2 From ideal to continuous collapses

Von Neumann shows how his theory of ideal collapses applies to the indirect measurement of any Hermitian observable [2]. To measure the position operator $q$ of our quantized system $Q$ in its quantum state $\psi(q)$, we let it to interact with the momentum $p_A$ of another quantum system $A$ (ancilla) whose coordinate $x_A$ will be the pointer to show the measurement outcome $\tilde{q}$. Accordingly, we assume a Gaussian wave function for $A$, centered at $x_A = 0$ with precision $\Delta$

$$\psi_A(x_A) = \left[2\pi\Delta^2\right]^{-1/4} \exp \left(\frac{-x_A^2}{4\Delta^2}\right). \quad (1)$$

We assume that the system $Q$ and the ancilla $A$ are uncorrelated initially, the initial wave function factorizes as $\psi_A(x_A)|\psi(q)\rangle$. The system and the ancilla will interact only a very short time so that the self-evolutions of $Q$ and $A$ can be ignored during the measurement. We shall approximate the total Hamiltonian by $\delta(t)q p_A$. We can integrate the Schrödinger equation during the measurement. The factorized initial wave function transforms unitarily into the correlated one:

$$\psi_A(x_A)|\psi(q)\rangle \to \psi_A(x_A - q)|\psi(q)\rangle. \quad (2)$$

The pointer $x_A$ has taken over the value of the system coordinate $q$. Let us read out the pointer’s coordinate with a precision much higher than $\Delta$ or any characteristic length of the system’s state $\psi(q)$. Hence we assume infinite precision formally. Then, according to von Neumann’s collapse theory, the wave function of the ancilla shrinks into a delta function $\delta(\tilde{q} - x_A)$, where $\tilde{q}$ is the measurement outcome, while the composite wave function (2) collapses into the product of the ancilla’s delta function and the system’s new wave function:

$$\psi_A(x_A - q)|\psi(q)\rangle \to \delta(x_A - q) \frac{\psi_A(\tilde{q} - q)|\psi(q)\rangle}{N(\tilde{q})}. \quad (3)$$

The factor $1/N(\tilde{q})$ normalizes the system’s new wave function:

$$N^2(\tilde{q}) = \int \psi_A(\tilde{q} - q)|\psi(q)\rangle^2 dq. \quad (4)$$

Furthermore, the probability distribution of the outcome $\tilde{q}$ is equal to the squared modulus of the overlap between the states respectively before (2) and after (3) the collapse:

$$p(\tilde{q}) = \left| \int \overline{\psi_A(x_A - q)} \psi_A(x_A - q) \psi_A(\tilde{q} - q)|\psi(q)\rangle d x_A dq \right|^2 = N^2(\tilde{q}). \quad (5)$$
In the second line we used Eq. (4). Now that the state (3) after the collapse factorizes again, we can summarize the net effect of the above standard measurement on the system $Q$, without any further reference to the ancilla $A$.

The system's original wave function $\psi(q)$ has become multiplied by a Gaussian factor [the ancilla's $\psi_A(\bar{q} - q)$, in fact] and then re-normalized:

$$\psi(q) \rightarrow N^{-1}(\bar{q})[2\pi \Delta^2]^{-1/4}\exp\left[-\frac{(\bar{q} - q)^2}{4\Delta^2}\right]\psi(q)$$  \hspace{1cm} (6)

where the center $\bar{q}$ of the Gaussian [the outcome of the standard measurement] is distributed according to the probability distribution $p(\bar{q}) = N^2(\bar{q})$ being equal to the squared norm of the unnormalized state after the collapse (6).

Without referring to the above derivation from standard measurement theory, this 'hitting-process' had been postulated in the eighties [3,4,11] in order to build up ad hoc models of continuous emergence of classicality under various titles like continuous measurement, dynamical collapse, spontaneous collapse e.t.c.. Only few physicists [12] emphasized that the hitting-process was formally derivable from standard measurement theory. On the contrary, many thought that the process represented a modification of standard quantum theory. This belief made the proponents (including me, among others) enthusiastic since we sensed the flavor of a heuristically innovated and successful theory. The opponents drew the negative conclusion, warning that the modification of the standard theory was completely groundless [15].

Meanwhile the same mathematical equations of continuous measurement were really obtained from standard quantum mechanics [13,14], still in Markov approximation. A few years later, however, it was possible to show that standard quantum theory of atom+radiation, when described in proper basis, led to exact stochastic equations for the atomic wave function [16]. These equations, equivalent to the fully quantized theory on one hand, turn out to reduce to the widely used phenomenological equations of continuous (dynamical, spontaneous, whatever) collapse (measurement) in the Markov limit. Indeed, the equations of continuous measurement follow from the hybrid representation of standard quantum mechanics.

3 Hybrid dynamics and the ideal collapse

The interaction between quantum and classical systems is called hybrid dynamics [15]. It was a long march from mean-field approximation [17] through its stochastic refinements [18,19] and attempts at canonical coupling [20,21] until the first mathematically consistent equations were written down [22,23].

We have finally obtained a general theory of hybrid dynamics [24].

Assuming a quantum system $Q$ in state $\rho_Q$ and a classical canonical system $C$ with phase space distribution $\rho_C(x, p)$, we form the hybrid system
\( Q \times C \). If the subsystems \( Q \) and \( C \) are uncorrelated then it is straightforward to construct the hybrid state

\[
\rho(x, p) = \rho_Q \rho_C(x, p)
\]

for the composite system. In general, we represent the state of the hybrid system by a hybrid "density" \( \hat{\rho}(x, p) \) which is a phase space dependent non-negative operator. Its trace is the phase space distribution \( \rho_C(x, p) \) of \( C \) while its phase space integral yields the density operator \( \rho_Q \) of \( Q \). When \( \hat{\rho}(x, p) \) is not factorable the unconditional quantum state \( \rho_Q \) must be distinguished from the conditional quantum states:

\[
\rho_{xy} = \frac{\hat{\rho}(x, p)}{\rho_C(x, p)}
\]

depending on the classical coordinates \( x, p \) as conditions. The Hamiltonian of the hybrid system takes this form:

\[
H(x, p) = H_Q + H_C(x, p) + H_{INT}(x, p)
\]

where, obviously, the interaction term is a phase space dependent Hermitian operator. One can construct the following canonical hybrid equation of motion \([20]\) for the hybrid state \( \hat{\rho}(x, p) \):

\[
\partial_t \hat{\rho} = -\hat{\{H, \hat{\rho}\}} + \frac{i}{\hbar} \hat{\{H, \hat{\rho}\}} - \frac{1}{\hbar^2} \hat{\{\hat{\rho}, H\}} P
\]

which is the naive combination of the Dirac \([~], ~\) and the Poisson \(\{ , \} P\) brackets. Unfortunately, this equation does not preserve the positivity of \( \hat{\rho}(x, p) \). So, the naive construction \((10)\) does not work. In fact, the hybrid dynamics cannot be a true reversible dynamics. We have to make a little compromise. This I found first for the special case when \( H_{INT} \) is linear in \( x \) and \( p \) \([22]\). One applies the following Gaussian coarse graining, over Planck cells, to the hybrid state:

\[
\rho(x, p) \rightarrow \int \exp[-(\xi^2 + \eta^2)] \rho(x + \xi, p + \eta) \frac{d\xi d\eta}{2\pi} .
\]

Applying this coarse-graining on both sides of the naive equation \((10)\), one obtains two new terms:

\[
\partial_t \rho = -\{H, \rho\} + \frac{i}{\hbar} \{H, \rho\} - \frac{1}{\hbar^2} \{\rho, H\} P - \frac{1}{\hbar^2} \{\partial_x H, \partial_x \rho\} - \frac{1}{\hbar^2} \{\partial_p H, \partial_p \rho\} .
\]

And this equation, as can be shown, preserves the positivity of the coarse-grained hybrid state. Of course, we cannot choose an arbitrary hybrid state as initial state, sharp values of \( x \) and \( p \), or wild fluctuations within single Planck cells are forbidden. The rigorous constraints for \( \rho(x, p) \) are given elsewhere \([16,24]\).
By taking the trace of Eq. (12), one can show that the evolution of the classical states is a flow:

$$\partial_t x = \left\langle \partial_p H(x, p) \right\rangle_{x_p}, \quad \partial_t p = -\left\langle \partial_x H(x, p) \right\rangle_{x_p},$$

(13)

where $\langle \ldots \rangle_{x_p}$ stands for the expectation values $tr(\ldots \rho_{x_p})$ in the current conditional quantum state (8). This flow generalizes the naive mean-field equations where $\rho_{x_p} \equiv \rho_Q$ and quantum fluctuations are ignored in the back-reaction of the quantum system $Q$ on the classical $C$.

But there are other similar concepts which are recovered by hybrid dynamics. Quantum Brownian motion is one. The exact non-Markov stochastic Schrödinger–equation [25, 26] of the Caldeira–Leggett-type open systems (which include, e.g., the atom + radiation systems) follows automatically from the corresponding hybrid equations (12) [16, 24]. This means that, in particular, the phenomenological Itô-Schrödinger–equations of continuous (dynamical) collapse (measurement) follow from the hybrid equations in the Markov limit.

Finally I demonstrate the "presence" of collapse mechanism in the hybrid dynamics. To this end, I show that the hybrid dynamical equations (12) describe the Stern–Gerlach measurement, including the collapse of the spin's state and the corresponding motion of the classical pointer. Our quantum system $Q$ is the electron's spin and initially it is in the superposition

$$|\text{in}\rangle = c_+|+\rangle + c_-|\rangle = \sum_{\alpha = \pm 1} c_\alpha |\alpha\rangle$$

(14)

of the two eigenstates $|\pm\rangle$ of $\sigma_3$. Our classical system $C$ is the pointer. Let it be a harmonic oscillator with Hamiltonian $\frac{1}{2}(x^2 + p^2)$, shortly but strongly coupled to the measured spin component $\sigma_3$ by the interaction $H_{INT} = g\delta(t)p\sigma_3$, where $\Delta = 1/g$ will be the precision of the measurement and we assume $\Delta \ll 1$. Since we are interested in the states just before and, respectively, after the measurement, only the interaction Hamiltonian is relevant and the hybrid equation (12) will take this form:

$$\partial_t \rho = -ig\delta(t)p[\sigma_3, \rho] - \frac{g}{2}\delta(t)[\sigma_3, \partial_x \rho] + \frac{g}{2}\delta(t)[\sigma_3, \partial_p \rho].$$

(15)

As it follows from this dynamics, $\bar{\sigma}_3 = x/g = x\Delta$ will play the role of the pointer variable to indicate the value of the spin operator $\sigma_3$ after the measurement. We assume the following factorized initial state for the hybrid system:

$$|\text{in}\rangle\langle\text{in}| \exp[-\frac{g}{2}(x^2 + p^2)] = |\text{in}\rangle\langle\text{in}| \rho_C(x, p; \text{in}),$$

(16)

where $\rho_C(x, p; \text{in})$ corresponds to the pointer's initial position $x = 0 \pm 1$, i.e. to $\bar{\sigma}_3 = 0 \pm \Delta$. The evolution (14) acts on matrix elements of the initial state.
(16) as follows:

\[
\exp \left( -i g p [\sigma_3, ] - \frac{g^2}{\hbar} [\sigma_3 \partial_x, ] - \frac{g^2}{\hbar} [\sigma_3 \partial_y, ] \right) |\alpha \rangle \langle \beta| \rho_C(x, p; \text{in}) = \\
|\alpha \rangle \langle \beta| \exp \left( -\frac{(\alpha - \beta)^2}{4g^2} - i(\alpha - \beta)g p \right) \\
\rho_C \left( x - \frac{\alpha + \beta}{2} g, p + \frac{\alpha - \beta}{2i} g; \text{in} \right)
\]

(17)

We see that the off-diagonal terms are heavily damped, so the initial state (16) transforms into a diagonal final state:

\[
\sum_{\alpha = \pm 1} |\alpha \rangle \langle \alpha| \rho_C(x - \alpha g, p; \text{in}) = |+\rangle \langle +| \rho_C(x, g, p; \text{in}) + |-\rangle \langle -| \rho_C(x, -g, p; \text{in})
\]

(18)

This result clearly shows that the pointer’s coordinate shifts either to the right \((x = g \pm 1)\) with probability \(|c_+|^2\) and then the spin’s state is \(|+\rangle\), or it moves to the left \((x = -g \pm 1)\) in the complementary cases:

\[
(|\text{out}, \vec{\sigma}_3|) = \left\{ \begin{array}{cl} 
|+\rangle, & \tilde{\sigma}_3 = -1 \pm \Delta \\
|\text{out}, \vec{\sigma}_3| = +1 \pm \Delta & \text{with probability } |c_+|^2 \\
\end{array} \right.
\]

(19)

where \(\Delta \ll 1\). This scheme of the final quantum and classical pointer states is, regarding to the initial state (16) with the superposed spin (14), identical to the result of the corresponding ideal [2] Stern–Gerlach quantum measurement.

4 Summary

As I argued in Sec. 2, all phenomenological stochastic Schrödinger equations, however sophisticated they are, remain in the framework of standard quantum mechanics (whose part is the von Neumann measurement theory, too). This shall of course question part of the criticism that these proposals are groundless modifications of quantum mechanics since they are not modifications after all. Rather they are indicating the natural presence of continuous collapse mechanisms within standard quantum theory.

In Sec. 3 I illustrated that the concept of canonically interacting classical and quantum systems automatically implies the emergence of classicality in a way which is definitely more general than the concept of collapse (measurement). Ideal collapses, continuous (Markov or non-Markov) collapses follow from the hybrid dynamics. The paradigmatic (and controversial) mean-field approach can naturally be identified and improved within the hybrid dynamics.

Like all continuous collapse models, also hybrid dynamics is equivalent mathematically with a certain enlarged unitary dynamics. Hybrid dynamics
is a powerful unified framework to describe the variety how classicality ‘appears’ [27] from quantum, yet this new phenomenology is in itself unlikely to innovate our knowledge about the foundations. We are being captured in the old castle of standard quantum mechanics. Sometimes we think that we have walked into a new wing. It belongs to the old one, however.

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