Mode-Coupling in Rotating Gravitational Collapse of a Scalar Field

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(June 15, 1999)

Abstract

We present an analytic study of the mode-coupling phenomena for a scalar field propagating on a rotating Kerr background. Physically, this phenomena is caused by the dragging of reference frames, due to the black-hole (or star’s) rotation. We find that different modes become mixed during the evolution and the asymptotic late-time tails are dominated by a mode which, in general, has an angular distribution different from the original one. We show that a rotating Kerr black hole becomes ‘bald’ slower than a spherically-symmetric Schwarzschild black hole.

I. INTRODUCTION

The radiative tail of gravitational collapse decays with time leaving behind a Kerr-Newman black hole characterized solely by the black-hole mass, charge and angular-momentum. This is the essence of the no-hair conjecture, introduced by Wheeler in the early 1970s [1]. The relaxation process of neutral fields was first analyzed by Price [2] for a nearly spherical collapse. Price demonstrated that the fields decay asymptotically as an inverse power of time. Physically, these inverse power-law tails are associated with the backscattering of waves off the effective curvature potential at asymptotically far regions.
The existence of these inverse power-law tails, which characterize the asymptotic late-time evolution of test fields on a Schwarzschild background was later verified by other authors [4–7] using independent analyses. The late-time evolution of natural test fields on different spherically symmetric spacetimes was studied in Refs. [8,5,9]. In addition, the late-time tails of gravitational collapse were demonstrated by full numerical computations [10,11]. The physical mechanism by which a charged black hole, which is formed during a gravitational collapse of a charged matter, dynamically sheds its charged hair was first studied analytically in [12,13], and a full non-linear numerical analysis was performed in [14].

However, all the above mentioned analyses were restricted to spherically symmetric backgrounds (in particular, to the Schwarzschild and Reissner-Nordström black holes). On the other hand, astrophysical objects are usually rotating. Hence, the physical process of stellar core collapse to form a black hole is expected to be non-spherical in nature. The corresponding problem of wave dynamics outside a realistic, rotating black hole is much more complicated due to the lack of spherical symmetry. Originally, this problem was addressed numerically by Krivan et. al. [15,16], followed by Ori [17] and Barack [18] who provided preliminary analytical results for the evolution of a test scalar field on a Kerr background.

Evidently, the most interesting situation from a physical point of view is the propagation of gravitational waves on a rotating Kerr background. The evolution of higher-spin fields (and in particular gravitational perturbations) outside a realistic, rotating black hole was studied analytically only recently [19]. This was done by analyzing the asymptotic late-time solutions of Teukolsky’s master equation [20,21], which governs the evolution of gravitational, electromagnetic, neutrino and scalar perturbations fields on Kerr spacetimes.

The main result presented in Ref. [19] is the existence of inverse power-law tails at the asymptotic regions of timelike infinity \( i_+ \), null infinity \( \text{scri}_+ \) and at the black-hole outer horizon \( H_+ \) (where the power-law behaviour is multiplied by an oscillatory term, caused by the dragging of reference frames at the event horizon). This late-time behaviour is similar to the late-time behaviour of massless fields on a spherically-symmetric Schwarzschild
However, there is one important feature (besides the oscillatory behaviour along the black-hole horizon) which is unique to rotating collapse: Active coupling of different modes. The phenomena of mode-coupling has been observed in numerical solutions of Teukolsky’s equation [15,16] (a similar phenomena is known in the case of rotating stars [22]). Physically, this phenomena is caused by the dragging of reference frames, due to the rotation of the black hole (or the star). The issue of mode-coupling was discussed in Ref. [19], where preliminary considerations and results were presented. In this paper we give a complete mathematical analysis of this important phenomena for the case of a scalar field propagating on a realistic Kerr background, and we focus on its physical implications. We consider all the various possible cases of mode-coupling, which result in different types of asymptotic late-time evolutions (different damping exponents).

The main difficulty in analyzing wave dynamics in rotating Kerr background is the nontrivial dependence on the angular variable $\theta$. The separation of the $r$ variable from the $\theta$ variable by the Teukolsky equation is only applicable to the Fourier-decomposed field, i.e., in the frequency domain. This is caused by the fact that the spheroidal harmonics used for the separation of the $\theta$ variable explicitly depend on the temporal frequency $\omega$. It is convenient to use the basis of the ($\omega$-dependent) spheroidal wave functions and carry out the analysis in the frequency domain; these functions allow us to separate the $\theta$-dependence from the radial dependence, and, more important, allow us to separate the different modes (different values of $l$) so each mode evolves independently.

The final goal is, however, to calculate the late-time behaviour of the field, along with its angular dependence, in terms of the time $t$ (or $u,v$). Thus, the final answer for the field should depend on $t$ and $\theta$, and is, of course, $\omega$-independent – a final expression which involves the $\omega$-dependent spheroidal harmonics would be useless. Therefore, the final expression for the field is most naturally expressed in terms of the $\omega$-independent spherical harmonics.

The plan of the paper is as follows. In Sec. II we give a short description of the physical system and summarize the main analytical results presented in Ref. [19]. In Sec. III we
discuss the effects of rotation and the mathematical tools needed for the physical analysis are derived. In Sec. IV we study the active coupling of different modes during a rotating gravitational collapse, with pure initial data. In Sec. V we consider the coupling of different modes, with generic initial data. In the appendix we study the coupling of the real and the imaginary parts of the radial field. We conclude in Sec. VI with a summary of our analytical results.

II. REVIEW OF RECENT ANALYTICAL RESULTS

We consider the evolution of a massless scalar field outside a rotating star or a black hole. The external gravitational field of a rotating object of mass $M$ and angular-momentum per unit-mass $a$ is given by the Kerr metric, which in Boyer-Lindquist coordinates takes the form

$$ds^2 = \left(1 - \frac{2Mr}{\Sigma}\right) dt^2 + \left(\frac{4Mar^2\sin^2\theta}{\Sigma}\right) dt d\varphi - \frac{\Sigma}{\Delta} dr^2 - \Sigma d\theta^2 - \sin^2\theta \left(r^2 + a^2 + \frac{2Ma^2r\sin^2\theta}{\Sigma}\right) d\varphi^2,$$

where $\Sigma = r^2 + a^2\cos^2\theta$ and $\Delta = r^2 - 2Mr + a^2$ (we use gravitational units in which $G = c = 1$).

In Boyer-Lindquist coordinates the wave equation for a massless scalar field reads [23]

$$B_1(r, \theta) \frac{\partial^2 \psi}{\partial t^2} + B_2(r, \theta) \frac{\partial \psi}{\partial t} - \frac{\Delta}{\sin^2\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \psi}{\partial \theta}\right) = 0,$$

Resolving the field in the form

$$\psi = (r^2 + a^2)^{-1/2} \sum_{m=-\infty}^{\infty} \Psi^m e^{im\varphi},$$

where $m$ is the azimuthal number, one obtains a wave-equation for each value of $m$:

$$B_1(r, \theta) \frac{\partial^2 \Psi^m}{\partial t^2} + B_2(r, \theta) \frac{\partial \Psi^m}{\partial t} - \frac{\Delta}{\sin^2\theta} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \Psi^m}{\partial \theta}\right) = 0.$$
where the tortoise radial coordinate $y$ is defined by $dy = \frac{r^2 + a^2}{\Delta} dr$. The coefficients $B_i(r, \theta)$ are given by

$$B_1(r, \theta) = 1 - \frac{\Delta a^2 \sin^2 \theta}{(r^2 + a^2)^2},$$  \hspace{1cm} (5)$$

$$B_2(r) = \frac{4iMmar}{(r^2 + a^2)^2},$$  \hspace{1cm} (6)$$

and

$$B_3(r, \theta) = \left[ 2(r - M)r(r^2 + a^2)^{-1} - m^2 \left( \frac{a^2}{\Delta} - \frac{1}{\sin^2 \theta} \right) \right] \frac{\Delta}{(r^2 + a^2)^2}. \hspace{1cm} (7)$$

The time-evolution of a wave-field described by Eq. (4) is given by

$$\Psi(z, t) = 2\pi \int \int_0^\pi \left\{ B_1(z') \left[ G(z, z'; t)\Psi(z', 0) + G_t(z, z'; t)\Psi(z', 0) \right] + B_2(z')G(z, z'; t)\Psi(z', 0) \right\} \sin \theta' d\theta' dy', \hspace{1cm} (8)$$

for $t > 0$, where $z$ stands for $(y, \theta)$, and $G(z, z'; t)$ is the (retarded) Green’s function, which is defined by

$$\left[ B_1(r, \theta) \frac{\partial^2}{\partial t^2} + B_2(r, \theta) \frac{\partial}{\partial t} - \frac{\Delta}{(r^2 + a^2)^2} \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) \right] G(z, z'; t) = \delta(t) \delta(y - y') \delta(\theta - \theta') \frac{1}{2\pi \sin \theta}. \hspace{1cm} (9)$$

The Green’s function (which is associated with the existence of a branch cut in the complex frequency plane [4], usually placed along the negative imaginary $\omega-$ axis) was calculated in Ref. [19] [see Eq. (31) of [19]]:

$$G^C(z, z'; t) = \sum_{l=|m|}^{\infty} \frac{(-1)^{l+1}2l}{\pi A^2[(2l + 1)!]^2} \int_0^{-\infty} \frac{\Gamma(l + 1 - 2i\omega M)}{\Gamma(-l - 2i\omega M)} \tilde{\Psi}_1(y, \omega) \tilde{\Psi}_1(y', \omega) S_l(\theta, a\omega) S_l(\theta', a\omega) \omega^{2l+1} e^{-i\omega t} d\omega, \hspace{1cm} (10)$$

where the function $\tilde{\Psi}_1(y', \omega)$ is given by (see Ref. [19] for additional details):

$$\tilde{\Psi}_1 = Ar^{l+1} e^{i\omega r} M(l + 1 - 2i\omega M, 2l + 2, -2i\omega r), \hspace{1cm} (11)$$

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and $A$ is a normalization constant. This expression for the Green’s function is obtained using the large-$r$ (or equivalently, the law-$\omega$ approximation [19]). The functions $S_{l}^{m}(\theta, a\omega)$ are the spheroidal wave functions which are solutions to the angular equation [24, 21]

\[
\left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + a^2 \omega^2 \cos^2 \theta - \frac{m^2}{\sin^2 \theta} + A_{l}^{m} \right] S_{l}^{m} = 0 .
\]  

(12)

For the $a\omega = 0$ case, the eigenfunctions $S_{l}^{m}(\theta, a\omega)$ reduce to the well-known spherical harmonics $Y_{l}^{m}(\theta, \phi) = S_{l}^{m}(\theta)e^{im\phi}$, and the separation constants $A_{l}^{m}(a\omega)$ are simply $A_{l}^{m} = l(l + 1)$.

**III. ROTATION EFFECTS**

The rotational dragging of reference frames, due to the rotation of the black hole (or the star) produces an active coupling (during the gravitational collapse) between different modes. Thus, even if the initial data is made of a pure mode (i.e., it is characterized by a particular spherical harmonic $Y_{l}^{m}$) other modes would be generated dynamically during the evolution. Mathematically, the coupling of different modes is attributed to the $\theta$-dependence of the spheroidal wave functions $S_{l}^{m}(\theta, a\omega)$ and of the coefficient $B_{1}(r, \theta)$.

It is well known that the late-time behaviour of massless perturbations fields is determined by the backscattering from asymptotically far regions [3, 2]. Thus, the late-time behaviour is dominated by the low-frequencies contribution to the Green’s function, for only low frequencies will be backscattered by the small effective curvature potential (for $r \gg M$). Thus, as long as the observer is situated far away from the black hole and the initial data has a considerable support only far away from the black hole, a small-$\omega$ approximation is sufficient in order to study the asymptotic late-time behaviour of the field [6].

If the angular equation (12) is written in the form

\[
(L^{0} + L^{1})S_{l}^{m} = -A_{l}^{m} S_{l}^{m} ,
\]  

(13)

with
\[ L^0(\theta) = \frac{1}{\sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d}{d\theta} \right) - \frac{m^2}{\sin^2 \theta}, \tag{14} \]

and

\[ L^1(\theta, a\omega) = (a\omega)^2 \cos^2 \theta, \tag{15} \]

it is immediately amenable to a perturbation treatment for small \( a\omega \). The spherical functions \( Y_l^m \) are used as a representation. They satisfy \( L^0 Y_l = -A_l^{(0)} Y_l \) with \( A_l^{(0)} = l(l + 1) \) (we suppress the index \( m \) on \( A_l \) and \( Y_l \)). For small \( a\omega \) standard perturbation theory yields (see, for example, [25])

\[ A_l = A_l^{(0)} + L^1_{ll} + \sum_{k \neq l} \frac{|L^1_{lk}|^2}{A_l^{(0)} - A_k^{(0)}} + \cdots \tag{16} \]

and

\[ S_l = Y_l + \sum_{k \neq l} \frac{L^1_{kl}}{A_l^{(0)} - A_k^{(0)}} Y_k + \left[ \sum_{k \neq l} \sum_{n \neq l} \frac{L^1_{kln} L^1_{nl}}{(A_l^{(0)} - A_k^{(0)})(A_l^{(0)} - A_n^{(0)})} Y_k - \sum_{k \neq l} \frac{L^1_{kl} L^1_{lk}}{(A_l^{(0)} - A_k^{(0)})^2} Y_k \right] + \cdots \tag{17} \]

where

\[ L^1_{lk} \equiv \langle 0lm|L^1|0km \rangle \equiv \int Y_l^{\ast m} L^1 Y_k^m d\Omega. \tag{18} \]

Equation (17) says that the black-hole rotation mixes various spherical harmonics functions.

The spherical harmonics are related to the rotation matrix elements of quantum mechanics [26]. Hence, standard formulae are available for integrating the product of three such functions. The one we need in order to evaluate the integral Eq. (18) is

\[ \langle 0lm | \cos^2 \theta | 0km \rangle = \frac{1}{3} \delta_{kl} + \frac{2}{3} \left( \frac{2k + 1}{2l + 1} \right)^{1/2} \langle k2m0|lm\rangle \langle k200|l0 \rangle, \tag{19} \]

where \( \langle j_1j_2m_1m_2|JM \rangle \) is a Clebsch-Gordan coefficient. The well-known properties of the Clebsch-Gordan coefficients [27]:

1. \( \langle j_1j_2m_1m_2|JM \rangle \) vanishes unless \( j_1, j_2 \) and \( J \) satisfy the triangle selection rule \( |j_1 - j_2| \leq J \leq j_1 + j_2 \).
2. $\langle j_1 j_2 00 | J0 \rangle = 0$ if $j_1 + j_2 + J$ is an odd integer (this is a parity conservation law).

imply that $L^1_{lk} \neq 0$ for $l = k, k \pm 2$ (all other matrix elements vanish). In general, if we choose the spherical harmonics as a representation, so that

$$S_l(\theta, a\omega) = \sum_{k=|m|}^{\infty} C_{lk}(a\omega)^{|l-k|} Y_k(\theta),$$

then, to leading order in $(a\omega)^2$, the coefficients $C_{lk}(a\omega)$ take the form (see, for example, [25])

$$C_{lk} = \prod_{n=l,l+2,...}^{k-2} \frac{\langle 0n+2m | \cos^2 \theta | 0nm \rangle}{A_l^{(0)} - A_{n+2}^{(0)}},$$

if $k \geq l + 2$ and $k - l$ is even, and

$$C_{lk} = \prod_{n=k,k+2,...}^{l-2} \frac{\langle 0nm | \cos^2 \theta | 0n+2m \rangle}{A_l^{(0)} - A_{n}^{(0)}},$$

if $k \leq l - 2$ and $l - k$ is even. In addition, $C_{lk} = 0$ if $|l - k|$ is odd, and $C_{ll} = 1$.

The time-evolution of a wave-field described by Eq. (4) is given by Eq. (8). The coefficient $B_1(r, \theta)$ appearing in Eqs. (4) and (8) depend explicitly on the angular variable $\theta$ through the rotation of the black hole (no such dependence exist in the non-rotating $a = 0$ case). To elucidate the coupling between different modes we should evaluate the integral [see Eqs. (5) and (8)]

$$\langle 0lm | \sin^2 \theta | 0km \rangle = \frac{2}{3} \delta_{kl} - \frac{2}{3} \left( \frac{2k+1}{2l+1} \right)^{1/2} \langle k2m0|lm\rangle \langle k200|l0 \rangle.$$  \hspace{1cm} (23)

Clearly, this integral vanishes unless $l = k, k \pm 2$. This, togerher with the expressions for the coefficients $C_{lk}$ imply that if a mode $Y_l^m$ presents in the initial data, then all modes $Y_l^m$ with $|l^* - l|$ even (and $l \geq |m|$) will be present at late times.

As long as we are interested in the asymptotic late-time behaviour (the $\omega \rightarrow 0$ limit) of the solution, we may use the approximation

$$\frac{\Gamma(l + 1 - 2i\omega M)}{\Gamma(-l - 2i\omega M)} \simeq 2i\omega M (-1)^{l+1} l^2,$$

which is valid for $\omega \rightarrow 0$. Thus, we obtain [see Eq. (10)]
A note is needed here on the separation of variables in the Kerr background. The main difficulty in analyzing wave dynamics in rotating Kerr background is the nontrivial dependence on the angular variable $\theta$. The separation of the $r$ variable from the $\theta$ variable by the Teukolsky equation is only applicable to the Fourier-decomposed field, i.e., in the frequency domain. This is caused by the fact that the spheroidal harmonics used for the separation of the $\theta$ variable explicitly depend on the temporal frequency $\omega$. The final goal is, however, to calculate the late-time behaviour of the field, along with its angular dependence, in terms of the time $t$ (or $u,v$) – at the end of the analysis we should recover the explicit temporal dependence of the field by integrating over all frequencies in the frequency plane (leading to the final answer which is, of coarse, $\omega$-independent). Obviously, as the final expression for the field should depend on $t$ and $\theta$, an expression of this angular dependence in terms of the $\omega$-dependent spheroidal harmonics would be useless.

For a generic initial data in the time domain, one is able to write these data as a sum over $l,m$, and integration over $\omega$. Namely, $\psi(r,\theta,\varphi,t) = \sum_{m=-\infty}^{\infty} \sum_{l=-m}^{m} e^{im\varphi} \int_{-\infty}^{\infty} R_{l}^{m}(r,t)S_{l}^{m}(\theta,a\omega)e^{-i\omega t}d\omega$ with $t = 0$. Then, each mode [characterized by a particular spheroidal wave function $S_{l}^{m}(\theta,a\omega)$] evolves independently. Nevertheless, Eq. (20) implies that each spheroidal wave function can be expressed as a sum over the ($\omega$-independent) spherical harmonics (were the $\omega$-dependence of the spheroidal wave function is now moved to the expansion coefficients). That is, the initial data can be expressed also as $\psi(r,\theta,\varphi,t) = \sum_{m=-\infty}^{\infty} \sum_{l=-m}^{m} R_{l}^{m}(r,t)Y_{l}^{m}(\theta)e^{im\varphi}$ with $t = 0$.

It is convenient to use the basis of the ($\omega$-dependent) spheroidal wave functions and carry out the analysis in the frequency domain; these functions allow us to separate the $\theta$-dependence from the radial dependence, and, more important, allow us to separate the different modes (different values of $l$) so each mode evolves independently, see Eq. (12) of
The final goal is, however, to calculate the late-time behaviour of the field, along with its angular dependence, in terms of the time $t$ (or $u, v$). Thus, the final answer for the field should depend on $t$ and $\theta$, and is, of course, $\omega$-independent – a final expression which involves the $\omega -$dependent spheroidal harmonics would be useless. Therefore, the final expression for the field is most naturally expressed in terms of the $\omega$-independent spherical harmonics.

We shall consider two kinds of initial data: A generic initial data, which can be decomposite in terms of the basis of the spheroidal wave functions. Then each spheroidal wave function evolves independently. In view of Eq. (20), and as explained above, this corresponds to the assumption that the (generic) initial pulse consists of all allowed modes (all spherical harmonics functions with $l \geq |m|$).

On the other hand, in order to compare our analytical results with the numerical results of Krivan et. al. [15], one should consider an initial data which is made of a single mode. This corresponds to the assumption that the initial angular distribution is characterized by a particular spherical harmonic function $Y_{l}^{m}$. We begin with this case, which can be compared with the numerical results of [15].

IV. PURE INITIAL DATA

A. Asymptotic behaviour at timelike infinity

We consider first the behaviour of the scalar field at the asymptotic region of timelike infinity $i_+$. As was explained, the late-time behaviour of the field should follow from the low-frequency contribution to the Green’s function. Actually, it is easy to verify that the effective contribution to the integral in Eq. (25) should come from $|\omega|=O(\frac{1}{t})$. Thus, in order to obtain the asymptotic behaviour of the field at timelike infinity (where $y, y' \ll t$), we may use the $|\omega|r \ll 1$ limit of $\Psi_1(r, \omega)$. Using Eq. 13.5.5 of [27] one finds

$$\Psi_1(r, \omega) \simeq Ar^{l+1}.$$ (26)
Substituting this in Eq. (25) we obtain

\[ G^C(z, z'; t) = \sum_{l=|m|}^{\infty} \frac{2iM}{\pi[(2l + 1)!!]^2} (yy')^{l+1} \int_0^{-i\infty} S_l(\theta, a\omega) S_l(\theta', a\omega) \omega^{2l+2} e^{-i\omega t} d\omega . \]  

(27)

Using the representation for the spheroidal wave functions \( S_l \), Eq. (20), together with the fact that the integral Eq. (23) vanishes unless \( l = k, k \pm 2 \), we find that the asymptotic late-time behaviour of the \( l \) mode (where \( |l^* - l| \) is even and \( l \geq |m| \)) is dominated by the following effective Green’s function:

\[ G^C_l(z, z'; t) = \sum_{k=|m|+p}^{L} \frac{2iM}{\pi[(2k + 1)!!]^2} (yy')^{k+1} C_{kl} C_{kl}^* Y_l(\theta) Y_{l^*}^*(\theta') \int_0^{-i\infty} (a\omega)^{l^*+l-2k-q} \omega^{2k+2} e^{-i\omega t} d\omega , \]  

(28)

where \( p = 0 \) if \( l^* - |m| \) is even and \( p = 1 \) otherwise, and \( q = 2 \) if \( l^* \geq |m| + 2 \) and \( q = 0 \) otherwise. Here, \( L = l^* - q \) for \( l \geq l^* \) modes, and \( L = l \) for \( l \leq l^* - 2 \) modes. Performing the integration, we obtain

\[ G^C_l(z, z'; t) = \sum_{k=|m|+p}^{L} \frac{2M(-1)^{(l^*+l+2-q)/2}(l^* + l + 2 + q)!}{\pi[(2k + 1)!!]^2} (yy')^{k+1} C_{kl} C_{kl}^* Y_l(\theta) Y_{l^*}^*(\theta') a^{l^*+l-2k-q} t^{-(l^*+l+3-q)} . \]  

(29)

Thus, the late-time behaviour of a scalar field at the asymptotic region of timelike infinity \( i_+ \) is dominated by the lowest allowed mode, i.e., by the \( l = |m| \) mode if \( l^* - |m| \) is even, and by the \( l = |m| + 1 \) mode if \( l^* - |m| \) is odd. The corresponding damping exponent is \(- (l^* + |m| + p + 1) \) if \( l^* \geq |m| + 2 \), and \(- (2l^* + 3) \) if \( l^* = |m|, |m| + 1 \).

These analytical results should be compared with the numerical results of Krivan et. al. [15]. The comparison can be made for the specific initial data considered in [15], and is presented in Table III – we find a perfect agreement between our analytical results and the numerical results of Krivan et. al. [15].
We further consider the behaviour of the scalar field at the asymptotic region of future null infinity \( \text{scri}^+ \). It is easy to verify that for this case the effective frequencies contributing to the integral in Eq. (25) are of order \( O(\frac{1}{u}) \). Thus, for \( y - y' \ll t \ll 2y - y' \) one may use the \( |\omega|y' \ll 1 \) asymptotic limit for \( \tilde{\Psi}_1(y', \omega) \) and the \( |\omega|y \gg 1 \) \((Im\omega < 0)\) asymptotic limit of \( \tilde{\Psi}_1(y, \omega) \). Thus,

\[
\tilde{\Psi}_1(y', \omega) \simeq Ay^{l+1},
\]

and

\[
\tilde{\Psi}_1(y, \omega) \simeq Ae^{i\omega(2l + 1)} \frac{e^{-\frac{i\pi}{4}(l+1-2i\omega M)}(2\omega)^{-l-1+2i\omega M}y^{2i\omega M}}{\Gamma(l + 1 + 2i\omega M)},
\]

where we have used Eqs. 13.5.5 and 13.5.1 of [27], respectively. Substituting this in Eq. (25) we obtain (using the \( M\omega \to 0 \) limit)

\[
G^C(z, z'; t) = \sum_{l=|m|}^{\infty} \frac{M2^{-l}(2l + 1)!(-1)^{(l+2)/2}}{\pi l![(2l + 1)!!]^2} y^{l+1} - \int_{-i\infty}^{i\infty} S_{l}(\theta, a\omega)S_{l}(\theta', a\omega)\omega^{l+1}e^{-i\omega(t-y)}d\omega.
\]

Using the representation Eq. (20) for the spheroidal wave functions \( S_l \), together with the fact that the integral Eq. (23) vanishes unless \( l = k, k \pm 2 \), we find that the behaviour of the \( l \) mode (where \( |l^* - l| \) is even and \( l \geq |m| \)) at the asymptotic region of null infinity \( \text{scri}^+ \) is dominated by the following effective Green’s function:

\[
G^C_l(z, z'; t) = \sum_{k=l^*-q_1}^{l^*+q_2} \frac{M2^{-k}(2k + 1)!(-1)^{(k+2)/2}}{\pi k![(2k + 1)!!]^2} y^{k+1} C_{kl}Y_{l}(\theta)Y^{*}_{k}(\theta') \int_{0}^{-i\infty} (a\omega)^{l-k}\omega^{k+1}e^{-i\omega(t-y)}d\omega,
\]

where \( q_2 = 0 \) for the \( l = l^* \) mode and \( q_2 = 2 \) for \( l \geq l^* + 2 \) modes \((q_1 = 2 \) if \( l^* \geq |m| + 2 \) and \( q_1 = 0 \) otherwise), and

\[
G^C_l(z, z'; t) = \frac{M2^{-l}(2l + 1)!(-1)^{(l+2)/2}}{\pi l![(2l + 1)!!]^2} y^{l+1} C_{l}Y_{l}(\theta)Y^{*}_{l}(\theta') \int_{0}^{-i\infty} (a\omega)^{l-2-l}\omega^{l+1}e^{-i\omega(t-y)}d\omega,
\]

B. Asymptotic behaviour at future null infinity
for \( l \leq l^* - 2 \) modes. Performing the integration in Eqs. (33) and (34), one finds

\[
G^C_l(z, z'; t) = \sum_{k=l^*-q_1}^{l+q_2} \frac{M^{2-k}(-1)^{(l+k+2)/2}(2k+1)!(l+1)!}{\pi k![(2k+1)!!]^2} y^{k+1} C_k Y_l(\theta) Y^*_k(\theta') a^{l-k} u^{-(l+2)},
\]

for \( l \geq l^* \) modes, and

\[
G^C_l(z, z'; t) = \frac{M^{2-l}(-1)^{(l^*+1)/2}(l^* - 1)!(2l+1)!}{\pi l![(2l+1)!!]^2} y^{l+1} C_{l^*-2} Y_l(\theta) Y^*_l(\theta') a^{l^*-2} u^{-l^*},
\]

for \( l \leq l^* - 2 \) modes. Thus, we find that the late-time behaviour of a scalar field at the asymptotic region of null infinity \( \text{scri}_+ \) is dominated by the \( l \leq l^* - 2 \) modes provided that \( l^* \geq |m| + 2 \) (where \( |l^* - l| \) is even and \( l \geq |m| \)). These modes decay as an inverse power of the retarded time \( u \), with a damping exponent equals to \(-l^*\). If \( l^* = |m|, |m| + 1 \), then the late-time evolution will be dominated by the \( l = l^* \) mode, which has a damping exponent equals to \(-(l^* + 2)\).

**C. Asymptotic behaviour at the black-hole outer horizon**

Finally, we consider the behaviour of the field at the black-hole outer-horizon \( H_+ \). The asymptotic solution at the horizon \((y \to -\infty)\) is given by [21] (see Ref. [19] for additional details)

\[
\tilde{\Psi}_1(y, \omega) = C(\omega) e^{-i(w - mw_+)} y,
\]

where \( w_+ = a/(2Mr_+) \). In addition, we use Eq. (30) for \( \tilde{\Psi}_1(y', \omega) \). In order to match the \( y \ll -M \) solution with the \( y \gg M \) solution we assume that the two solutions have the same temporal dependence (this assumption has been proven to be very successful for neutral [5] and charged [12] perturbations on spherically symmetric backgrounds). In other words, we take \( C(\omega) \) to be \( \omega \)-independent.

Using the representation Eq. (20) for the spheroidal wave functions \( S_l \), together with the fact that the integral Eq. (23) vanishes unless \( l = k, k \pm 2 \), we find that the asymptotic
behaviour of the \( l \) mode (where \(|l^* - l|\) is even and \( l \geq |m|\)) at the black-hole outer horizon \( H_+ \) is dominated by the following effective Green’s function:

\[
G^C_l(z, z'; t) = \sum_{k = |m| + p}^L \frac{2M(-1)^{(l^*+l+2-q)/2}(l^* + l + 2 - q)!}{\pi[(2k + 1)!!]^2} \gamma^{k+1} C_{kl}C_{l^*+q}Y_l(\theta)Y_{l^*+q}(\theta')a^{l^*+l-2k-q}e^{imw+uy}e^{-(l^*+l+3-q)} ,
\]

where \( q, p \) and \( L \) are defined as before, and \( \Gamma_k \) are constants. Hence, we find that the late-time behaviour of a scalar field at the black-hole outer horizon is dominated by the lowest allowed mode, i.e., by the \( l = |m| \) mode if \(|l^* - m|\) is even, and by the \( l = |m| + 1 \) mode if \(|l^* - m|\) is odd.

V. GENERIC INITIAL DATA

In this section we consider the generic case. That is, we assume that the initial pulse consists of all the allowed (\( l \geq |m| \)) modes. The analysis here is very similar to the one presented in IV. We consider first the behaviour of the scalar field at the asymptotic region of timelike infinity \( i_+ \). Using Eq. (27), together with the representation Eq. (20) for the spheroidal wave functions, we find that the asymptotic late-time behaviour of the \( l \) mode (where \( l \geq |m| \)) is dominated by the following effective Green’s function:

\[
G^C_l(z, z'; t) = \frac{2iM}{\pi[(2k + 1)!!]^2} (yy')^{k+1} C_{kl}Y_l(\theta)Y_l(\theta') \int_0^{i\infty} (a\omega)^{-k} \omega^{2k+2} e^{-i\omega t} d\omega ,
\]

where \( k = |m| + p \), and \( p = 0 \) if \( l - |m| \) is even, and \( p = 1 \) otherwise. Performing the integration, we obtain

\[
G^C_l(z, z'; t) = \frac{2M(-1)^{(l+|m|+p+2)/2}(l+|m|+p+2)!}{\pi[(2|m|+2p+1)!!]^2} \gamma^{|m|+p+1} C_{|m|+p}Y_l(\theta)Y_{|m|+p}(\theta')a^{-|m|-p}e^{-(l+|m|+p+3)} .
\]

We emphasize that the power indices \( l + |m| + p + 3 \) (in absolute value) in rotating Kerr spacetimes are smaller than the corresponding power indices (the well known \( 2l + 3 \)) in
spherically symmetric Schwarzschild spacetimes. (There is an equality only for the \(l = |m|, |m|+1\) modes). This implies a slower decay of perturbations in rotating Kerr spacetimes. From Eq. (40) it is easy to see that the time scale \(t_c\) at which the late-time tail of rotating gravitational collapse is considerably different from the corresponding tail of non rotating collapse (for the \(l > |m|+1\) modes) is 
\[
t_c = \frac{y y' a}{(2k+1)!!},
\]
where \(y'\) is roughly the average location of the initial pulse.

We further consider the asymptotic behaviour of the field at future null infinity \(\text{scri}_+\). Using Eq. (32), together with the representation Eq. (20) for the spheroidal wave functions, one finds that the behaviour of the \(l\) mode at the asymptotic region of null infinity is dominated by the following effective Green’s function:

\[
G^C_l(z, z'; t) = \sum_{k=|m|+p}^{l} \frac{M 2^{-k} (2k+1)! (-1)^{(k+2)/2} y^{k+1}}{\pi k! [(2k+1)!!]^2} C_{kl} Y_l(\theta) Y^*_k(\theta') \int_0^{-i\infty} (a\omega)^{-l-k} \omega^{k+1} e^{-i\omega(t-y)} d\omega, \quad (41)
\]
where \(p\) is defined above. Performing the integration we obtain

\[
G^C_l(z, z'; t) = \sum_{k=|m|+p}^{l} \frac{M 2^{-k} (-1)^{(l+k+2)/2} (2k+1)! (l+1)! y^{k+1}}{\pi k! [(2k+1)!!]^2} C_{kl} Y_l(\theta) Y^*_k(\theta') u^{-l-k} a^{-(l+2)}. \quad (42)
\]

Finally, we consider the behaviour of the field at the black-hole outer-horizon \(H_+\). Following an analysis similar to the one presented in Sec. IV C, we find that the asymptotic behaviour of the \(l\) mode (where \(l \geq |m|\)) at the black-hole outer horizon \(H_+\) is dominated by the following effective Green’s function:

\[
G^C_l(z, z'; t) = \Gamma_k \frac{2 M (-1)^{(l+|m|+p+2)/2} (l+|m|+p+2)!}{\pi [(2|m|+2p+1)!!]^2} \frac{1}{y^{|m|+p+1}} C_{|m|+p} Y_l(\theta) Y^*_{|m|+p}(\theta') u^{-|m|-p} e^{inw} y^{-(l+|m|+p+3)} , \quad (43)
\]

VI. SUMMARY OF RESULTS

The asymptotic late-time evolution of a test scalar field on a realistic Kerr background is characterized by inverse power-law tails at the three asymptotic regions: timelike infinity.
future null infinity $scri_+$, and the black-hole outer-horizon $H_+$ (where the power-law behaviour is multiplied by an oscillatory term, caused by the dragging of reference frames at the event horizon). This late-time behaviour is similar to the the late-time evolution of massless fields on a spherically-symmetric Schwarzschild background, originally analyzed by Price [2].

However, there is one important feature (besides the oscillatory behaviour along the black-hole horizon) which is unique to rotating collapse: Active coupling of different modes. Physically, this phenomena is caused by the dragging of reference frames, due to the black-hole (or star’s) rotation (this phenomena is absent in the non-rotating $a = 0$ case). Hence, the late-time tails of realistic rotating collapse are dominated by a mode, which, in general, is different from the initial one.

As shown in our analysis, there are various cases, which result in different types of asymptotic late-time evolutions (different damping exponents). These distinct cases are characterized by the values of $l^*$ and $|m|$, and the parity of $l^* - |m|$. The dominant modes at asymptotic late-times and the values of the corresponding damping exponents (for pure initial data, characterized by a particular spherical harmonic function $Y^{\ell^*m}_l$) are summarized in Tables I (for $l^* \geq |m| + 2$) and II (for $l^* = |m|, |m| + 1$). These analytical results should be compared with the numerical results of [15] – we find a complete agreement between the two, as can be seen in Table III.

In summary, the rotation of a black hole (or a star) results in an active coupling of different modes. Thus, new modes, which are different from the original one, would be generated during a rotating collapse and may play an important role at asymptotic late times. Hence, the radiative tail of rotating gravitational collapse decays slower than the corresponding tail of a non-rotating collapse. In general, a rotating Kerr black hole becomes ‘bald’ slower than a spherically-symmetric Schwarzschild black hole.

**ACKNOWLEDGMENTS**

I thank Tsvi Piran for discussions. This research was supported by a grant from the
APPENDIX A: COUPLING OF THE REAL AND IMAGINARY PARTS OF $\Psi$

In general, the $\Psi^m$ functions are complex quantities. Owing to the purely real character of the $B_1(r, \theta)$ and $B_2(r)$ coefficients, the real and imaginary parts of each $\Psi$ are decoupled for axially-symmetric ($m = 0$) modes (where $B_2$ vanishes and the Green’s function is purely real). However, for non-axially symmetric ($m \neq 0$) modes, the imaginary coefficient $B_2(r)$ mixes the real and imaginary parts of the same mode. Hence, the real part of a particular mode will always generate the corresponding imaginary part of the same mode, and vice versa (note, however, that the coefficient $B_2(r)$ is $\theta$–independent. Hence, contrasted with the $B_1(r, \theta)$ coefficient, it cannot mix nearest neighbors modes). We now assume that the initial data for $\Psi$ is purely real, and (without loss of generality) we assume that its angular distribution is characterized as before by a particular spherical harmonic function $Y^m_m$.

We consider first the asymptotic behaviour of the imaginary part of $\Psi^m$ at timelike infinity $i_+$. Following an analysis similar to the one presented in Sec. IV A, one finds that the asymptotic late-time behaviour of the imaginary part of the $l$ mode (where $|l^*-l|$ is even and $l \geq |m| > 0$) is dominated by the following effective Green’s function:

$$G^{C}_{l^*}(z, z'; t) = \sum_{k = |m| + p}^{L} \frac{2M(-1)^{(l^*+l+2)/2}(l^* + l + 2)!}{\pi[(2k+1)!!]^2} (yy')^{k+1} C_{kl}C_{kl^*}Y_l(\theta)Y_{l^*}^*(\theta') a^{l^*-l+2k} t^{-(l^*+l+3)} ,$$

where $p = 0$ if $l^*-|m|$ is even and $p = 1$ otherwise. Here, $L = l^*$ for $l \geq l^*$ modes and $L = l$ for $l \leq l^* - 2$ modes. Hence, we find that the evolution at timelike infinity $i_+$ of the imaginary part of $\Psi$ is dominated by the lowest allowed mode.

We further consider the asymptotic behaviour of the imaginary part of $\Psi$ at future null infinity $scri_+$. Following an analysis similar to the one presented in Sec. IV B, one finds that the asymptotic late-time behaviour of the imaginary part of the $l$ mode is dominated by the following effective Green’s function:
\begin{equation}
G^C_l(z, z'; t) = \frac{M(-1)^{(l+l^*+2)/2}2^{-l^*}(2l^* + 1)!(l + 1)!}{\pi l^*![(2l^* + 1)!!]^2} y^{l^*+1} C_{l^*l} Y_l(\theta) Y_{l^*}^*(\theta') a^{-l^*} u^{-(l+2)} ,
\end{equation}

for \( l \geq l^* \) modes, and

\begin{equation}
G^C_l(z, z'; t) = \frac{M(-1)^{(l+l^*+2)/2}2^{-l^*}(2l^* + 1)!(l + 1)!}{\pi l![(2l + 1)!!]^2} y^{l+1} C_{ll^*} Y_l(\theta) Y_{l^*}^*(\theta') a^{-l} u^{-(l^*+2)} ,
\end{equation}

for \( l \leq l^* - 2 \) modes. Thus, we find that the evolution at null infinity \( \text{scri}_+ \) of the imaginary part of \( \Psi \) is dominated by the \( l \leq l^* \) modes (provided that \(|l - l^*| \) is even and \( l \geq |m| > 0 \)).

Finally, we note that the Green’s function Eq. (38) is already a complex function for non-axially symmetric \( (m \neq 0) \) modes. Thus, the real and imaginary parts of the \textit{same} mode are coupled by both the \( B_1(r, \theta) \) and \( B_2(r) \) coefficients. Since the coupling can occur by the \( \theta \)—dependent coefficient \( B_1(r, \theta) \) (which couples nearest neighbors modes), the asymptotic behaviour at the black-hole outer horizon \( H_+ \) of the imaginary part of the \( l \) mode is dominated by the effective Green’s function Eq. (38).
REFERENCES


### TABLE I. Dominant modes and asymptotic damping exponents for \( l^* \geq |m| + 2 \)

<table>
<thead>
<tr>
<th>Asymptotic Region</th>
<th>Dominant Mode(s)</th>
<th>Damping Exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Timelike Infinity</td>
<td>(</td>
<td>m</td>
</tr>
<tr>
<td>Null Infinity</td>
<td>(</td>
<td>m</td>
</tr>
<tr>
<td>Outer Horizon</td>
<td>(</td>
<td>m</td>
</tr>
</tbody>
</table>

### TABLE II. Dominant mode and asymptotic damping exponents for \( l^* = |m|, |m| + 1 \)

<table>
<thead>
<tr>
<th>Asymptotic Region</th>
<th>Dominant Mode</th>
<th>Damping Exponent</th>
</tr>
</thead>
<tbody>
<tr>
<td>Timelike Infinity</td>
<td>(l^*)</td>
<td>(-(2l^* + 3))</td>
</tr>
<tr>
<td>Null Infinity</td>
<td>(l^*)</td>
<td>(-(l^* + 2))</td>
</tr>
<tr>
<td>Outer Horizon</td>
<td>(l^*)</td>
<td>(-(2l^* + 3))</td>
</tr>
</tbody>
</table>

### TABLE III. Analytical vs. numerical results – the dominant damping exponent at \( i_+ \)

<table>
<thead>
<tr>
<th>Initial Data</th>
<th>Analytical Result</th>
<th>Numerical Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>(l = m = 1)</td>
<td>-5</td>
<td>-4.87</td>
</tr>
<tr>
<td>(l = m = 2)</td>
<td>-7</td>
<td>-6.98</td>
</tr>
<tr>
<td>(l = 2, m = 0)</td>
<td>-3</td>
<td>(\sim -2.9)</td>
</tr>
</tbody>
</table>

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