Exact Solutions of the Schrödinger Equation with Inverse-Power Potential in Two Dimensions

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Abstract

By applying a factorization ansatz for the eigenfunctions, an exact analytic solution of the stationary Schrödinger equation in two dimensions is obtained with the inverse-power potential $V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1}$.

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1. Introduction

It is well known that exact solutions of the fundamental dynamical equations play a crucial role in different fields of physics. Indeed, the exact solutions of the Schrödinger equation are possible only for certain potentials, and some approximation methods are frequently used to arrive at the solutions. In particular, the inverse-power potential, proportional to $r^{-n}$, has been studied in different areas of classical as well as quantum mechanics. For instance, the interatomic potential in molecular physics [1, 2], the potentials $V(r) = -Z^2 \alpha/r^4$ [3] (interaction between an ion and a neutral atom) and $V(r) = -d_1d_2/r^3$ [4] (interaction between a dipole $d_1$ and another dipole $d_2$) are often used to describe the interaction between a kind of matter and another one. The interaction in one-electron atoms, muonic and hadronic and Rydberg atoms also requires considering the inverse-power potentials [5]. Actually, the interaction potentials mentioned above are only special cases of the inverse-power potential $V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1}$, when some parameters of the potential vanish.

The reasons we write this paper are as follows. In the light of the wide interest in lower-dimensional field theory in the recent literature, it appears necessary to solve the two-dimensional Schrödinger equation with the inverse-power potential, an investigation which, to our knowledge, has never appeared in the literature. Furthermore, two-dimensional models are often applied to make the more involved higher-dimensional systems tractable. Therefore, it seems reasonable to study the two-dimensional Schrödinger equation with the inverse-power potential. On the other hand, by virtue of an ansatz for the eigenfunctions, the first two authors have recently succeeded in dealing with the Schrödinger equation in the presence of anharmonic potentials, such as the singular potential in two dimensions [9], the sextic potential [10] and the octic potential [11]. The purpose of this paper is to study the two-dimensional Schrödinger equation with this potential by this way.

Our paper is organized as follows. In section 2 we analyze the two-dimensional stationary Schrödinger equation with the inverse-power potential, making use of a factorization ansatz for the eigenfunctions, and we obtain an exact solution of the
Schrödinger equation, with application to the ground state and the first excited state. The structure of such calculations with arbitrary values of the angular momentum quantum number is derived in section 3. Some particular cases and concluding remarks are presented in section 4.

2. An ansatz for the eigenfunctions

We consider the two-dimensional Schrödinger equation for stationary states with a potential \( U \) depending only on the distance \( r \) from the origin, i.e.

\[
\left( -\frac{\hbar^2}{2\mu} \Delta + U(r) \right) \psi = \tilde{E} \psi.
\] (1a)

On setting \( V(r) \equiv \frac{2\mu}{\hbar^2} U(r), \) \( E = \frac{2\mu}{\hbar^2} \tilde{E} \), this reads

\[
\left[ \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} + V(r) \right] \psi(r, \varphi) = E \psi(r, \varphi)
\] (1b)

where hereafter the potential is taken to be

\[
V(r) = Ar^{-4} + Br^{-3} + Cr^{-2} + Dr^{-1} \quad A > 0 \quad D < 0.
\] (2)

The choice of \( r, \varphi \) coordinates reflects a model where the full Hilbert space is the tensor product of the space of square-integrable functions on the positive half-line with the space of square-integrable functions on the circle. We therefore write

\[
\psi(r, \varphi) = r^{-1/2} R_m(r) e^{\pm i m \varphi} \quad m = 0, 1, 2, \ldots
\] (3)

and this factorization leads to a second-order equation for the radial function \( R_m \) with vanishing coefficient of the first derivative, i.e.

\[
\left[ \frac{d^2}{dr^2} + E - V(r) - \left( \frac{m^2}{r^2} - \frac{1}{4} \right) \right] R_m(r) = 0
\] (4)

where \( m \) and \( E \) are the angular momentum quantum number and the energy, respectively. The work in [6–11] suggests to express the radial function \( R_m \) in the product form

\[
R_m(r) = h_m(r) \exp[k(r)]
\] (5)
where
\[ h_m(r) = \begin{cases} 1 & \text{when } m = 0 \\ \prod_{j=1}^{m}(r - \sigma_j^{(m)}) & \text{when } m = 1, 2, 3, \ldots \end{cases} \] (6a)

and
\[ k(r) = \frac{a}{r} + br + c \log r \quad a < 0 \quad b < 0. \] (6b)

The insertion of the ansatz (5) into Eq. (4) leads to the equation
\[ R''_m(r) - \left[ k''(r) + (k'(r))^2 + \left( \frac{h''_m(r) + 2h'_m(r)k'(r)}{h_m(r)} \right) \right] R_m(r) = 0 \] (7)

where the prime denotes the derivative with respect to the variable \( r \). We now express \( R''_m \) from Eq. (4) arriving at the equation
\[ -E + V(r) + \frac{m^2 - 1/4}{r^2} = k''(r) + (k'(r))^2 + \frac{h''_m(r) + 2h'_m(r)k'(r)}{h_m(r)} \] (8)

which is the most fundamental formula for the following analysis.

First, we study the ground state. In such a case the quantum number \( m \) vanishes, and we take \( h_0(r) = 1 \) in agreement with (6a). The equations (2), (6b) and (8) lead therefore to an algebraic equation where we equate coefficients of \( r^p \), for all \( p = -4, -3, -2, -1, 0 \). Hence we find
\[ a^2 = A \quad 2a(1 - c) = B \] (9a)
\[ c(c - 1) - 2ab = \lambda \] (9b)
\[ 2bc = D \quad b^2 = -E \] (9c)

where hereafter
\[ \lambda \equiv C + m^2 - \frac{1}{4}. \] (9d)

The first equation in (9a) is solved by \( a = \pm \sqrt{A} \), but only the latter root yields a regular wave function at \( r = 0 \), and hence we choose \( a = -\sqrt{A} \). One then finds, from Eq. (9b) and the first equation in (9c),
\[ \lambda = \mu(1 + \mu) + \frac{D\sqrt{A}}{1 + \mu} \] (10)

where we have defined \( \mu \equiv B/\sqrt{4A} \).
Equations (9a)–(9c) lead also to a very useful formula for the energy, upon remarking that

\[ 8Ab^2 - 4\sqrt{A}b + BD = 0. \]  

(11)

This algebraic equation has two roots, i.e.

\[ b = \frac{\lambda \pm \sqrt{\lambda^2 - 2BD}}{4\sqrt{A}} \]  

(12)

and hence we find the energy of the fundamental state in the form

\[ E_0^\pm = -\frac{1}{16A} \left[ \lambda \pm \sqrt{\lambda^2 - 2BD} \right]^2. \]  

(13)

Moreover, by virtue of Eq. (5), the corresponding eigenfunction can be written as

\[ R_0(r) = N_0 r^c \exp \left[ \frac{a}{r} + br \right] \]  

(14)

where \( N_0 \) is a normalization constant.

In the first excited state the quantum number \( m \) is equal to 1. Following (6a) we write \( h_1(r) = r - \sigma_1^{(1)} \), and Eq. (8) leads to

\[ -E + \frac{A}{r^4} + \frac{B}{r^3} + \frac{C}{r^2} + \frac{D}{r} + \frac{3}{4} \frac{1}{r^2} = \frac{2a}{r^3} - \frac{c}{r^2} \]

\[ + \left( \frac{-a}{r^2} + b + \frac{c}{r} \right)^2 + 2 \left( \frac{-a}{r^2} + b + \frac{c}{r} \right) \left( r - \sigma_1^{(1)} \right). \]  

(15)

We now multiply both sides of Eq. (15) by \( r - \sigma_1^{(1)} \) and equate the coefficients of \( r^p \), for all \( p = -4, -3, -2, -1, 0, 1 \). Such a procedure yields the equations

\[ -A\sigma_1^{(1)} = -a^2\sigma_1^{(1)} \]  

(16)

\[ A - B\sigma_1^{(1)} = -2a(1 - c)\sigma_1^{(1)} + a^2 \]  

(17)

\[ B - \left( C + \frac{3}{4} \right) \sigma_1^{(1)} = -2ac + \left[ c(1 - c) + 2ab \right] \sigma_1^{(1)} \]  

(18)

\[ C + \frac{3}{4} - D\sigma_1^{(1)} = c(c + 1) - 2b \left( a + c\sigma_1^{(1)} \right) \]  

(19)

\[ D + E\sigma_1^{(1)} = 2b(c + 1) - b^2\sigma_1^{(1)} \]  

(20)

\[ -E = b^2. \]  

(21)
The solution of the above system is hence found to be

\[ A = a^2 \quad E = -b^2 \]  
\[ B = 2a(1 - c) \quad D = 2b(c + 1) \]  
\[ \lambda = c(c + 1) - 2b\left(a - \sigma_1^{(1)}\right) \]  
\[ b\left(\sigma_1^{(1)}\right)^2 + c\sigma_1^{(1)} - a = 0. \]  

In particular, Eq. (25) provides a restriction on the parameter \( \sigma_1^{(1)} \) and is obtained upon inserting the formula for \( \lambda \) (see (24)) into Eq. (18), which reads

\[-\left(C + \frac{3}{4}\right)\sigma_1^{(1)} = \left[c(1 - c) + 2ab\right]\sigma_1^{(1)} - 2a\]

by virtue of the first equality in (23). Of course, we choose again \( a = -\sqrt{A} \) to ensure regularity of the wave function, and we use Eqs. (22)–(24) to find

\[ c = 1 + \mu \]  
\[ 2b = \frac{D}{c + 1} = \frac{D\sqrt{A}}{2\sqrt{A} + \frac{B}{2}} \equiv \eta \]  
\[ \lambda = (1 + \mu)(2 + \mu) + \eta\left(\sqrt{A} + \sigma_1^{(1)}\right) \]  

while Eq. (25) may be re-expressed as

\[ \frac{\eta}{2}\left(\sigma_1^{(1)}\right)^2 + (1 + \mu)\sigma_1^{(1)} + \sqrt{A} = 0. \]

Moreover, Eqs. (22)–(24) imply that the following algebraic equation for \( b \) holds (cf (11)):

\[ 2\left(\sigma_1^{(1)} - a\right)b^2 - \lambda b + \frac{1}{2}Dc = 0 \]

which is solved by

\[ b = \frac{\lambda \pm \sqrt{\lambda^2 - 4Dc(\sigma_1^{(1)} - a)}}{4(\sigma_1^{(1)} - a)} \]

so that the energy eigenvalue reads (see (22))

\[ E_1^\pm = -\left(\frac{\lambda \pm \sqrt{\lambda^2 - 4D(\sigma_1^{(1)} + \sqrt{A})(1 + \mu)}}{4(\sigma_1^{(1)} + \sqrt{A})}\right)^2. \]
The corresponding radial function $R_1$ can be written as (see (5) and (6))

$$R_1(r) = N_1 \left( r - \sigma_1^{(1)} \right) r^c \exp \left[ \frac{a}{r} + br \right]$$  \hspace{1cm} (33)

where $N_1$ is a normalization constant.

3. Arbitrary values of $m$

It is now possible to understand the general structure of the calculation with arbitrary values of the angular momentum quantum number $m$. For this purpose, one first needs the explicit form of the product in the ansatz (6a). This is found to read

$$h_m(r) = \sum_{k=0}^{m} a_k r^k$$  \hspace{1cm} (34)

where

$$a_m = 1$$  \hspace{1cm} (35)

$$a_{m-1} = -\sum_{i=1}^{m} \sigma_i^{(m)}$$  \hspace{1cm} (36)

$$a_{m-2} = \sum_{i<j}^{m} \sigma_i^{(m)} \sigma_j^{(m)}$$  \hspace{1cm} (37)

$$a_{m-3} = -\sum_{i<j}^{m} \sigma_i^{(m)} \sigma_j^{(m)} \sum_{k<l}^{m} (\sigma_k^{(m)} + \sigma_l^{(m)})$$  \hspace{1cm} (38)

and so on. Thus, the insertion of (6a) into Eq. (8) leads to the same formulae for $A$, $E$ and $B$ obtained in (22) and (23), whereas we find for the other parameters

$$D = 2b(c + m)$$  \hspace{1cm} (39)

$$\lambda = c(c + 2m - 1) + m(m - 1) - 2b \left( a - \sum_{i=1}^{m} \sigma_i^{(m)} \right)$$  \hspace{1cm} (40)

and (cf (29))

$$m\sqrt{A} + (m + \mu) \sum_{i=1}^{m} \sigma_i^{(m)} + \frac{\eta}{2} \sum_{i=1}^{m} (\sigma_i^{(m)})^2 = 0$$  \hspace{1cm} (41)

$$(m - 1)\sqrt{A} \sum_{i=1}^{m} \sigma_i^{(m)} + (2\mu + 2m - 1) \sum_{i<j}^{m} \sigma_i^{(m)} \sigma_j^{(m)}$$

$$+ \frac{\eta}{2} \sum_{i<j}^{m} \sigma_i^{(m)} \sigma_j^{(m)} \sum_{k<l}^{m} (\sigma_k^{(m)} + \sigma_l^{(m)}) = 0$$  \hspace{1cm} (42)
\[(m - 2)\sqrt{A} \sum_{i<j}^{m} \sigma_{i}^{(m)} \sigma_{j}^{(m)} + 3(\mu + m - 1) \sum_{i<j<l}^{m} \sigma_{i}^{(m)} \sigma_{j}^{(m)} \sigma_{l}^{(m)} + \frac{\eta}{2} \sum_{i<j<l}^{m} \sigma_{i}^{(m)} \sigma_{j}^{(m)} \sum_{p<q<r}^{m} (\sigma_{p}^{(m)} + \sigma_{q}^{(m)} + \sigma_{r}^{(m)}) = 0. \tag{43} \]

Equations (41)–(43) are the full set of restrictions on the parameters of the potential, and Eq. (40) may be re-expressed in the form (cf (28))

\[\lambda = (1 + \mu)(2m + \mu) + m(m - 1) + 2b \left( \sqrt{A} + \sum_{i=1}^{m} \sigma_{i}^{(m)} \right) \tag{44} \]

because \(c = 1 + \mu\) (see (26)). On defining

\[\omega_1 \equiv \lambda + m(2m + \mu) - m(m - 1) \tag{45} \]

the general formula for energy eigenvalues reads (cf (32))

\[E_{m}^{\pm} = - \left( \omega_1 \pm \sqrt{\omega_1^2 - 4D(2m + \mu)(\sum_{i=1}^{m} \sigma_{i}^{(m)} + \sqrt{A})^{1/2}} \right)^2 m = 1, 2, 3, \ldots \tag{46} \]

Furthermore, the radial function is factorized in the form \((m = 1, 2, 3, \ldots)\)

\[R_{m}(r) = N_{m} \left( \prod_{i=1}^{m} (r - \sigma_{i}^{(m)}) \right) r^{c} \exp \left[ \frac{a}{r} + br \right] \tag{47} \]

where all normalization constants can be evaluated from the condition

\[\int_{0}^{\infty} |R_{m}(r)|^2 dr = 1. \tag{48} \]

For this purpose, one needs the following standard integral \((K_{\nu}\) being the modified Bessel function of second kind and order \(\nu\) [12]):

\[\int_{0}^{\infty} r^{\nu-1} \exp[-(\lambda_{1}r + \lambda_{2}r^{-1})]dr = 2 \left( \frac{\lambda_{2}}{\lambda_{1}} \right)^{\nu/2} K_{\nu}(2\sqrt{\lambda_{1}\lambda_{2}}) \tag{49} \]

which holds if the real parts of \(\lambda_{1}\) and \(\lambda_{2}\) are positive and \(\arg(2\sqrt{\lambda_{1}\lambda_{2}}) < \frac{\pi}{2}\). For example, the explicit calculation shows that

\[|N_{0}| = \left[ 2\tau_{v_{0}/2}K_{v_{0}}(s) \right]^{-1/2} \tag{50} \]

\[|N_{1}| = \left[ 2\tau^{1/2}K_{v_{2}}(s) - 4\sigma_{1}^{(1)} \tau^{1/2}K_{v_{1}}(s) + 2(\sigma_{1}^{(1)})^{2} \tau^{v_{2}/2}K_{v_{0}}(s) \right]^{-1/2} \tag{51} \]
where we have defined
\[
\tau \equiv \frac{a}{b} \quad \text{(52)}
\]
\[
s \equiv 4\sqrt{ab} \quad \text{(53)}
\]
\[
\nu_0 \equiv 2c + 1 \quad \text{(54)}
\]
\[
\nu_1 \equiv 2c + 2 \quad \text{(55)}
\]
\[
\nu_2 \equiv 2c + 3 \quad \text{(56)}
\]

4. Concluding remarks

Some particular cases can be easily obtained from our general formulae. They are as follows.

(i) The coefficient \(D\) vanishes provided that \(c = -m\) in Eq. (39). One then finds, from (46), the corresponding energy in the form
\[
E_m = -\frac{\lambda^2}{4 \left( \sum_{i=1}^{m} \sigma_i^{(m)} + \sqrt{A} \right)^2} \quad \text{(57)}
\]
and the parameters \(A\) and \(B\) are related by (see the first of (23))
\[
B + 2(1 + m)\sqrt{A} = 0. \quad \text{(58)}
\]

(ii) If \(a\) vanishes then \(A\) and \(B\) vanish as well, and if \(C > 0\) one deals with a potential of Mie-type, i.e.
\[
V(r) = \frac{C}{r^2} + \frac{D}{r}. \quad \text{(59)}
\]
The corresponding energy eigenvalue can be written as (see (46))
\[
E_m = -\left( \omega_1 \pm \frac{\omega_1^2}{4} - 4D(2m + \mu) \right) \sum_{i=1}^{m} \sigma_i^{(m)} \right) \quad m = 1, 2, 3, \ldots \quad \text{(60)}
\]

(iii) If both \(a\) and \(C\) vanish, then \(A\) and \(B\) vanish as well, and the ‘effective’ potential \(V(r) + \frac{m^2 - \frac{1}{4}}{r^2}\) in Eq. (8) reduces to
\[
V(r) + \frac{m^2 - \frac{1}{4}}{r^2} = \frac{D}{r} + \frac{m^2 - \frac{1}{4}}{r^2}. \quad \text{(61)}
\]
The discrete spectrum is then expressed by an equation formally analogous to Eq. (60), but where now the sum of all $\sigma_i^{(m)}$ can be obtained by using Eqs. (9d) and (40) after setting $C = 0$ in Eq. (9d) and bearing in mind Eq. (39), i.e.

$$
\sum_{i=1}^{m} \sigma_i^{(m)} = \frac{(c + m)}{D} \left[ m - \frac{1}{4} - c(c + 2m - 1) \right].
$$

(62)

To sum up, we have found the exact solution of the stationary Schrödinger equation in two dimensions with the inverse-power potential defined in Eq. (2), by using the factorization ansatz (3) and (5) for the eigenfunctions. It now remains to be seen whether the structure of the ansatz can be extended to higher dimensions and to other classes of inverse-power potentials.

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