vanishes when \( \mu = + \).

The procedure to evaluate Eq. (3.9) is relatively straightforward. We first perform the \( K^- \) integration via contour integration (inserting the appropriate factors of \( i\epsilon \)). We are only interested in obtaining all the \( \log \) (and \( \log^2 \)) divergent terms. These arise as dependence on the ultraviolet cutoff, \( \Lambda \), and on the infrared cutoff, \( \epsilon \). By investigating these divergences separately and combining, we obtain all the divergent pieces.

For convenience we introduce the following compact notation

\[ w^* = \frac{w^2 + m^2}{w^+}, \]

where \( w \) is a brane 2-vector such that \( w^2 = -2w^+w^- \). Thus the result for the triangle graph on the left-side of Fig. 5 is

\begin{align*}
\Gamma_{\text{Triangle}}^{\text{left}} &= \frac{ig^4N_c\gamma_1^+\gamma_2^+}{4\pi^2Q^2Q^+} \left\{ Q^2 \left[ \left( 2 \ln \frac{Q^+}{\epsilon} - 1 \right) \ln \frac{\Lambda^2}{Q^2} + \ln^3 \frac{Q^+}{\epsilon} \right] \\
&+ Q^- Q^+ \left[ \left( 2 \ln \frac{Q^+}{\epsilon} + \ln \frac{Q^-}{\epsilon} + \ln \frac{p^+ + Q^+}{\epsilon} \right) \ln \frac{\Lambda^2}{Q^2} + 2 \ln^2 \frac{Q^+}{\epsilon} \right] \\
&+ \ln \frac{Q^+}{\epsilon} \left[ \frac{(Q^2 + 2Q^+Q^-)(p + Q^+)^*}{(p + Q^+)^* - Q^2/Q^+} \ln \frac{Q^2/Q^+}{(p + Q^+)^*} \\
&+ \frac{(Q^2 + 2Q^+Q^-)p^*}{p^* + Q^2/Q^+} \ln \frac{Q^2/Q^+}{p^*} \right] + \text{Finite} \right\}. \quad (3.12)
\end{align*}

A similar result corresponding to the Feynman diagram on the right-side of Fig. 5 may be obtain from Eq. (3.12) by the substitution, \( p \rightarrow q - Q^\parallel \).

### 3.4 Box Graph

We proceed to the box graph contributing to the four-point amplitude which is portrayed in Fig. 6. Thus the Feynman integral corresponding to this is given by

\begin{align*}
\Gamma_{\text{Box}} &= N_c \int \frac{d^4K}{(2\pi)^4} \frac{-i}{\gamma_1^+(p - K^\parallel)\alpha + m (ig\gamma_1^+)(ig\gamma_2^+)} \frac{-i}{\gamma_2^-(Q^\parallel + K^\parallel)\alpha + m (ig\gamma_2^+)} \\
&\times D^{-\alpha}(K)D^{-\alpha}(K + Q) \quad (3.13)
\end{align*}

This integral can be evaluated in a similar manner to the triangle integral encountered in subsection 3.3. We again first perform the contour integration over \( K^- \). We then individually
target the divergences associated with the transverse integration, \( dK \) and those associated with the \( K^+ \) integration. Combining these contributions yields all the divergent pieces of this integral. Then again using the notation adopted in Eq. (3.11) we get

\[
\Gamma_{\text{Box}} = -i g^4 N_c \gamma_1^+ \gamma_2^+ \left\{ 2 \ln \frac{Q^+}{\epsilon} \ln \frac{\Lambda^2}{Q^2} + \ln^2 \frac{Q^+}{\epsilon} + \ln \frac{Q^+}{\epsilon} \right. \\
\left. \frac{1}{(p+Q\parallel)^*+(q-Q\parallel)^*} \left( \frac{(q-Q\parallel)^*((q-Q\parallel)^*-2Q^-)}{(q-Q\parallel)^*+Q^2/Q^+} \ln \frac{Q^2/Q^+}{(q-Q\parallel)^*} + \frac{(p+Q\parallel)^*((p+Q\parallel)^*+2Q^-)}{(p+Q\parallel)^*-Q^2/Q^+} \ln \frac{Q^2/Q^+}{(p+Q\parallel)^*} \right) \\
+ \frac{1}{p^*+q^*} \left( \ln \frac{Q^2/Q^+}{p^*} + \frac{q^*(q^*+2Q^-)}{q^*-Q^2/Q^+} \ln \frac{Q^2/Q^+}{q^*} \right) \right\} + \text{Finite} \right. \\
\] (3.14)

It is noteworthy that after multiplying by \( e^{Q \cdot R} \) and integrating over \( Q \) as prescribed in Eq. (3.2), the ultraviolet divergences \( \ln \Lambda^2 \) and the \( \ln^2 \epsilon \) divergences disappear because they are multiplied by \( \delta(R) = 0 \) at finite separation. Thus the box is not needed to obtain asymptotic freedom in this branion scattering process. The \( \ln \epsilon \) divergences, however, remain and are necessary to eventually obtain \( \epsilon \) independent on-shell scattering.

### 3.5 Divergence structure of the One Loop 4 Point Function

We have assembled the various pieces necessary to construct the divergent structure of the off-shell one-loop four-point Green function. The one-loop four-point amplitude is given by: four-point trees with a factor of \( \sqrt{Z_2} \), see Eq. (3.7), for each external leg; a four-point exchange diagram with the one-loop gluon propagator, Eq. (3.8); the triangle graphs of Fig. 5 (the solution for the graph on the right can be obtained by a simple substitution in the solution for the graph on the left, Eq. (3.12)); the box graph, Eq. (3.14). Since we have taken \( N_c \to \infty \), we have of course not included non-planar contributions, e.g. from the crossed box graph.

When we simplify the four-point function all terms proportional to \( \ln \Lambda^2/Q^2 \) cancel up to the expected \( \frac{11}{3} \ln \Lambda^2/Q^2 \) term which is the correct asymptotic behavior for QCD. Also all terms proportional to \( \ln^2 Q^+/% mathcal{E} \) cancel. Note that these cancelations occur before integrating over \( Q \), but do involve \textit{all} of the one loop diagrams. As mentioned in subsection 3.4 the box diagram shows no \( \ln \Lambda \) or \( \ln^2 \epsilon \) dependence after integrating over \( Q \) as prescribed in Eq. (3.2).
The remaining divergent structure of the off-shell four-point Green function is given by

\[ \frac{ig^4 N_c \gamma_1 \gamma_2 Q^-}{4\pi^2 Q^+ Q^2} \left[ \frac{11}{3} \ln \frac{\Lambda^2}{Q^2} + \frac{1}{Q^+ Q^-} \ln \epsilon \left\{ \frac{Q^2 p^* q^* + Q^2 Q^- (p^* - q^*) + Q^+ Q^-(p^* + q^*)}{(p^* + q^*)(p^* + Q^2/Q^+)} \ln \frac{Q^2/Q^+}{p^*} \right. \\
+ \frac{Q^2 (p + Q\|)^* (q - Q\|)^* + Q^2 Q^- ((q - Q\|)^* - (p + Q\|)^*) + Q^+ Q^- (p + Q\|)^* ((p + Q\|)^* + (q - Q\|)^*)}{((p + Q\|)^* + (q - Q\|)^*)^2 - Q^2/Q^+} \ln \frac{Q^2/Q^+}{(p + Q\|)^*} \\
+ \frac{Q^2 (p + Q\|)^* (q - Q\|)^* + Q^2 Q^- ((q - Q\|)^* - (p + Q\|)^*) + Q^+ Q^- (q - Q\|)^* ((p + Q\|)^* + (q - Q\|)^*)}{((p + Q\|)^* + (q - Q\|)^*)^2 + Q^2/Q^+} \ln \frac{Q^2/Q^+}{(q - Q\|)^*} \\
+ \left. \frac{Q^2 p^* q^* + Q^2 Q^- (p^* - q^*) + Q^+ Q^- q^* (p^* + q^*)}{(p^* + q^*)(q^* - Q^2/Q^+)} \ln \frac{Q^2/Q^+}{q^*} \right\} \right\} + \text{Finite} \right] . \tag{3.15} \]

Since this off-shell amplitude is not, as it stands, a physical quantity, it is not necessary that the \( \ln \epsilon \) divergences cancel. Even the on-shell limit is not quite physical because of the usual IR divergences associated with the possibility of soft gluon bremsstrahlung. In particular the terms we have labeled “Finite” are cutoff independent and finite off-shell, but display IR divergences in the on-shell limit. Focusing on the \( \ln \epsilon \) terms, we see that the on-shell limit involves ambiguous terms of the form \( 0/0 \). Dropping the terms that unambiguously vanish on-shell, these terms simplify to:

\[ \frac{ig^4 N_c \gamma_1 \gamma_2 Q^-}{4\pi^2 Q^+ Q^2} \ln \frac{\Lambda^2}{Q^2} \ln \frac{p^* - q^*}{p^* + q^*} \ln \frac{q^*}{p^*} + \frac{(p + Q\|)^* - (q - Q\|)^*}{(p + Q\|)^* + (q - Q\|)^*} \ln \frac{(q - Q\|)^*}{(p + Q\|)^*} . \tag{3.16} \]

Since \( \omega^* = m^2/w^+ - 2w^- \) we see that by taking the branions in either the initial or final states equally off energy shell these ambiguous terms can be eliminated. Note that this restriction still allows the total energy to be off-shell, an important flexibility for bound state problems. While we are free to make this restriction on the initial and final states, it would not be allowed if this off-shell amplitude were part of a larger diagram. In that case, there must be other \( \ln \epsilon \) terms for them to cancel against. We have not yet managed to show that this actually happens. Even so we have achieved our goal in this section of demonstrating asymptotic freedom, which did involve an intricate cancelation of \( \ln \epsilon \) divergences in the coefficient of \( \ln \Lambda^2 \).

4 Discussion and Conclusion

In this paper we have proposed a light-front friendly way to extract the interaction energy between two sources in a gauge theory. We have analyzed the method in several ways at weak coupling. But the larger motivation for this proposal is to develop a framework within a light-cone quantization approach for establishing quark confinement in non-abelian gauge theories.

Consider, for example, the ’t Hooft limit of \( N_c \to \infty \), which is well known to reduce to summing planar graphs. Another version of this limit interprets the sum of planar diagrams on a light front as the quantum dynamics of a gluonic chain [15]. Without the notion of pinned sources, these chains would be dynamical bound states, and the concept of a confining force would have to be indirectly inferred from the excitation spectrum of these chains. In ordinary equal time quantization, the concept of a Wilson loop allows a direct definition of the confining force, however the possibility of vacuum fluctuations clouds the chain interpretation of the large \( N_c \) limit. Our goal was to modify the Wilson criterion to make it suitable for light front physics.

In light-cone quantization the sum of planar diagrams coupled to two separated 1-branes can be interpreted as the dynamics of a nonrelativistic chain of gluons vibrating in the transverse space and stretched between the points in transverse space marking the locations of the 1-branes. Since
the $p^+$ of each gluon and branion is dynamical, the Newtonian mass of each of these objects varies; but in such a way that the total mass is conserved [5]. This picture is made more concrete when the $p^+$ is discretized, $p^+ \rightarrow Mm$, where $M$ is a large integer, and $m$ is the discrete unit of $p^+$. We hope that pinning the ends of the chain to points will offer conceptual and technical simplifications to the problem of assessing whether the dynamics of the chain leads to a confining force.

Although the dynamics of continuum gauge theories generically involves a scale dependent coupling which is presumably never actually much larger than unity, the introduction of an ultraviolet cutoff allows a formal strong coupling limit. For example, the discretization of $x^+$ (in addition to $p^+$) provides such a cutoff [8]. In that context the strong 't Hooft coupling limit favors large planar fishnet diagrams, which should behave as seamless world sheets. In this limit then the sum of planar diagrams coupled to our 1-branes would correspond to a fundamental light-cone string stretched between them, and therefore a confining force would emerge. However, the precise nature of such a “fishnet” approximation to QCD is yet to be determined.

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References