Simple Connection Between Atmospheric and Solar Neutrino Vacuum Oscillations

Ernest Ma

Department of Physics
University of California
Riverside, California 92521

Abstract

Extending the minimal standard model of particle interactions (without right-handed singlet neutrinos) to include a heavy scalar triplet $\xi$ to obtain nonzero Majorana neutrino masses, I derive the following simple realistic connection between atmospheric and solar neutrino vacuum oscillations: 

\[
(\Delta m^2)_{\text{sol}}(\Delta m^2)_{\text{atm}}/m_\nu^4 \sin^2 2\theta_{\text{atm}} = 2I^2,
\]

where $m_\nu$ is the assumed common approximate mass of each neutrino (which may be suitable for hot dark matter) and 

\[ I = \left( \frac{3}{16\pi^2} \right) \frac{(G_F/\sqrt{2})}{m^2} \ln\left( \frac{m^2_\xi}{m^2_W} \right) \]

comes from the radiative splitting of the degeneracy due to the charged leptons.
There is now a vast literature on models of neutrino oscillations [1]. Most try to understand why atmospheric neutrino oscillations [2] of $\nu_\mu(\bar{\nu}_\mu) \to \nu_\tau(\bar{\nu}_\tau)$ require near-maximal mixing [3]. Many also suggest that solar neutrino oscillations [4] of $\nu_e$ to a linear combination of $\nu_\mu$ and $\nu_\tau$ should have near-maximal mixing as well [5]. Both are possible in the context of three nearly mass-degenerate neutrinos [6, 7] which could then be considered as candidates for hot dark matter [8].

Recently it has been pointed out [9] that if all three neutrinos obtain equal Majorana masses of order 1 eV from the canonical seesaw mechanism [10], then their splitting due to the different charged-lepton masses from the two-loop exchange of two $W$ bosons [11] is of the right magnitude for solar neutrino vacuum oscillations. However, the inclusion of atmospheric neutrino oscillations has to be rather ad hoc in this case. In fact, it is rare indeed that any bona fide model of neutrino masses even gets a relationship between the mass difference of one oscillation and that of another. [One exception is the recently proposed model [12] of radiative masses for $\nu_e, \nu_\mu, \nu_\tau$, plus a singlet (sterile) neutrino $\nu_s$, which explains atmospheric and solar neutrino oscillations as well as the $\bar{\nu}_\mu(\nu_\mu)$ to $\bar{\nu}_e(\nu_e)$ data of the LSND (Liquid Scintillator Neutrino Detector) experiment [13]. It has the successful relationship $(\Delta m^2)_{\text{atm}} \simeq 2[(\Delta m^2)_{\text{sol}}(\Delta m^2)_{\text{LSND}}]^{1/2}$, where $(\Delta m^2)_{\text{sol}}$ refers to the matter-enhanced solution [14] of the solar neutrino deficit.]

In this note I will present the most economical model to date of neutrino masses which has the following simple realistic connection between atmospheric and solar neutrino vacuum oscillations:

$$\frac{(\Delta m^2)_{\text{sol}}(\Delta m^2)_{\text{atm}}}{m^4_\nu (\sin^2 2\theta)_{\text{atm}}} = 2I^2 = 4.9 \times 10^{-13} \left( \ln \frac{m^2_\xi}{m^2_W} \right)^2,$$  

(1)

where $m_\nu$ is the assumed common approximate mass of each neutrino, $m_\xi$ is the mass of a heavy scalar triplet, and

$$I = \frac{3G_F m^2_\tau}{16\pi^2 \sqrt{2}} \ln \frac{m^2_\xi}{m^2_W},$$

(2)
comes from the one-loop radiative splitting of the degeneracy due to the charged leptons, as explained below. Numerically, let $m_\nu = 0.6$ eV, $(\sin^2 2\theta)_{atm} = 1$, and $m_\xi = 1$ TeV, then Eq. (1) is satisfied with the best fit values of $(\Delta m^2)_{sol} = 4.0 \times 10^{-10}$ eV$^2$ and $(\Delta m^2)_{atm} = 4.0 \times 10^{-3}$ eV$^2$.

To start with, the minimal standard model (without right-handed singlet neutrinos) is extended to include a heavy scalar triplet $\xi = (\xi^+, \xi^+, \xi^0)$, where $m_\xi^2 >> m_W^2$ is assumed. This provides the three neutrinos $\nu_e, \nu_\mu, \nu_\tau$ with small Majorana masses [15]. As emphasized recently [16], such an alternative is as simple and natural as the canonical seesaw mechanism [10] which was used in Ref. [9]. Now let there be a discrete $S_3$ symmetry (which has irreducible representations $2, 1, \text{and } 1'$) such that $\xi$ is a $1$ and the standard Higgs doublet $\Phi = (\phi^+, \phi^0)$ is also a $1$, whereas two of the lepton doublets form a $2$ and the third is a $1$ or $1'$. The relevant terms in the interaction Lagrangian are then given by

$$L_{int} = \xi^0[f_0(\nu_1\nu_2 + \nu_2\nu_1) + f_3\nu_3\nu_3] + \mu\xi^0\phi^0\phi^0 + h.c.$$  

The field $\xi^0$ acquires a naturally small vacuum expectation value [15] $u \simeq -\mu\langle\phi^0\rangle^2/m_\xi^2$ and the $3 \times 3$ Majorana neutrino mass matrix is of the form

$$M_\nu = \begin{pmatrix} 0 & m_0 & 0 \\ m_0 & 0 & 0 \\ 0 & 0 & m_3 \end{pmatrix},$$  

where $m_0 = 2f_0u$ and $m_3 = 2f_3u$. Actually, the difference between $m_0$ and $m_3$ will be assumed small compared to either $m_0$ or $m_3$ in the following, i.e. each neutrino is accorded an approximate common mass $m_\nu$.

The neutrinos are now identified with their charged-lepton partners as follows:

$$\nu_1 = \nu_e, \quad \nu_2 = c\nu_\mu - s\nu_\tau, \quad \nu_3 = c\nu_\tau + s\nu_\mu,$$  

where $s \equiv \sin \theta$ and $c \equiv \cos \theta$. This construction is made to accommodate the atmospheric data [2] as $\nu_\mu - \nu_\tau$ oscillations with $\sin^2 2\theta = 4s^2c^2$ and $\Delta m^2 = m_0^2 - m_3^2$. At this point,
the eigenvalues of $M_\nu$ of Eq. (4) are $-m_0$, $m_0$, and $m_3$. However, since the charged-lepton masses break the assumed $S_3$ symmetry, the two-fold degeneracy of the $\nu_1 - \nu_2$ sector is broken radiatively in one loop. There are two effects. One is a finite correction to the mass matrix, as shown in Figure 1. The other is a renormalization of the coupling matrix [17] from the shift in mass scale from $m_\xi$ to $m_W$. As expected, the dominant contribution comes from the $\tau$ Yukawa coupling. The two contributions are naturally of the same texture and are easily calculated to be $4I/3$ and $-I/3$ respectively, where $I$ is already given by Eq. (2).

The mass matrix $M_\nu$ is now corrected to read

$$M_\nu = \begin{pmatrix} 0 & m_0(1 + s^2 I) & -scm_0 I \\ m_0(1 + s^2 I) & 0 & -scm_3 I \\ -scm_0 I & -scm_3 I & m_3(1 + 2c^2 I) \end{pmatrix}. \quad (6)$$

The two-fold degeneracy of the $\nu_1 - \nu_2$ sector is then lifted, with the following mass eigenvalues:

$$-m_0(1 + s^2 I) - \frac{s^2 c^2 (m_0 - m_3) I^2}{2(m_0 + m_3)}, \quad m_0(1 + s^2 I) + \frac{s^2 c^2 (m_0 + m_3) I^2}{2(m_0 - m_3)}, \quad (7)$$

where $I^2 << (m_0 - m_3)^2/(m_0 + m_3)^2$ has been used, being justified numerically. Hence their mass-squared difference is

$$\Delta m^2 \simeq s^2 c^2 m_0 I^2 \left[ \frac{(m_0 + m_3)^2}{m_0 - m_3} - \frac{(m_0 - m_3)^2}{m_0 + m_3} \right] \simeq \frac{8s^2 c^2 I^2 m_\nu^4}{m_0^2 - m_3^2}, \quad (8)$$

where $m_\nu \simeq m_0 \simeq m_3$ has been used. Identifying this with solar neutrino vacuum oscillations then yields Eq. (1).

In the above, the choice $\nu_1 = \nu_e$ leads to $(\sin^2 2\theta)_{sol} = 1$. The eigenstates of $M_\nu$ from Eq. (4) or Eq. (6) are the same to first order:

$$\frac{1}{\sqrt{2}}(\nu_e - cv_\mu + sv_\tau), \quad \frac{1}{\sqrt{2}}(\nu_e + cv_\mu - sv_\tau), \quad sv_\mu + cv_\tau. \quad (9)$$

For $s = c = 1/\sqrt{2}$, the so-called bimaximal mixing solution [5] of neutrino oscillations is obtained. With the assumed form of Eq. (4), it is also worth noting that renormalization
effects due to the $\tau$ and $\mu$ Yukawa couplings do not affect the degeneracy of the $\nu_1 - \nu_2$ sector to first order. This is why $(\Delta m^2)_{\text{sol}}$ can be small enough here to be suitable for vacuum oscillations. The zero $\nu_e - \nu_e$ entry in the neutrino mass matrix is crucial for the validity of Eq. (7) and has been chosen to avoid neutrinoless double beta decay [18]. This is an important constraint as long as $m_\nu$ is greater than about 1 eV, which used to be a desirable feature as a component of dark matter [8]. However, with the recent observation of a nonzero cosmological constant [19], whereas $m_\nu$ is probably still needed for large-scale structure formation in the universe, its magnitude can be much smaller. In general, $\nu_1$ may be a linear combination of $\nu_e, \nu_\mu,$ and $\nu_\tau$, but it has to be predominantly $\nu_e$. Otherwise, $m_\tau$ (and $m_\mu$) radiative contributions would appear in the diagonal entries of Eq. (6) and modify Eqs. (7) and (8). For illustration, the values $m_\nu = 0.6$ eV and $m_\xi = 1$ TeV have been used.

It may be argued that $m_\xi$ is naturally of order $10^{13}$ GeV or greater [15], in which case $m_\nu$ should be somewhat smaller. More precisely,

$$m_\nu \sim 1.3 \text{ eV} \left[ \ln \frac{m^2_\xi}{m^2_W} \right]^{-\frac{1}{2}}. \quad (10)$$

For $m_\xi = 10^{13}$ GeV, the required $m_\nu$ is then about 0.18 eV.

The charged-lepton mass matrix which accompanies $M_\nu$ of Eq. (4) is not uniquely defined, because only the left-handed fields are correlated with it. Nevertheless, $S_3$ is clearly violated. So far, I have not identified the origin of this violation. It may simply be explicit, or it may be spontaneous, in the sense that it occurs only when the electroweak gauge symmetry is broken. An example of the latter is the following model. Under $S_3$, let

$$\left[ \left( \begin{array}{c} \nu_1 \\ l_1 \\ \phi_1^0 \end{array} \right)_L, \left( \begin{array}{c} \nu_2 \\ l_2 \\ \phi_2^0 \end{array} \right)_L, \left( \begin{array}{c} \nu_3 \\ l_3 \\ \phi_3^0 \end{array} \right)_L \right] \sim 2, \quad \left[ \begin{array}{c} \phi_1^0 \\ \phi_1^- \end{array} \right] \sim 2, \quad \left[ \begin{array}{c} \phi_2^0 \\ \phi_2^- \end{array} \right] \sim 2, \quad \left[ \begin{array}{c} \phi_3^0 \\ \phi_3^- \end{array} \right] \sim 1, \quad \left[ \begin{array}{c} \xi^+ \\ \xi^0 \end{array} \right] \sim 1. \quad (11)$$
With $\langle \xi^0 \rangle \neq 0$, $M_\nu$ of Eq. (4) is obtained, whereas $M_l$ is now given by
\[
M_l = \begin{pmatrix}
h_1 \langle \phi_0^2 \rangle & h_2 \langle \phi_0^3 \rangle & h_3 \langle \phi_0^4 \rangle \\
h_2 \langle \phi_0^3 \rangle & h_1 \langle \phi_0^1 \rangle & h_3 \langle \phi_0^4 \rangle \\
h_4 \langle \phi_0^2 \rangle & h_4 \langle \phi_0^3 \rangle & h_5 \langle \phi_0^5 \rangle
\end{pmatrix},
\]
(13)
where $h_{1,2,3,4,5}$ are the couplings of all possible Yukawa terms invariant under $S_3$. Before electroweak symmetry breaking, charged-lepton masses as well as neutrino masses are zero, as in the standard model. After electroweak symmetry breaking, let $\langle \phi_0^2 \rangle << \langle \phi_0^1 \rangle$, then $S_3$ is also broken in $M_l$ at tree level but not in $M_\nu$. Radiative corrections then break $S_3$ in $M_\nu$ as shown in this paper. In the limit $h_2 \to 0$ and $\langle \phi_0^0 \rangle \to 0$, $e_L$ is indeed separated from the $\mu_L - \tau_L$ sector and Eq. (5) holds as desired. In terms of fine tuning, this model is no worse than the standard model which also requires arbitrary Yukawa couplings to fix the charged-lepton masses.

In conclusion, I have presented in this note a new and economical extension of the minimal standard model, where a heavy scalar triplet $\xi$ is added to provide the three known neutrinos with nonzero Majorana masses. This replaces the usual method of adding three heavy right-handed neutrino singlets. A discrete $S_3$ symmetry is then assumed so that two neutrinos are degenerate in mass, with their splitting controlled by one-loop radiative corrections. This results in a simple realistic connection between atmospheric and solar neutrino vacuum oscillations as given by Eq. (1). It is consistent with the present data and will get tested further as more data become available in the near future from planned experiments in neutrino oscillations, neutrinoless double beta decay, neutrino mass, searches for dark matter and for new particles in high-energy accelerators.

ACKNOWLEDGEMENT

This work was supported in part by the U. S. Department of Energy under Grant No. DE-FG03-94ER40837.
References


[18] For a review, see for example H. V. Klapdor-Kleingrothaus, hep-ex/9901021.

Fig. 1. One-loop radiative breaking of neutrino mass degeneracy.