\( N = 1 \) Supersymmetric Yang-Mills on the lattice at strong coupling

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ABSTRACT

We study \( N = 1 \) supersymmetric \( SU(N_c) \) Yang-Mills theory on the lattice at strong coupling and large \( N_c \). Our method is based on the hopping parameter expansion in terms of random walks, resummed for any value of the Wilson parameter \( r \) in the small hopping parameter region. Results are given for the mesonic (2-gluino) and fermionic (3-gluino) propagators and spectrum.

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1 Introduction

The non-perturbative aspects of the strongly interacting supersymmetric (SUSY) gauge theories were intensely investigated in the past [1] and recently they have been the object of renewed interest [2, 3]. These theories are interesting from a phenomenological point of view as their non-perturbative properties might play a crucial role in the understanding of the SUSY breaking mechanism [1]. However, besides the relevant phenomenological implications, the SUSY gauge theories have intrinsic importance as their very nature allows the calculation of some exact non-perturbative solutions [2, 3].

In this paper we concentrate our attention on the most simple SUSY gauge theory: the pure $N = 1$ SUSY Yang-Mills (SYM) with $SU(N_c)$ gauge group. This contains the purely gluonic action, plus one flavour of Majorana fermions in the adjoint representation of the colour group. It is believed that this theory is characterised by the same non-perturbative phenomena as QCD: colour confinement and chiral symmetry breaking [4]. Nevertheless, noticeable differences between the SYM theory and QCD appear to exist even at the fundamental level. Due to supersymmetry there is indeed a new anomalous SUSY current which belongs to the same supermultiplet together with the anomalous chiral and energy-momentum tensor currents [2].

The fundamental question of the breaking of the supersymmetry in $N = 1$ SYM theory was addressed in [4, 5]. According to the general argument of the Witten index [5] or the Veneziano-Yankielowicz low energy effective theory [4], the supersymmetry does not break. Nevertheless, here the chiral symmetry breaks and the gluino condensate acquires a vacuum expectation value [4, 6]. However, the authors of Ref. [7] argue that in this theory the spontaneous breaking of chiral symmetry implies the spontaneous breaking
of supersymmetry due to non-perturbative effects.

From [4, 6] we learn that the low energy supermultiplet contains three degenerate massive colourless bound states: a scalar, a pseudoscalar and a fermion field (where the appearance of the fermion field in the low-energy supermultiplet is a consequence of the colour adjoint representation for the gluino). Moreover, there is no pseudo-goldstone boson (or pion) associated with the chiral symmetry breaking, as the latter is broken by the anomaly.

Some time ago Curci and Veneziano [8] suggested that the SYM theories can be studied non-perturbatively on the lattice by using numerical Monte Carlo simulations. This is clearly analogous to the approach used in QCD theory. They argued [8] that, even if the lattice breaks explicitly supersymmetry, it is possible to recover the SUSY and chiral Ward identities on the continuum limit.

Recently two different collaborations [9]-[12] studied non-perturbatively on the lattice the spectrum of the SYM ($N = 1$) theory following the guidelines suggested in [8]. In [11], because of the limitations deriving from the use of computing resources, the quenched approximation was used to study the spectrum. This approximation consists in neglecting the internal gluino loops. In other words, in the correlation functions of fermion composite operators, the fermion determinant is not included. In SYM theories, if general arguments are taken into account [4], the quenched approximation cannot be justified on the basis of large $N_c$ dominance, since gluinos are in the adjoint representation of the colour group. However, in [11] it is observed that, within the statistical errors of the Monte Carlo simulation, the spectrum seems to show no deviations from the supersymmetry expectations under the approximations considered there. In connection with this result, in [13], by means of an effective lagrangian approach, the splitting in the low energy
supermultiplet induced by the quenched approximation has been analysed, and it is argued that the splittings from the supersymmetry predictions are small.

In any case, in [10, 12] unquenched results for an SU(2) SYM gauge theory were obtained by using numerical simulations with dynamical gluinos. It is likely that new results in this field will quickly follow.

In this paper we study the spectrum of the SYM \((N = 1)\) on the lattice at strong coupling and in the large \(N_c\) limit. The lattice strong coupling expansion technique (see [14] and references therein) has been extensively used as an analytical probe to test qualitative properties both of the continuum and of the lattice theory by itself. Up to now, however, the main part of the results refers to QCD, and –to our knowledge– no work deals with a supersymmetrised version of it. In detailed computations the strong coupling expansion of \(SU(N_c)\) theories is frequently combined with the large \(N_c\) one.

Our reference comes from works in which issues such as the phase structure of QCD or the computation of meson and baryon masses are addressed [15]-[21]. The most frequent computational frameworks split into two types:

- **Effective actions**: The Wilson-Dirac lattice action is considered at large \(N_c\) and small \(\beta\). The large \(N_c\) expansion can be recognised as a saddle-point expansion of the gauge functional integral, previously simplified by the \(\beta \to 0\) limit [15]-[17]. The method allows the study of the large and small hopping parameter regions at the expense of introducing assumptions on the form of the saddle points. The main disadvantage of this method for our purposes is that the one matrix integration formulae, a basic ingredient of the construction, are not available for matrices in the adjoint representation, and the generalization does not seem straightforward.
• **Path resummation:** The fermion matrix $\mathbf{M}$ is inverted by using the standard hopping parameter expansion [14, 16, 21]. This allows to compute propagators $\mathbf{M}^{-1}$ in terms of sums over paths on the lattice (random walks); the objects to be summed are traces of products of spin matrices ($r\mathbf{I} \pm \gamma_\mu$), $r$ being the Wilson parameter. We will see that provided the hopping parameter $\kappa$ is small enough and some constraints on the parameters are imposed, no assumptions are required to perform this computation in the $N_c \to \infty$, $\beta \to 0$ limit. In some of the previous references the analysis has been performed only for the case $r = 1$, where the structure of the series is considerably simplified [16]. The main difficulty of path combinatorics arising for $r \neq 1$ was addressed in Ref. [22]. In Ref. [23] an independent derivation of the main resummation formulas for $r \neq 1$ is given.

In our case, it will be shown that the hopping parameter expansion in terms of random walks is valid, with slight changes, for the case of gauge fields in the adjoint representation. Thus, using the formulas of Ref. [23] we will be able to give expressions for propagators and masses for any value of $r$ and small enough $\kappa$. We considered that keeping $r$ arbitrary could be very important. This allows the possibility of searching for multicritical points, where several lattice masses vanish. In particular, we investigated whether one could find a critical point in the $\kappa - r$ plane where the massless modes would form a supermultiplet. This would signal a possible candidate for a supersymmetric continuum limit, in the spirit of the chiral restoration of ordinary QCD. For example, if a low energy theory of the type described in [8] would take place, one would find a massless pseudoscalar meson, together with a scalar meson and a fermion. We should warn the reader that by massless we refer to the vanishing of the lattice masses, not necessarily the
renormalised physical masses. However, in our approximations the diagrams contributing to the anomaly, and giving mass to the pseudoscalar, are sub-leading, indicating that the physical masses are also zero, as predicted for a Goldstone boson associated to the spontaneous breaking of chiral symmetry. This situation contrasts with what one obtains in perturbation theory and constitutes one of the most salient features of our result.

In addition to the path resummation formulae, we also need to analyse the behaviour in the large $N_c$ limit of integrals over the group of products of matrices in the adjoint representation. These are studied in [25]. In this paper we will only need to know the order in $N_c$ of certain integrals. Given this, the actual leading order results are very simple.

To increase the usefulness of our paper, many of our formulas will be given for arbitrary space-time dimensionality $d$. In addition, we will indicate the necessary changes to make the formulae valid for matter fields in the fundamental representation and for Dirac, rather than Majorana, fermions. Specific attention will be paid, however, to the supersymmetric cases of $d = 3, 4$.

The paper is organised as follows: in section 2 we present the formalism and general formulae for the expectation values and correlation functions of two and three gluino operators in the strong coupling large $N_c$ limit. In section 3 we analyse the specific channels and present the results for the spectrum in the meson and fermion sector. Finally in section 4 we summarise our conclusions, discuss the physical implications and explain the prospects for possible future extensions of our results. The paper is completed by collecting in the Appendix the terminology and formulae on lattice paths that we will use along the text.
2 General formalism

We begin by fixing our notation and terminology. Let us consider a $d$-dimensional hypercubic lattice $\mathcal{L} \equiv \mathbb{Z}^d$. We introduce an index set $I$ having $2d$ elements. To any direction in space-time we associate two indices $\mu$ and $\bar{\mu}$ corresponding to the two opposite senses (forward and backward) associated to each direction $\mu$. Given an element $\alpha \in I$, the element $\bar{\alpha}$ labels the one with reverse orientation ($\bar{\mu} = \mu$). To any element $\alpha \in I$, we can associate a vector $\mathcal{V}(\alpha)$ as follows:

$$
\mathcal{V}(\mu) = e_\mu, \quad \mathcal{V}(\bar{\mu}) = -e_\mu,
$$

with $e_\mu$ the unit vector in the $\mu$ direction (the lattice spacing is set to 1). In the Appendix we give some additional results and terminology that we will be using in the following sections.

Let us now write down the lattice version of the SUSY Yang-Mills action:

$$
S = \beta S_g + \frac{1}{2} \Psi_i \Psi_j M_{ij},
$$

where $\beta S_g$ is the pure gauge part and $\Psi_i$ is a Grassman variable representing the field of a Majorana fermion. The index $i$ is a short form for the three indices $n$, $a$ and $A$. The index $n$ specifies a point in the $d$-dimensional hypercubic lattice $\mathcal{L}$. The index $a$ takes $N_c^2 - 1$ values corresponding to the dimension of the Lie Algebra of $SU(N_c)$. The index $A$ is a spinorial index taking $2^{[d/2]}$ values (the symbol $[x]$ stands for integer part of $x$). Without much problem, but at the expense of breaking supersymmetry, we could add a flavour index ranging over a finite number of values. The matrix $M$ must be antisymmetric and its form is given by

$$
M = C \mathcal{M}
$$

$$
\mathcal{M} = (\mathbf{I} - \sum_{\alpha \in I} \Delta_\alpha),
$$
where $I$ is the unit matrix and $C_{AB}$ is the charge conjugation matrix, satisfying:

$$
\gamma^t_{\mu} C = -C \gamma_{\mu} \\
C^t = -C,
$$

where the superscript $t$ stands for transpose.

Finally the matrix $\Delta_{\alpha}$ is given by:

$$(\Delta_{\alpha})_{ij} = \kappa \delta_{mn+\nu(\alpha)} U_{\alpha}^{ab}(n) (rI - \gamma_{\alpha})_{AB},$$

with $i = (n, a, A)$, $j = (m, b, B)$; $\kappa$ is the hopping parameter, and $r$ is the Wilson parameter. For $\alpha = \mu$, $U_{\mu}(n)$ is the gauge field link variable (which is in the adjoint representation) and $\gamma_{\mu}$ the Dirac matrix, while for $\alpha = \bar{\mu}$ we have $U_{\bar{\mu}}(n) = U^t_{\mu}(n - e_{\mu})$ and $\gamma_{\bar{\mu}} = -\gamma_{\mu}$. Notice that we have:

$$\Delta_{\alpha} \Delta_{\bar{\alpha}} = \kappa^2 (r^2 - 1) I.$$

It is easy to show that given the conditions Eqs. (5)-(6), the matrix $M$ is indeed antisymmetric.

Customarily, the value of $r$ is taken in the interval $[0, 1]$. This follows from the requirement of Osterwalder-Schrader positivity. However, at strong coupling this is not a necessary condition for the physical correlation functions to admit an analytical continuation to unitary Minkowski Green functions. For that reason we will be working for arbitrary $r$ and comment at the relevant places whether the $p$-gluino correlation functions satisfy the positivity conditions.

The constraint imposed by the existence of the matrix $C$, together with the needed matching of fermionic and bosonic degrees of freedom, makes the naive continuum limit of the above lagrangian supersymmetric ($N =$
1) in $d = 3, 4$. The same lagrangian is supersymmetric in $d = 10$ if $\Psi$ is a Majorana-Weyl field, and in $d = 6$ if $\Psi$ is a Weyl spinor. However, the requirement of Weyl character conflicts with the well-known absence of chirality on the lattice, which prevents us from directly writing a lattice version of the required lagrangian (indeed, our Wilson-type action breaks chirality explicitly). On the other hand, the general argument by Curci and Veneziano [8] linking the recovery of chirality and supersymmetry could play a role in the interpretation of these cases, as of course does in the (more transparent) $d = 3, 4$ as well.

Having in mind this caveats regarding the interpretation of the results, in the rest of our exposition we will try to work as much as possible in arbitrary dimension $d$ without specifying more. This has the advantage that it allows our formulae to be used with little changes by other researchers with other physical interests. For example, links can be established with the technique of $1/d$ expansions [24].

We will concentrate upon gauge invariant operators of the form:

$$O_i(x) = \Psi_{A_1}^{a_1}(x) \cdots \Psi_{A_p}^{a_p}(x) S_{i}^{A_1 \cdots A_p} C_{i}^{a_1 \cdots a_p},$$

(9)

where $C_{i}^{a_1 \cdots a_p}$ is an invariant colour tensor and $S_{i}^{A_1 \cdots A_p}$ a spin tensor. The index $i$ labels different possibilities for the definition. The restriction to operators obtained by multiplying gluino fields at the same lattice point (ultralocal operators) will not affect our spectrum results which will be general. In particular, this affects the composite fermion operator which belongs to the same supermultiplet as the scalar and pseudoscalar:

$$\tilde{\chi}^A(x) = \frac{1}{2} (\sigma_{\mu\nu})^{AB} F_{\mu\nu}^{a}(x) \Psi_{B}^{a}(x),$$

(10)

where in this formula the fermion fields $\tilde{\chi}$, $\Psi$ and the Yang-Mills field $F_{\mu\nu}$ live in the continuum spacetime, and $\sigma_{\mu\nu} = \frac{1}{2} [\gamma_{\mu}, \gamma_{\nu}]$. Later on, we will
argue that its corresponding minimal mass will be contained within the set of masses corresponding to 3-gluino operators.

Note that the operators in Eq. 9 are non-vanishing only when the gluino fields are combined in a completely antisymmetric way. By putting this together with the requirement of $C_i$ being an invariant tensor, we arrive at certain constraints on the possible operators; for example, in the case of meson-like operators ($p = 2$) it turns out that, necessarily, $C_i = I$ and $S_i$ is an antisymmetric matrix. As an example, Table 1 displays a complete basis of 2-gluino operators in $d = 4$.

In this paper we will be interested in computing the expectation values of these operators and products of these operators:

$$\langle O_i(x) \rangle = \frac{1}{Z} \prod_{n \in \mathcal{L}} \left( \int d\Psi(n) \prod_{\mu} \int dU_{\mu}(n) \right) O_i(x) e^{-S}$$  \hspace{1cm} (11)

$$G_{ij}(x - y) \equiv \langle O_i(x) O_j(y) \rangle = \frac{1}{Z} \prod_{n \in \mathcal{L}} \left( \int d\Psi(n) \prod_{\mu} \int dU_{\mu}(n) \right) O_i(x) O_j(y) e^{-S}$$  \hspace{1cm} (12)

at strong coupling. We will be able to accomplish this goal for $\beta = 0$, and in the large $N_c$ limit. Corrections to the formulae in powers of $\beta$ and $\frac{1}{N_c}$ are in principle feasible and will be considered elsewhere.

As usual, one can explicitly integrate out the fermions in Eq. (11)-(12). The main formula that one uses is:

$$\prod_i \left( \int d\Psi_i \right) \exp\left\{-\frac{1}{2} \Psi_i \Psi_j M_{ij} + J_i \Psi_i \right\} = Pf(M) \exp\left\{-\frac{1}{2} J_i J_j (M^{-1}C^{-1})_{ij} \right\} ,$$  \hspace{1cm} (13)

where $Pf(M)$ stands for the Pfaffian of the matrix $M$. The square of the Pfaffian is the determinant, so that up to a sign we can write:

$$Pf(M) = \sqrt{\det(C) \det(\mathcal{M})} = \exp\left\{\frac{1}{2} Tr(\log(\mathcal{M})) \right\} ,$$  \hspace{1cm} (14)
where we have used the standard exponential representation of a determinant, and the fact that we can choose $\sqrt{\det(C)} = 1$. Concerning the sign, it is clear that what matters is the relative sign for different values of the gauge field configuration. However, for very small $\kappa$, the matrix $M$ is close to the unit matrix and our representation (Eq. (14)) is valid. Problems can only arise when one of the eigenvalues of $M$ crosses zero. Using the Schwartz inequality and our expression for $M$, one can conclude that this problem never occurs provided $|\kappa| < \frac{1}{2d(|r|+1)}$.

What we will do now is to expand the quantities entering in Eq. (13) as a sum of paths. Using the terminology of the Appendix, we obtain:

\begin{equation}
(M^{-1}(x,y))^{ab}_{AB} = \sum_{\gamma \in S(x \rightarrow y)} W^{ab}(\gamma) \Gamma_{AB}(\gamma)
\end{equation}

(15)

\begin{equation}
Pf(M) = \exp\left\{ \frac{1}{2} \sum_{x \in L} \sum_{L=1}^{\infty} \sum_{\gamma \in S_L(x \rightarrow x)} \frac{1}{L} Tr(W(\gamma)) Tr(\Gamma(\gamma)) \right\},
\end{equation}

(16)

where $x, y$ are lattice points, $W(\gamma)$ is the path ordered product (along the path $\gamma$) of the gauge field link variables, and $\Gamma(\gamma)$ denotes the appropriate product of the spin matrices:

\begin{equation}
\Gamma(\gamma \equiv (x, \vec{a})) = \kappa^L (r - \gamma_{a_1}) \cdots (r - \gamma_{a_L}).
\end{equation}

(17)

Let us first consider 2-body operators:

\begin{equation}
O_i(x) = S_i^{AB} \Psi_A^a(x) \Psi_B^a(x) = \overline{\Psi}(x) \hat{S}_i \Psi(x)
\end{equation}

(18)

with $\hat{S}_i = C^{-1} S_i$.

(19)

The second expression in (18) has the same form as for Dirac fermions, the difference being that for Majoranas we have the relation $\overline{\Psi} = \Psi^t C$. Notice that $S_i$ is an antisymmetric matrix. Using the formula (13) and expanding
by using Wick’s theorem we obtain:

\[
\langle O_i(x) \rangle = -\frac{1}{\langle Pf(M)_g \rangle} \langle Tr \left( M^{-1}(x,x) C^{-1} S_i \right) Pf(M) \rangle_g \quad (20)
\]

\[
G_{ij}(x-y) = \frac{1}{\langle Pf(M)_g \rangle} \langle Pf(M) \times \left[ -2 Tr \left( C^{-1} S_i M^{-1}(x,y) C^{-1} S_j M^{-1}(y,x) \right) + Tr \left( M^{-1}(x,x) C^{-1} S_i \right) Tr \left( M^{-1}(y,y) C^{-1} S_j \right) \right] \rangle_g , \quad (21)
\]

where \( Tr \) denotes the trace over colour and spin indices and the symbol \( \langle \rangle_g \) denotes expectation value with respect to the pure gauge action part.

The first term on the right hand side of Eq. (21) represents the so-called \textit{OZI} contribution, while the second term contains the disconnected and non-\textit{OZI} contributions [8]. Now using the formulae (15)-(16) we can write back expressions (20)-(21) in terms of products of sums of paths.

Up to now the expressions have been completely general, but now we will consider the simplification arising from considering \( \beta = 0 \) and \( N_c \to \infty \). What we need to know is the expectation value of the product of traces of Wilson loops in this limit. The main result that we will use [25] is that for a collection of closed paths \( \{ \gamma_1, \ldots, \gamma_s \} \) which are not pure spikes (see the Appendix) we have:

\[
\langle Tr(W(\gamma_1)) \ldots Tr(W(\gamma_s)) \rangle_g = O(1) . \quad (22)
\]

This is true both for connected and disconnected expectation values. However, for a pure spike path \( W(\gamma) = I \), and therefore its trace is \( N_c^2 - 1 \). We see that the connected correlation between a pure spike Wilson loop and any other operator vanishes. Our first conclusion is then that, at leading order in \( N_c \), the factor \( Pf(M) \) cancels between numerator and denominator. This is precisely the quenched approximation, which turns out to be exact in this
limit. In principle, the result is surprising since fermions carry indices taking \( N_c^2 - 1 \) values (as gluons), and the usual arguments why fermion loops are subleading in the fundamental representation do not apply here. This adds to the results obtained by other methods [11, 13], pointing towards the fact that the deviations introduced by the quenched approximation are not too large.

Other conclusion that follows from formula (22) is the suppression in the large \( N_c \) limit of the non-OZI contributions to the connected correlation functions. This includes the mass induced by the anomaly on the pseudoscalar Goldstone boson. We should, hence, expect such a massless state signaling the recovery of chiral symmetry and its spontaneous breaking. As mentioned in the Introduction this result, as well as the exactness of the quenched approximation, contrasts with the behaviour obtained in perturbation theory to leading order in \( N_c \).

Our results have been obtained at \( \beta = 0 \). However, one can introduce the pure gauge action in various ways having the same naive continuum limit. If one chooses to write the Wilson action in the adjoint representation of the group, formula (22) implies that to the order we are working (and in the small \( \beta \) region) there are no corrections to any order in \( \beta \). This is not the case if the customary fundamental representation Wilson action is chosen.

The conclusion of the previous paragraphs is that to leading order in \( N_c \) all that we have to take into account are closed loops which are pure spikes. For that purpose the results of Ref.[23] and collected in the Appendix are needed. Notice that for a pure spike path of length \( L \), the spinor matrix is given by:

\[
\Gamma(\gamma) = (\kappa^2(r^2 - 1))^{L/2} \mathbf{I}.
\]  

(23)
Thus, using the formulae given in the Appendix we can conclude:

\[ \langle O_i(x) \rangle = -(N_c^2 - 1) F(0, \kappa^2(r^2 - 1)) Tr(C^{-1} S_i) = -(N_c^2 - 1) Tr(\hat{S}_i) \frac{1}{1 - \frac{2d}{2d-1} \xi}, \]  

(24)

with

\[ \xi \equiv 1 - \sqrt{1 - 4(2d-1)\kappa^2(r^2 - 1)} \frac{1}{2}. \]  

(25)

This is the contribution of closed paths which are pure spikes. The corrections coming from other paths are order 1, and thus subleading. The previous formulae are obtained by resummation of a series. The radius of convergence is given by the closest singularity. Thus the formulae are strictly speaking only valid in the region \( |\kappa^2(r^2 - 1)| < \frac{1}{4(2d-1)} \). It is possible that, by analytical continuation, the formulae could be valid in some points beyond this region, such as for larger negative \( \kappa^2(r^2 - 1) \).

Expression (24) is also valid for Dirac fermions. It is also valid if the fields (Dirac or Majorana) transform in the fundamental representation of the colour group, provided \( (N_c^2 - 1) \) is replaced by \( N_c \), the dimension of the representation in question.

Now we look at the correlation function of two-fermion operators. In this case we have two factors of \( \mathcal{M} \) and hence we have an expansion in terms of paths \( \gamma \) going from \( x \) to \( y \), and paths \( \gamma' \) going from \( y \) to \( x \). Nevertheless, the integration over the gauge group forces the combined path to be a pure spike path. To take this into account we proceed as follows. We replace the summation over paths by a summation over paths with no spikes, resumming all paths which have such a path as reduced path. Thus, each term in the new expansion corresponds to a reduced path \( \hat{\gamma} \) going from \( x \) to \( y \), and another one \( \hat{\gamma}' \) that returns to \( x \). However, now the condition imposed by the integration over the gauge group is simply that \( \hat{\gamma}' \) is the reverse path of \( \hat{\gamma} \).
(which we label $\hat{\gamma}^{-1}$). In this way the double summation reduces to a single summation. Summing up all that we have just expressed in words, we can give the following formula for the connected correlation function:

$$G_{ij}^{(conn.)}(x - y) = -\eta \, D_R \sum_{L=0}^{\infty} \sum_{\hat{\gamma} \in \hat{S}_L(x-y)} (F(L, \kappa^2(r^2 - 1)))^2 \times \text{Tr} \left( \hat{S}_i \Gamma(\hat{\gamma}) \hat{S}_j \Gamma(\hat{\gamma}^{-1}) \right).$$

(26)

The previous formula has been written in a way which makes it valid for Majorana ($\eta = 2$) or Dirac ($\eta = 1$) fermions. The symbol $D_R$ stands for the dimension of the gauge group representation ($N_c^2 - 1$ for the adjoint and $N_c$ for the fundamental). Now we can use the expression for $F(L, \kappa^2(r^2 - 1))$ and the formulae for resumming over paths that are given in the Appendix, to conclude:

$$G_{ij}^{(conn.)}(x - y) = R_2(\xi) \prod_{\mu} \left( \int \frac{d\varphi_{\mu}}{2\pi} \right) e^{\varphi_{\mu}(x-y)} \langle S_i | [\Theta_2(\xi) \mathbf{I} - \hat{\mathbf{A}}_2(\varphi)]^{-1} \tilde{\mathbf{C}}_2^{-1} | S_j \rangle,$n

(27)

where we have:

$$R_2(x) \equiv -\eta \, D_R \frac{1 - \frac{2d-2}{2d-1}x}{1 - \frac{2d-2}{2d-1}x}$$

(28)

$$\Theta_2(x) \equiv (1 - x)^2 + \frac{x^2}{(2d-1)}$$

(29)

$$\hat{\mathbf{A}}_2(\varphi) \equiv \kappa^2 \sum_{\alpha \in I} e^{i\varphi_{\alpha}} (r - \gamma_\alpha) \otimes (r - \gamma_\alpha)$$

(30)

$$\tilde{\mathbf{C}}_2 \equiv C \otimes C.$$  

(31)

$|S_i\rangle$ is just given by the matrix $S_i$, but considered as a $2^{[\frac{d}{2}]} \cdot 2^{[\frac{d}{2}]}$ dimensional vector. It is useful to express the matrix elements of the $2^{[\frac{d}{2}]} \times 2^{[\frac{d}{2}]}$ matrix $\hat{\mathbf{A}}_2$ between $S_i$ states in terms of the matrices $\hat{S}_i$ defined in Eq. (19). We
have:
\[ \langle S | \tilde{A}_2(\varphi) \tilde{C}_2^{-1} | S_j \rangle = \kappa^2 \sum_{\alpha \in I} e^{i \varphi_\alpha} Tr[\tilde{S}_i(r - \gamma_\alpha)\tilde{S}_j(r + \gamma_\alpha)] . \] (32)

With this interpretation, formula (27) is valid for Dirac fermions as well. We will leave to the next section the evaluation of this expression and the study of the properties of the resulting propagator.

We now turn our attention to 3-gluino fermion operators of the form:
\[ O_i(x) = \Psi_{a_1}^{A_1}(x)\Psi_{a_2}^{A_2}(x)\Psi_{a_3}^{A_3}(x)C_{a_1a_2a_3}S_i^{A_1A_2A_3} . \] (33)

In this case there are two possible invariant colour tensors: \( d_{abc} \) and \( f_{abc} \).

The main result that we will need on the group integration at large \( N_c \) is the following: given three paths without spikes \( \gamma_1, \gamma_2 \) and \( \gamma_3 \), we have:
\[ C_{abc}C_{a'b'c'} \langle W^{a_1a_2}(\gamma_1)W^{b'b'}(\gamma_2)W^{c'c'}(\gamma_3) \rangle_g = N_c^3 \delta(\gamma_1 = \gamma_2 = \gamma_3) \]
\[ + \text{subleading terms} , \] (34)

where \( C_{abc} \) is either \( f \) or \( d \) (the antisymmetric and symmetric \( SU(N_c) \) invariant tensors). The mixed terms \( f - d \) are subleading in \( N_c \). Also subleading are contributions in which the three paths are non-equal. Using this expression and the formulae derived in the Appendix, it is possible to compute the expectation values of products of 3-gluino operators. Now the Wick expansion gives a total of 6 terms (once the \( N_c \)-subleading ones are discarded).

These terms ensure that if the colour invariant tensor is \( f \) or \( d \), the spin matrix \( S_j \) can be chosen completely symmetric or antisymmetric respectively, as required by Fermi statistics. With this choice the 6 terms give the same contribution, and the correlation function can be written as:
\[ \langle O_i(x)O_j(y) \rangle = R_3(\xi) \prod_{\mu} \left( \int \frac{d\varphi_\mu}{2\pi} e^{i\varphi(x-y)} \langle S_i | [\Theta_3(\xi)I - \tilde{A}_3(\varphi)]^{-1} \tilde{C}_3^{-1} | S_j \rangle , \right. \]
where now:

\[ R_3(x) \equiv -6N_c^3 \frac{(1-x)^3 - (\frac{x}{2d-1})^3}{(1-(\frac{2d}{2d-1})x)^3} \]  

\[ \Theta_3(x) \equiv (1-x)^3 + \frac{x^3}{(2d-1)^2} \]  

\[ \tilde{A}_3(\varphi) \equiv \kappa^3 \sum_{\alpha \in I} e^{i\varphi_\alpha} (r - \gamma_\alpha) \otimes (r - \gamma_\alpha) \otimes (r - \gamma_\alpha) \]  

\[ \tilde{C}_3 \equiv C \otimes C \otimes C. \]

As before, the vector \(|S_i\rangle\) is the one constructed from the corresponding spin matrix. Explicitly:

\[ \langle S_i | \tilde{A}_3(\varphi) \tilde{C}_3^{-1} | S_j \rangle = \kappa^3 \sum_{\alpha \in I} e^{i\varphi_\alpha} \times \]  

\[ S_i^{A_1 A_2 A_3} ((r - \gamma_\alpha) C^{-1})^{A_1 B_1} ((r - \gamma_\alpha) C^{-1})^{A_2 B_2} ((r - \gamma_\alpha) C^{-1})^{A_3 B_3} S_j^{B_1 B_2 B_3}. \]

The class of operators considered (Eq. (33)) does not include the lattice counterpart of that in Eq. (10). A possible candidate would be:

\[ \chi^A(x) = \frac{1}{2} (\sigma_{\mu\nu})^{AB} P_{\mu\nu}^{ab}(x) f^{abc} \Psi^c_B(x), \]  

where \( P_{\mu\nu}(x) \) is an appropriate combination of adjoint plaquettes in the \((\mu, \nu)\) plane whose naive continuum limit is, up to a convenient multiplicative factor, the adjoint gauge field \( F_{\mu\nu}(x) \). This has the advantage of including only gauge variables in the adjoint representation, allowing the use of our integration formulas. However, by examining the strong coupling large \( N_c \) expansion of the correlation of 2 such operators, one easily realises that it is given by combinations of triple paths joining the two operators. Thus, we expect that the mass spectrum following from 3-gluino ultralocal operators would include also the states coupled to (41).
To conclude we simply want to mention that in a similar way one can obtain expressions for expectation values and correlations of \( p \)-gluino operators. No additional difficulty arises, and the final expression looks just like Eq. (35) but with corresponding functions and matrices \( R_p, \Theta_p, \tilde{C}_p, \) and \( \tilde{A}_p \).

In particular:

\[
\Theta_p(x) \equiv (1 - x)^p + \frac{x^p}{(2d - 1)^{p-1}} \quad (42)
\]

\[
\tilde{A}_p(\varphi) \equiv \kappa^p \sum_{\alpha \in I} e^{r\varepsilon_{\alpha}} (r - \gamma_{\alpha}) \otimes \ldots \otimes (r - \gamma_{\alpha}) . \quad (43)
\]

3 Explicit results on the propagators and spectra

In this section we will analyse the results on the expectation values presented in the previous one. Our main goal will be the extraction of the spectrum of the theory at \( \beta = 0 \) and leading order in the \( 1/N_c \) expansion.

We will first of all look at the expectation values of single 2-gluino operators. Our main result is formula (24). The only independent operator \( S_i \) giving a non-vanishing spinorial trace can be chosen as \( S_i = C I \). On physical terms, it corresponds to a non-vanishing gluino scalar condensate for the full range of values where our resummation is valid \( (1 - \sqrt{2}/2 \leq \xi \leq 1/2) \). Once the different normalisations of the fields and operators, and the appropriate colour factors are taken into account, this expression coincides with the result given in Ref. [17] for this expectation value when the fermion field is in the fundamental representation.

Our next step will be to analyse the results for the correlations of two fermion operators \( G_{ij}(x) \), and the corresponding (meson) spectrum. The expression given in the preceding section for the correlation (formula (27))
requires the inversion of a $2^d \times 2^d$ matrix. (For odd space-time dimensions one must replace $d$ by $2^{[d/2]}$). This matrix is $\Theta_2(\xi) I - \tilde{A}_2(\varphi)$ defined in Eqs. (29,30).

Thus, in even space-time dimensions, it is convenient for the study to choose as a basis of the $2^d$-dimensional space of meson operators, those corresponding to $\hat{S}_i$ being the standard basis of the $d$-dimensional Clifford Algebra, for which we adopt the writing:

$$\hat{S}(n) \equiv e^{i\varepsilon(n)} \gamma_0^{n_0} \cdots \gamma_{d-1}^{n_{d-1}}, \quad (44)$$

where $\varepsilon(n)$ is an appropriate phase which we will choose equal to zero in what follows. Thus, one state of the basis is the scalar operator $\hat{S}_s$ corresponding to the unit matrix, other elements are the vector operators $\hat{S}_v(\mu)$ corresponding to the gamma matrices $\gamma_\mu$, and so on.

For odd space-time dimensions, one could also consider the operators $\hat{S}(n_\mu)$ associated to the standard basis of the Clifford algebra (which is basically the same as for dimension $d - 1$), but one must take into account that they are not independent. They are constrained by the identity:

$$\gamma_0 \cdots \gamma_{d-1} = K_d I, \quad (45)$$

where $K_d$ is a phase depending on the space-time dimension.

Going back to the even-dimensional case, we have to express the matrix elements of the matrix $\tilde{A}_2(\varphi)$ within this basis. To do so it is convenient to view the $2^d$-dimensional space in question as the Fock space of a system of fermions: the *gamma-fermions*. Each integer $n_\mu \in \{0, 1\}$ entering Eq. (44) can be interpreted as the occupation number of the state $\mu$. It is convenient to add an additional one-particle state labelled ‘$-1$’ whose usefulness will be clear in what follows. Thus, in the standard second quantization notation we can write:

$$\hat{S}(n_\mu) \equiv |n_{-1}, n_0, \ldots n_{d-1} \rangle, \quad (46)$$
with \( \hat{S}(n_\mu) \) defined in (44). The extra occupation number \( n_{-1} \) is fixed to be a \textit{parity bit} state, taking the value 1 when the total number of gamma-fermions in the other states is odd and the value 0 if it is even. Hence, in both cases, this imposes the constraint that the total number of fermions must be even. With this convention it is possible to express the matrix \( \tilde{A}_2(\varphi) \) in terms of creation and annihilation operators of these fermions as follows:

\[
\tilde{A}_2(\varphi) = \kappa^2 (2\tilde{\sigma}(r_2 - 1 + 2a_+^\dagger a_{-1}) - 2(4a_+^\dagger a_{-1} - 2)\sum_{\mu=0}^{d-1} \cos(\varphi_\mu)a_\mu^\dagger a_\mu - 4ir\sum_{\mu=0}^{d-1} \sin(\varphi_\mu)(a_\mu^\dagger a_{-1} + a_{-1}^\dagger a_\mu)) , \tag{47}
\]

with \( \tilde{\sigma} = \sum_{\mu=0}^{d-1} \cos(\varphi_\mu) \).

We see that the operator conserves the number of gamma-fermions. Hence, each even number of gamma-fermions \( 2p \) characterises a block in which \( \tilde{A}_2(\varphi) \) can be diagonalised or inverted. Within each block one has two subspaces corresponding to \( n_{-1} = 1 \) and \( n_{-1} = 0 \), which correspond to the product of \( 2p - 1 \) and \( 2p \) gamma matrices respectively. The matrix \( \tilde{A}_2(\varphi) \) mixes these 2 subspaces.

Let us clarify the previous formulae by looking at a few examples. If \( p = 0 \), we have the gamma-fermionic Fock vacuum state, which is an eigenstate of \( \tilde{A}_2(\varphi) \) with eigenvalue \( 2\tilde{\sigma}\kappa^2(r^2 - 1) \). This state is precisely the scalar meson operator. Next, we consider the space of \( 2p = 2 \) gamma-fermions. The subsector \( n_{-1} = 1 \) corresponds to the vector operators \( (\hat{S}_V(\mu) = \gamma_\mu) \) and the \( n_{-1} = 0 \) to tensor states \( (\hat{S}_T(\mu\nu) = \gamma_\mu\gamma_\nu) \). The conclusion is that vector and tensor states mix between themselves but not with other states. Considering the space of \( 2p \) gamma-fermions, we conclude that the operators associated to the product of \( 2p - 1 \) Dirac gamma matrices mix with those involving \( 2p \) gamma matrices, but with no other states. The inversion or diagonalisation problem has been considerably simplified with this technique. This is the
generalisation of the block structure found by previous authors studying QCD at strong coupling [16].

In the previous analysis we have not taken into account the restriction imposed by the fact that our gluinos are Majorana. As mentioned before, in this case if the operators involve the product of the gluino fields at the same point (ultralocal operators), the matrices $S_i$ can be chosen antisymmetric. This restriction translates in our language into $p$ being an even number: the number of gamma fermions must be a multiple of 4. Thus, for instance, the only relevant blocks in $d = 4$ for this ultralocal operators are the scalar singlet and the $p = 2$ containing the pseudoscalar ($\gamma_5$) and the axial vector ($\gamma_5\gamma_\mu$).

We now introduce an important symmetry of the operator given in (47). This is the unitary transformation $C$ related to the charge conjugation of gamma-fermions:

$$C a_\mu^+ C^\dagger = a_\mu$$  \hspace{1cm} (48)
$$C a_{-1}^+ C^\dagger = - a_{-1}$$  \hspace{1cm} (49)
$$C |0\rangle = K_d' |1, \ldots, 1\rangle .$$  \hspace{1cm} (50)

Up to a phase the operation exchanges occupied by empty for all states. One can easily see that the previous transformation commutes with the operator $\tilde{A}_2(\varphi)$ given in expression (47). Notice, that with this change a state with $2p$ gamma-fermions changes into one with $(d+1-2p)$, which for even space-time dimensions ($d = 2s$) is an odd number. We see that the spaces with an odd number of gamma fermions are useful after all. Thus, there is a hierarchy of complexity in the Fock space of gamma-fermions as the number grows. The Fock vacuum corresponding to the scalar operator is an eigenstate of $\tilde{A}_2(\varphi)$. Next, comes the 1-particle space, which through charge conjugation
of gamma-fermions corresponds to the operator involving the matrix \( \gamma \equiv \gamma_0 \cdots \gamma_{d-1} \) and \( \gamma_\mu \). In what follows we will proceed to invert the matrix \( \Theta_2(\xi)I - \tilde{A}_2(\varphi) \) and obtain the propagator for these simplest cases.

Before proceeding to the inversion, let us comment about the necessary changes to deal with an odd-space time dimension \( d \). In this case we might introduce gamma-fermions as well, but due to the constraint (45) there are actually 2 states corresponding to the same operator. However, with a bit of effort one can show that the two states are precisely the 2 states that are mapped by the transformation \( C \), provided \( K'_d \) is chosen equal to \( K^*_d \) entering in Eq. (45). With this choice, one can see that for odd space-time dimension expression (47) remains valid, but that the physical space of meson operators has to be identified with the subspace of the gamma-fermion Fock space which is invariant under \( C \) and has an even number of gamma-fermions. With this in mind all that follows can be applied to even and odd space-time dimensions equally.

Now we consider the scalar state (corresponding to the unit matrix) first. Indeed, to comply with the normalisation chosen for the \( d = 4 \) case in Table 1, we take \( \tilde{S}_S = 2^{-\frac{1}{2}} I \). The matrix \( \tilde{A}_2(\varphi) \) reduces here to the constant \( 2\tilde{\sigma} \kappa^2 (r^2 - 1) \). Then we can directly write the momentum-space propagator explicitly:

\[
\tilde{G}_{SS}(\varphi) = \frac{H(\xi)}{\Phi_2(\xi) - \sum_{\mu=0}^{d-1} \cos(\varphi_\mu)},
\]

where we have defined the following functions:

\[
H(x) = \frac{-\eta \, D_R(2d - 1) \left( 1 + \frac{2d-2}{2d-1} x \right)}{2x(1-x) \left( 1 - \frac{2d}{2d-1} x \right)},
\]

\[
\Phi_2(x) = \frac{(1-x)^2(2d-1) + x^2}{2x(1-x)},
\]

and \( \eta, \xi \) and \( D_R \) are the ones defined in section 2. The function \( \Phi_2 \) is
decreasing for all $\xi \neq 0$ in the convergence interval, and in addition satisfies the following properties:

\[
\Phi_2(1/2) = d
\]
\[
\xi \in (0, 1/2) \Rightarrow \Phi_2(\xi) > d
\]
\[
\xi < 0 \Rightarrow \Phi_2(\xi) < -d .
\]

Next, for even space-time dimension, we proceed to study the space of 1 gamma-fermion. As mentioned previously it corresponds to the matrix $\gamma (\gamma_5 \text{ in 4 dimensions})$ and $\gamma_5 \gamma_\mu$. Expression (47) reduces in this case to:

\[
\tilde{\mathcal{A}}_2(\varphi) = 2\kappa^2(r^2 - 1)\tilde{\sigma} + \kappa^2 \sum_{\alpha,\beta = -1}^{d-1} T_{\alpha\beta} a_\alpha^+ a_\beta ,
\]

where

\[
T_{-1-1} = 4\tilde{\sigma}
\]
\[
T_{-1\mu} = T_{\mu-1} = 4r \sin(\varphi_{\mu})
\]
\[
T_{\mu\nu} = 4 \cos(\varphi_{\mu}) \delta_{\mu\nu} .
\]

Now the propagator can be obtained by inverting the matrix $\Theta_2 \mathbf{I} - T$. This can be done by making a non-unitary change of variables which brings $T$ to a diagonal matrix up to a $2 \times 2$ block. In this way, one obtains the expression for the propagator in momentum space $\tilde{G}^{(PA)}$ in the 1-gamma fermion sector, i.e., the axial vector and the pseudoscalar block in $d = 4$. To make a contact with the usual conventions, we change the normalisation of the operators to the forms (remember $d$ is even) $P = 2^{-d/4}\gamma$ and $A(\rho) = i2^{-d/4}\gamma_\rho \gamma$, generalising again the 4-dimensional ones given in Table 1. The result is:

\[
\tilde{G}_{PP}(\varphi) = H(\xi) \left( \alpha_d + \sum_{\mu=0}^{d-1} \beta_{\mu}^2 \right)^{-1}
\]

23
\[
\hat{G}_{PA}(\varphi) = \hat{G}_{\rho \rho}(\varphi) = -\frac{\beta}{\alpha_d} \hat{G}_{PP}(\varphi)
\]
\[
\hat{G}_{\rho A}(\varphi) = \frac{1}{\alpha_d} \left( \alpha_d + \sum_{\mu=0, \mu \neq \rho}^{d-1} \frac{\beta^2}{\alpha_d} \hat{G}_{PP}(\varphi) \right)
\]
\[
\hat{G}_{A\rho A}(\varphi) = \hat{G}_{A\rho A}(\varphi) = -\frac{\beta}{\alpha_d} \hat{G}_{PP}(\varphi)
\]

where \( H \) and \( \Phi_2 \) are the quantities defined in Eqs. (52), and the functions \( \alpha_d, \alpha_\mu \), and \( \beta_\mu \) have the following expression:

\[
\alpha_d = \Phi_2(\xi) - \frac{r^2 + 1}{r^2 - 1} \sum_{\mu=0}^{d-1} \cos(\varphi_\mu)
\]

\[
\alpha_\mu = \Phi_2(\xi) - \sum_{\mu=0}^{d-1} \cos(\varphi_\mu) - \frac{2}{r^2 - 1} \cos(\varphi_\mu)
\]

\[
\beta_\mu = \frac{2r}{r^2 - 1} \sin(\varphi_\mu)
\]

Obtaining momentum-space propagators for other blocks is feasible, but the expressions become more and more complicated. Furthermore, for ultralocal operators in 4 dimensions, the previous propagators are the only non-vanishing ones. Thus, we will focus in what follows on the analysis of the meson spectrum.

The lattice masses are the minima of the lattice energies as we vary the spatial momentum \( \vec{\varphi} \). These minima can only occur at \textit{special momenta} \( \vec{\varphi} = \varphi^{(\text{special})} \) (\( \varphi_i^{(\text{special})} = 0, \pi \)). The advantage is that now \( \sin(\varphi_i^{(\text{special})}) = 0 \), which simplifies expression (47) considerably. The procedure to obtain the masses is the following: Extract the eigenvalues of the matrix \( \Theta_2(\xi)I - \hat{A}_2(\vec{\varphi}^{(\text{special})}) \), which are functions of the temporal momentum \( \varphi_0 \). Then determine \( \varphi_0^{\text{pole}} \), the (complex) value of \( \varphi_0 \) for which the eigenvalue vanishes.
The lattice masses are now given by \( M = -\log(|e^{i\varphi_{pole}}|) \). This coincides with the definition of mass as the exponent controlling the decay of correlation functions at long times.

Now let us proceed to obtain the eigenvalues. By looking at expression (47), one sees that the matrix is diagonal except for the term proportional to \((a^+_0 a_{-1} + a^+_{-1} a_0)\). Thus, the occupation numbers of the spatial states \( n_i \) are not changed by the operator. Hence, for fixed values of these numbers, the operator reduces to a 4 by 4 matrix: an operator acting on the two-state fermion system labelled by \( n_{-1} \) and \( n_0 \). Furthermore, the states having \( n_{-1} = n_0 = 0 \) and \( n_{-1} = n_0 = 1 \) are eigenstates of the matrix \( \tilde{A}_2(\varphi^{(special)}) \). The other two states are mixed, but finding the eigenvalues and eigenvectors is trivial, since it is a \( 2 \times 2 \) matrix. We can summarise the results obtained in the following formulae:

\[
\begin{align*}
\tilde{A}_2(\varphi^{(special)})|0, 0, \vec{n}\rangle &= (2\kappa^2(r^2 - 1) \cos(\varphi_0) + 2\kappa^2r^2\sigma - 2\kappa^2\tau) |0, 0, \vec{n}\rangle \\
\tilde{A}_2(\varphi^{(special)})|1, 1, \vec{n}\rangle &= (2\kappa^2(r^2 - 1) \cos(\varphi_0) + 2\kappa^2r^2\sigma + 2\kappa^2\tau) |1, 1, \vec{n}\rangle \\
\tilde{A}_2(\varphi^{(special)})|\text{mixed, } \vec{n}\rangle &= 2\kappa^2(r^2\sigma + (r^2 + 1) \cos(\varphi_0) \pm \sqrt{\tau^2 - 4r^2\sin^2(\varphi_0)}) |\text{mixed, } \vec{n}\rangle 
\end{align*}
\]

where we have introduced the following notation:

\[
\begin{align*}
\sigma &= \sum_{i=1}^{d-1} \vartheta_i \\
\tau &= \sum_{i=1}^{d-1} (-1)^{n_i} \vartheta_i \\
\epsilon &= \frac{1}{r^2-1} \\
\theta &= \frac{r^2+1}{r^2-1} \\
\vartheta_i &= \cos(\varphi_i) \in \{-1, 1\} 
\end{align*}
\]

From these eigenvalues one can apply the previously described procedure.
and obtain the formulae for the masses $M(n_{-1}, n_0, n_1, \ldots, n_{d-1})$:

\begin{align*}
\cosh(M(0, 0, \vec{n})) &= |\Xi + \tau \epsilon| \quad (59) \\
\cosh(M(1, 1, \vec{n})) &= |\Xi - \tau \epsilon| \quad (60) \\
\cosh(M(\text{mixed } \pm, \vec{n})) &= \theta \Xi \mp \sqrt{(\theta^2 - 1)(\Xi^2 - 1) + \tau^2 \epsilon^2} \quad (61)
\end{align*}

with $\Xi = \Phi_2(\xi) - r^2 \epsilon \sigma$,

where $\Phi_2(\xi)$ is defined in Eq. (52) and the remaining symbols in (58).

Now let us discuss these results. All the dependence on the occupation numbers $n_i$ lies in the quantity $\tau$. On the other hand, both $\Xi$ and $\tau$ depend on the choice $\varphi_i = 0, \pi$. From the definition of $\tau$ one sees that its maximum positive value is obtained whenever all states having $\vartheta_i = 1$ are empty and those with $\vartheta_i = -1$ occupied. The maximum negative value is attained in the opposite situation. In both cases, the maximal absolute value is the same: $d - 1$. Let us now comment briefly on the main features of the spectrum formulae:

$n_{-1} = n_0$ sectors

This sector contains the scalar state corresponding to $n_{-1} = n_0 = n_i = 0$, whose mass is given by:

\begin{equation}
\cosh(M_S) = |\Phi_2(\xi) - \sigma| . \quad (62)
\end{equation}

For $|r| > 1 (\xi > 0)$ the scalar ground state corresponds to zero momentum $\vec{\varphi} = \vec{0}$. For $|r| < 1 (\xi < 0)$ it corresponds to the doubler state $\varphi_i = \pi$. The state with minimum mass within these sectors corresponds to $|\tau| = d - 1$ and $\varphi_i = 0$, which is the state associated to the matrix $\gamma_0$. Its corresponding mass is:

\begin{equation}
\cosh(M_V) = |\Phi_2(\xi) - \theta(d - 1)| . \quad (63)
\end{equation}
\[ n_{-1} + n_0 = 1 \text{ sector} \]

Let us consider first even space-time dimensions. In that case this sector has lighter states than the previous one. Notice that the argument of the square root in Eq. (61) can be written as \( (\theta \Xi - 1)^2 - (\Xi - \theta)^2 - \tau^2 \epsilon^2 \). Thus, the lightest state corresponds to the maximum value of \( |\tau| = d - 1 \). Furthermore, one can prove that the mass decreases with \( |\Xi| \), and hence the lightest state corresponds to \( \varphi_i = 0 \). Combining this with the maximal \( \tau \) one sees that this lightest state corresponds to \( \bar{\gamma} \) and \( \gamma_0 \bar{\gamma} \) (\( \gamma_5 \) and \( \gamma_0 \gamma_5 \) in 4 dimensions). Its mass is given by formula (61) with \( \Xi = \Phi^2(\xi) - (d - 1) \epsilon^2 \) and \( |\tau| = d - 1 \). The corresponding critical line, where this lightest meson becomes massless, marks the edge of the physical region. It also marks the boundary of validity of our formulae. The equation for this critical line is given by:

\[ \Phi^2(\xi) = d \theta . \tag{64} \]

Along this line, the other meson states have a positive definite mass. The only exception occurs for \( \theta = 1 \) (\( \epsilon = 0 \)). The latter is an interesting region, where all the meson masses are degenerate and dependent only on \( \xi \). All states become massless at \( \xi = \frac{1}{2} \). Notice that the region corresponds to the limit \( r \to \infty, \kappa \to 0 \) with \( \kappa r \) fixed. If we look at the expression of the action in this limit we see that the normal Dirac term becomes negligible with respect to the Wilson term. We have then essentially the masses corresponding to the scalar-gauge theory. Unphysical features could be expected to arise in this limit, because, as we commented in the previous section, \( |r| > 1 \) would be forbidden by the requirement of Osterwalder-Schrader positivity at the lattice level. However, it can be checked that the meson propagators in this limit satisfy the positivity condition.

If we solve in this even-dimensional case for the critical hopping \( \kappa^2 \) which
defines the critical line as a function of $r$ we get, in particular, the result
\[ \kappa_c^2(r = 1) = 1/4d, \]
in agreement with the well-known \[ \kappa_c^2(r = 1) = 1/16 \] for \[ d = 4 \] [16]-[19].

For odd space dimensions the previous analysis has to be modified. Now, it is not possible to satisfy $|\tau| = d - 1$ and $\varphi_i = 0$ simultaneously, due to the requirement that the number of gamma-fermions must be even. Thus, the lightest state has a higher mass. This state is a mixed vector-tensor state ($\gamma_i$ and $\gamma_0\gamma_i$). The critical line is given by $|\Phi_2(\xi) - (d - 1)\theta| = 1$ and corresponds to a massless vector meson. For $r > 1$ this massless state is obtained for $\varphi = 0$ and for $r < 1$ it corresponds to one direction being $\varphi_i = \pi$. There are no other massless particles except for $r \to \infty$.

After this general presentation, we will now restrict ourselves to the most interesting cases of 3 and 4 dimensions, where the continuum theory is supersymmetric. We will start by considering $d = 4$. The formulae for the propagators and masses follow from the general formulae obtained previously. The normalisation of our operators in this case is given in Table 1. As mentioned previously, if we consider correlation functions of ultralocal operators the only non-zero ones for Majorana fermions are the scalar (S) and the pseudoscalar-axial (PA) sectors. For ease of access to the results we now collect the expressions for the scalar mass $M_S$ and the lightest pseudoscalar mass $M_P$:

\begin{align}
\cosh(M_S) &= |\Phi_2(\xi) \pm 3| \quad (65) \\
\cosh(M_P) &= \theta \Xi - \sqrt{(\theta^2 - 1)(\Xi^2 - 1)} + 9\epsilon^2 ,
\end{align}

where the $\pm$ in $\cosh(M_S)$ is for $|r| < 1$ and $|r| > 1$, respectively. We recall
the expression for $\Phi_2$ and $\Xi$ in this dimension:

$$
\Phi_2(x) = \frac{8x^2 - 14x + 7}{2x(1 - x)}
$$

$$
\Xi = \Phi_2(\xi) - 3r^2 \epsilon.
$$

$\xi$, $\epsilon$ and $\theta$ are, as usual, the ones in Eqs. (25, 58).

The critical line where the pseudoscalar particle becomes massless is given by $\kappa = \kappa_c(r)$, where the critical hopping parameter $\kappa_c^2$ is given by:

$$
\kappa_c^2(r) = \frac{23r^2 + 9 + 3\sqrt{9r^4 + 46r^2 + 9}}{896r^4}.
$$

(66)

Our formulae are equal for fields in the fundamental and the adjoint representation of the group, so that up to a normalisation factor, they can be directly compared with the results obtained previously on the literature [16]-[19]. Once the appropriate normalisation of the fields is taken into account, we find perfect agreement with the results obtained by previous authors.

Now we discuss the $d = 3$ case, for which no explicit results have been given previously. The expression for the propagator does not follow from our previous formulae. In this case the basis of meson operators is given by the set of 2 by 2 matrices $\hat{S}_i = \{1, \sigma_i\}$, where $\sigma_i$ are the Pauli matrices. To comply with the conventional indexing of the Pauli matrices, we will adopt the names $(\varphi_1, \varphi_2, \varphi_3)$ for the lattice momentum coordinates, time being assigned by convention to coordinate 1. It is easy to see that here the propagator splits into two separate blocks: the scalar and the vectorial sector, corresponding to the unit matrix and Pauli matrices respectively. By inverting the matrix $\Theta_2(\xi)I - \tilde{A}_2(\varphi)$ we find the following results for the scalar $\hat{G}_{SS}(\varphi)$ and vectorial $\hat{G}_{VV}(\varphi)$ propagators:

$$
\hat{G}_{SS}(\varphi) = \frac{H(\xi)}{\Phi_2(\xi) - \sum_{i=1}^{3} \cos(\varphi_i)}
$$

(67)
\[ \hat{G}_{VV}(\varphi) = \frac{H(\xi)\hat{M}^V}{\hat{\alpha}_1\hat{\alpha}_2\hat{\alpha}_3 + \sum_{i=1}^{3} \beta_i^2\hat{\alpha}_i} . \]

where the matrix elements of \( \hat{M}^V \) on the \( \sigma_i \) basis are given by

\[ \hat{M}^V_{ii} = \tilde{\beta}_i^2 + \frac{1}{2} \sum_{j,k=1}^{3} (\epsilon_{ijk})^2\tilde{\alpha}_j\tilde{\alpha}_k \]

\[ \hat{M}^V_{i\neq j} = \tilde{\beta}_i\tilde{\beta}_j - \sum_{k=1}^{3} \epsilon_{ijk}\tilde{\alpha}_k\tilde{\beta}_k , \]

\[ \epsilon_{ijk} \text{ is the completely antisymmetric tensor and the functions } \tilde{\alpha}_i, \tilde{\beta}_j \text{ are given by} \]

\[ \tilde{\alpha}_i = \Phi_2(\xi) - \theta \sum_{i=1}^{3} \cos(\varphi_i) + 2\epsilon \cos(\varphi_i) \]

\[ \tilde{\beta}_i = 2\epsilon r \sin(\varphi_i) . \]

For \( d = 3 \), the functions \( H \) and \( \Phi_2 \) adopt the form:

\[ H(x) = -\eta D_R \left( \frac{5}{2} \right) \frac{4x - 5}{x(x - 1)(6x - 5)} \]

\[ \Phi_2(x) = \frac{6x^2 - 10x + 5}{2x(1 - x)} . \]

Let us now consider meson masses. The general formulae (59)-(61) are valid for this case. The restriction to gamma-fermion states which are eigenstates of the charge conjugation operator \( C \) simply eliminates the double degeneracy of all levels. Taking into account the necessary evenness of the number of gamma-fermions, we arrive at:

\[ \cosh(M_S) = |\Phi_2 - \sigma| = |\Xi + \sigma\epsilon| \]

\[ \cosh(M_{11}) = |\Phi_2 - \sigma\theta| = |\Xi - \sigma\epsilon| \]

\[ \cosh(M_{mix}^\pm) = \theta \Xi \pm \sqrt{(\Xi^2 - 1)(\theta^2 - 1) + \epsilon^2(4 - \sigma^2)} , \]
where, as usual, \( \sigma = \sum_{i=2}^{3} \vartheta_i \), and \( \vartheta_i \equiv \cos(\varphi_i) \) takes values \( \{-1, 1\} \). \( \Xi \) is defined below Eq. (61). The analysis of the lightest meson follows the general presentation done previously for odd space-time dimension.

Finally, we will consider correlation functions of the 3-gluino operators given in Eq. (33). We recall that depending on the choice of the invariant colour tensor \( (f \text{ or } d) \) the spin matrix \( \mathbf{S}_i \) can be chosen completely symmetric or completely antisymmetric under the exchange of the 3 spinorial indices. The general expression for the propagator was given in Section 2 (Eq. (35)), and involves the inversion of the matrix \( \Theta_3(\xi) \mathbf{I} - \tilde{\mathbf{A}}_3 \). Although performing this inversion explicitly in general seems a fairly complicated problem, we will be able to perform it inversion for the simplest case. This corresponds to a completely antisymmetric \( \mathbf{S}_i \) matrix. Actually, for \( d = 4 \) it gives the full antisymmetric subspace. We proceed by introducing the antisymmetric matrices \( \mathbf{A}_A \), where \( A \) is a spinorial index which can be looked at as the spin components of a composite fermion (a spin 1/2 fermion in 4 dimensions).

The form of \( \mathbf{A}_A \) is given by:

\[
(A_A)^{A_1A_2A_3} = C^{A_1A_2A_3} - C^{A_1A_3A_2} + C^{A_2A_3A_1},
\]

(74)

The advantage of these matrices is that the states \( \tilde{G}^{-1}_3 \langle A_A \rangle \) are the basis of an invariant subspace under the action of \( \tilde{\mathbf{A}}_3 \). This allows us to perform the appropriate inversion in this subspace. If we label by \( \tilde{G}^{(A)}_{AA'} \) the momentum space propagator between the states \( \mathbf{S}_i = \mathbf{A}_A \) and \( \mathbf{S}_j = \mathbf{A}_{A'} \), we can write:

\[
\tilde{G}^{(A)}_{AA'} = -R_3(\xi) (6 - 3 Tr(\mathbf{I}_{\text{spin}})) \times \\
\left( C[(\Theta_3(\xi) - 2\kappa^3 r(r^2 - 1)\tilde{\sigma})\mathbf{I} + 2i\kappa^3 (r^2 - 1) \sum_{\mu} \sin(\varphi_\mu) \gamma_\mu]^{-1} \right)_{AA'}
\]

(75)

where \( Tr(\mathbf{I}_{\text{spin}}) \) is the dimension of the spin space (equal to \( 2^{d/2} \) for even \( d \)) and \( \tilde{\sigma} = \sum_\mu \cos(\varphi_\mu) \). The inversion of the matrix contained in the previous
formula can be done in the standard way for a fermion propagator: \((a + b_μγ_μ)^{-1} = (a - b_μγ_μ)/(a^2 - b_μ^2)\).

Setting as usual the space-like momenta to 0 or \(π\), we might extract the mass of this fermion state from the position of the pole in the propagator. The result is:

\[
\cosh(M_1^2) = \left| r\Xi_3 \pm \sqrt{(\Xi_3)^2 - \epsilon} \right| ,
\]

where we recall that \(\epsilon = 1/(r^2 - 1)\) and the symbol \(\Xi_3\) stands for:

\[
\Xi_3 = \frac{\Theta_3(\xi)\epsilon^2}{2\kappa^3} - r\epsilon\sigma
\]

with \(\sigma = \sum_i \cos(φ_i^{(special)})\). It can be shown that, although the mass formula depends on the sign of \(κ\) and \(r\), the spectrum does not. Without loss of generality, we can set \(κ > 0\); then, the lowest mass is obtained for \(φ = 0\) when \(r > 1\) or \(-1 < r < 0\), and for \(φ = π \forall i\) when \(r < -1\) or \(0 < r < 1\).

Obtaining the remaining masses of 3 gluino states analytically in general is a complicated problem. Nevertheless, we do not need them for the physical conclusions of this paper. Indeed as we saw in the case of mesons the only point in parameter space were the scalar and the pseudoscalar became degenerate in mass occurred for \(κ → 0, r → ∞\) with \(κr = √\lambda\) fixed. Extracting the masses and propagators for all \(p\)-gluino states in this limit is fairly simple. As in the case of mesons all spin states are degenerate, and the masses only depend on \(σ\). By looking at the expressions (42,43) for the matrix to be inverted when considering a \(p\)-gluino operator, we immediately see that:

\[
Θ_pI - \tilde{A}_p \xrightarrow{r→∞} (Θ_p - 2κ^p p^p\tilde{σ})I .
\]

Now, the eigenvalues, whose zeroes gives the masses, are explicit. Their value
is given by:

\[
cosh(M_p) = \Phi \left( \left( \frac{(2d-1)(1-\xi)}{\xi} \right)^{p/2} \right) - \sigma , \tag{79}
\]

where we have used the definition (25) of \( \xi \) and introduced the function

\[
\Phi(x) = \frac{1}{2} \left( x + \frac{(2d-1)}{x} \right) . \tag{80}
\]

As \( p \) increases \( \left( \frac{(2d-1)(1-\xi)}{\xi} \right)^{p/2} \), the argument of the function \( \Phi \) in (79), increases. It is easy to see that within the relevant range \( x \geq 1 \) the function \( \Phi(x) \) is monotonously increasing. This allows us to prove the following inequality:

\[
M_p > M_q \quad \text{for} \quad p > q \tag{81}
\]

As a consequence for any value of \( \xi \) in the physical range \([0, \frac{1}{2}]\) and \( \varphi_i = 0 \), the mass of the 3-gluino states is higher than the mass of the 2-gluino states. In particular at the critical point \( \xi = \frac{1}{2} \), the 3-gluino states are massive.

It is also interesting to know whether there are points within the physical region (bounded by the 2-gluino critical lines) where some 3-gluino states become massless. For example, in Ref. [7] it is predicted that if chiral symmetry is broken spontaneously then so is supersymmetry, and a massless goldstino particle appears. We have explicitly checked that this does not occur within our framework for the 3 and 4-dimensional cases. The 3-gluino states are always massive.

### 4 Conclusions and future outlook

In this section we will summarise our results and discuss their implications. We have obtained the spectrum of N=1 SUSY Yang-Mills on the lattice at
large number of colours $N_c$ and strong coupling, by considering the hopping parameter expansion as a sum over lattice paths (random walks). We have resummed the expressions in the hopping parameter in a certain region enclosing the origin, for an arbitrary value of the Wilson parameter $r$. We have worked at zeroth order in $\beta$, the pure gauge coupling constant. However, Wilson’s action for the gauge part can be added either as a trace in the fundamental representation or as a trace in the adjoint one, with the corresponding couplings related to match the same naive continuum limit. Indeed, if we choose the adjoint version of Wilson action, our results (propagators and masses) are valid to all orders in $\beta$ (probably only within some region encircling the origin). This can be proven in the same way that one sees that the quenched approximation is exact in our case.

We have given formulae for the propagators and masses of 2 and 3 gluino states. The 2-gluino masses do coincide with the results for the meson spectrum in ordinary lattice QCD at strong coupling [17]-[19], and obtained by means of the effective potential method. Our method is based on the random walk resummation technique [22], [23]. This generalises the method of Kawamoto [16] for $r \neq 1$. Both methods have their relative advantages and disadvantages and occasionally there have been some conflicting conclusions (See the U(1) problem discussion in Refs. [19] and [21]). The random walk method does not rely on specific assumptions about the symmetries of the saddle point solution. It is rather based on the resummation of convergent series. Convergence is simple to see, since the number of lattice paths of length $L$ grows like a power of $L$ and the matrices involved are bounded. The radius of convergence is given by the distance to the nearest singularity. In our case, we have two resummations involved. One on the pure spike paths, which converges provided $\kappa^2 < \frac{1}{4(d-1)(r^2-1)}$, and a second one whose
border line in four dimensions, is given by the critical line (Eq. (66)) where the pseudoscalar becomes massless. Furthermore, in our case, the replacement of the Pfaffian by the square root of the determinant can be rigorously justified if $|\kappa| < \frac{1}{2d(|\kappa|+1)}$. This falls a bit too short compared to the other limits. Finally, we want to stress that we have provided formulae for arbitrary space-time dimensions $d$, which could be of interest for other researchers in the field.

Apart from the technical interest of our methods and results, we consider that the main usefulness of our results, is that they provide a guideline and boundary formulae for groups investigating this model by Monte Carlo simulations. Of course, our results are only valid for large $N_c$, but experience teaches us that this is frequently a numerically good approximation. There is one issue in which unfortunately our method could perhaps not help. It has been predicted, that this model should have a first order phase transition in $\kappa$ [26]. Our work predicts the presence of a second order phase transition associated to the vanishing of the pseudoscalar mass. It can be argued however, that a series expansion like ours can be seen as the expansion around one of the effective potential vacua. Thus, the mentioned second order transition could lie in the metastable phase. The point about the order of the phase transition should be settled by future Monte Carlo simulations.

Finally, it is tempting to speculate about the relevance of our results in the spirit of supersymmetry breaking. For that purpose one is interested in critical lines where a continuum limit can be defined. The states whose lattice mass vanish at the critical line, are the states that survive this continuum limit. If supersymmetry is recovered at this continuum limit one expects these states to form a supermultiplet. The analysis of Curci and Veneziano leads to a multiplet in which in addition to the pseudoscalar par-
particle, there is a scalar one and a spin $\frac{1}{2}$ fermion. These particles have equal positive continuum masses. Since the contribution to the mass of the pseudoscalar comes through the anomaly, which vanishes in our case, we should expect a massless multiplet. Nevertheless, it is doubtful that the analysis of Curci and Veneziano applies at strong coupling since it is based on the continuum SUSY Yang-Mills lagrangian. By power counting, this model has a single relevant parameter, the gluino mass, and hence fine tuning one of the bare couplings one could find a line corresponding to vanishing mass and restored supersymmetry. However, at strong coupling the gluonic degrees of freedom stay of the order of the cut-off. Hence, the low energy lagrangian, if supersymmetric, would rather coincide with the Wess-Zumino model. This has many more relevant operators (the different masses and couplings) and demands tuning of more bare parameters to approach it. In this respect the situation in 3 dimensions could be interesting since the model would be interacting. In 4 dimensions we would expect a free low energy lagrangian giving the physics of the continuum limit. With all these concerns in mind we did not want to loose the opportunity to explore the $\kappa - r$ parameter space in search for degenerate low energy multiplets. Actually, we concluded that the only point where several mesons become massless is in the limit $\kappa \to 0$, $r \to \infty$ and $r\kappa = \frac{1}{2\sqrt{2d-1}}$ (i.e., $\xi = \frac{1}{2}$). The masses at this point are the ones corresponding to a gauge-Higgs system: an interesting model in its own right. At this point we looked at the masses of the $p$-gluino states with $p > 2$. This includes fermionic degrees of freedom (for $p$ odd). However, we showed that these states remain massive (in cut-off units) at this critical point. So that the main conclusion is that there is no point in the $\kappa - r$ plane giving a possible candidate for a supersymmetric continuum limit.

We conclude the paper by mentioning a few possible improvements of our
results. First of all, the possibility of extending the calculations and formulae to next to leading order in $1/N_c$ seems a feasible one. The most important consequence of this extension could be in cases when the effects are absent to leading order, like the effect of quenching, anomalies, etc. Then one can try to include higher orders in $\beta_{\text{fundamental}}$, or combined $1/N_c$ and $\beta_{\text{adjoint}}$. Then, of course, it would be very good to rederive the results of this paper with the effective action method. This technique, as mentioned previously, would allow the discovery of possible first order phase transitions. Finally, it should be commented that our methods and results could be used to study other supersymmetric models, such as SUSY QCD.

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Appendix

In this appendix we will present the terminology and main results on random walks that we will need in the text. Not to conflict with other definitions, we will refer to these random walks as lattice paths and a precise definition will be provided. We will work in arbitrary dimension $d$ and will employ additional definitions given at the beginning of Section 2. Proofs will not be given. For that we address the reader to Ref. [23].
A lattice path of length \( L \) is an element \( \gamma \equiv (n, \bar{\alpha}) \in \mathcal{L} \times I^L \). The point \( n \in \mathcal{L} \) is the origin of the path, \( \bar{\alpha} \) is the path sequence, and \( m = n + \sum_{i=1}^{L} \gamma(\alpha_i) \) is the endpoint of the path. We can label the different path sets as follows. Let \( \mathcal{S}_L(n) \) be the space of all paths with origin \( n \) and length \( L \); \( \mathcal{S}_L(n \rightarrow m) \) is the space of all paths with origin \( n \), endpoint \( m \) and length \( L \). Now we will introduce the notion of a spike.

A path \( \gamma \equiv (n, \alpha_1, \ldots, \alpha_L) \) has spikes if there exist one integer \( 1 \leq i \leq L \) such that \( \alpha_{i+1} = \bar{\alpha}_i \). In the converse case one says that the path has no spikes. The set of all paths without spikes of length \( L \) and origin \( n \) is labelled \( \bar{\mathcal{S}}_L(n) \) (\( \bar{\mathcal{S}}_L(n \rightarrow m) \) if the endpoint is fixed to \( m \)). Now, we can associate to any path \( \gamma \) a corresponding path \( \bar{\gamma} \) called its reduced path, by simply eliminating in an orderly manner all pairs \( \alpha_{i+1} = \bar{\alpha}_i \) in its sequence. Notice that this procedure preserves the origin and endpoint of the path.

Our main strategy will be to convert the sum of paths into a sum over reduced paths. For that we would first need to compute the following function:

\[
F(\bar{L}, z) = \sum_{p=0}^{\infty} z^p N(\bar{L}, p)
\]

where \( N(\bar{L}, p) \) is the number of paths of length \( \bar{L} + 2p \) whose reduced path is a path (with no spikes) of length \( \bar{L} \). This number does not depend on the path itself but only on its length. In Ref. [23] it is shown that:

\[
F(\bar{L}, z) = \frac{1}{(1 - \frac{2d}{2d-1}\xi) \left(\frac{1 - \xi}{1 - \xi} \right)^{\bar{L}}}
\]

\[
\xi = \frac{1 - \sqrt{1 - 4(2d - 1)z}}{2}
\]

In addition, we would need to be able to perform sums over the set of reduced paths with origin in \( n \) of a product of matrices. To be specific, let \( A_\alpha \) for \( \alpha \in I \), be a collection of matrices satisfying \( A_\alpha A_{\bar{\alpha}} = \lambda I \), with \( I \) the
unit matrix. Then we want to compute the matrix:

\[ T(A) = 1 + \sum_{L=1}^{\infty} \sum_{(n,\vec{a}) \in S_L(n)} A_{\alpha_1} \cdots A_{\alpha_L}. \]  
(85)

It can be shown [23] that \( T(A) \) is given by the formula:

\[ T(A) = (1 - \lambda)(1 + (2d - 1)\lambda - \tilde{A})^{-1}, \]  
(86)

where \( \tilde{A} = \sum_{\alpha \in I} A_{\alpha} \).

In the text we will need to evaluate an expression like Eq. (85) with a slight variation. We would need to sum only over paths going from one lattice point \( x \) to another one \( y \). This modified situation can be reduced to the case given before by the following procedure. Instead of considering the matrix \( A_{\alpha} \) we will multiply it by a phase \( e^{i\varphi_{\alpha}} \), where \( \varphi_{\mu} = -\varphi_{\mu} \). Then, diagrams that go from \( x \) to \( y \) have coefficients that go like \( e^{i\varphi(x-y)} \). Thus, the required expression can be obtained by projecting onto this term:

\[ T_{x\to y}(A) = \prod_{\mu} \left( \int \frac{d\varphi_{\mu}}{2\pi} \right) e^{-i\varphi(x-y)} (1 - \lambda)(1 + (2d - 1)\lambda - \tilde{A}'(\phi))^{-1}, \]  
(87)

with \( \tilde{A}'(\phi) = \sum_{\alpha \in I} e^{i\varphi_{\alpha}} A_{\alpha} \).
References


Table 1: Nomenclature and normalisation for the spin $S_i$ and colour $C_i$ matrices used to define the four dimensional 2-gluino operators. The last column gives the number of components associated to each one of them. The vector and tensor spin matrices $V(\rho)$ and $T(\rho\sigma)$ are symmetric, thus forbidden when the field is Majorana.

<table>
<thead>
<tr>
<th>$i$ label</th>
<th>$S_i$</th>
<th>$C_i$</th>
<th># d.f.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S$</td>
<td>$\frac{1}{2}CI$</td>
<td>I</td>
<td>1</td>
</tr>
<tr>
<td>$V(\rho)$</td>
<td>$\frac{1}{2}C\gamma_\rho$</td>
<td>I</td>
<td>4</td>
</tr>
<tr>
<td>$T(\rho\sigma)$</td>
<td>$\frac{-\sqrt{2}}{4\sqrt{2}}C[\gamma_\rho, \gamma_\sigma]$</td>
<td>I</td>
<td>6</td>
</tr>
<tr>
<td>$A(\rho)$</td>
<td>$\frac{1}{2}C\gamma_\rho \gamma_5$</td>
<td>I</td>
<td>4</td>
</tr>
<tr>
<td>$P$</td>
<td>$\frac{1}{2}C\gamma_5$</td>
<td>I</td>
<td>1</td>
</tr>
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</table>