CONSISTENT OFF-SHELL TREE STRING AMPLITUDES

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Abstract

We give a construction of off-shell tree bosonic string amplitudes, based on the
operatorial formalism of the $N$-string Vertex, with three external massless states
both for open and closed strings by requiring their being projective invariant. In
particular our prescription leads, in the low-energy limit, to the three-gluon ampli-
tude in the usual covariant gauge.

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One of the main reasons for studying off-shell string amplitudes is the investigation of the field theory limit of string theories where the inverse string tension $\alpha' \to 0$ [1] [2]. A detailed analysis of the relation between string theory and field theory is non-trivial and potentially very powerful since string theory manages to organize scattering amplitudes in a very compact form and in a considerably low number of diagrams. In particular, closed strings can be used to shed light on perturbative quantum gravity [3].

Furthermore, off-shell string amplitudes are relevant in studying processes involving interactions among D-branes [4].

Off-shell extensions of such amplitudes have been studied a great deal until now [1] [2] [5] ÷ [10] and different prescriptions have been given according to the pursued approaches.

In this letter we use the operatorial formalism of the $N$-string Vertex $V_{N;g}$ [11] for computing tree open and closed string amplitudes with three external massless states. This simple case results to be very clarifying about the prescription to use for analyzing off-shell string physics.

In the framework of the same formalism, in two recent papers [12] [13], off-shell one-loop amplitudes with an arbitrary number of external massless particles have been computed both for open and closed strings.

The $N$-string Vertex $V_{N;g}$ depends on $N$ Koba-Nielsen variables $z_i$'s [11], complex in the closed string case and corresponding to the punctures of the external states, through $N$ conformal transformations $V_i(z)$'s, which define a local coordinate system vanishing around each $z_i$, i.e.:

$$V_i(0) = z_i.$$

A choice of such a coordinate system can be regarded as a sort of a gauge choice. When $V_{N;g}$ is saturated with $N$ physical string states satisfying the mass-shell conditions, the corresponding amplitude does not depend on the $V_i$'s. If these conditions are relaxed, the dependence of $V_{N;g}$ on them is transferred to the off-shell amplitude. This is analogous to what happens in gauge theories, where on-shell amplitudes are gauge invariant, while their off-shell counterparts are not.

From the gauge choice made for the local maps $V_i$'s it depends the field theory gauge-
fixed Lagrangian which generates the amplitudes to be compared with the ones obtained from string theory in the low-energy limit. In fact one does not know a priori what gauge-fixing is chosen by the string when the field theory limit is extracted from its amplitudes.

The results obtained in ref. [2] in the one-loop case show that it is possible to perform choices of the \( V_i \)'s which, in the case of open bosonic strings, reproduce the field theory amplitudes obtained by using the background gauge.

Hence it arises the necessity to define the assumptions underlying the choice of the functions \( V_i \)'s.

The basic assumption we make in this work is that \textit{off-shell amplitudes have to be projective invariant}; indeed projective invariance, or Möbius invariance, is a crucial property if off-shell finiteness and factorization are required [14]. The choice of the local maps around the punctures has therefore to guarantee this invariance.

In the specific case of the amplitude for three massless states it turns out that, in the limit \( \alpha' \to 0 \), it depends on the choice of the \( V_i \)'s only through the ratio \( V_i''(0)/(V_i'(0))^2 \). Requiring its being projective invariant in this limit allows to select a family of functions of the punctures \( (z_1, z_2, z_3) \) depending on one parameter. The value of this latter can be fixed by requiring that the sum over all the anticyclic permutations of the lowest order term in \( \alpha' \), providing the field theory tree scattering amplitudes for photons, be identically zero. In this way we univocally determine the ratio \( V_i''(0)/(V_i'(0))^2 \).

Furthermore, requiring that the whole amplitude, and not only its low-energy limit, be projective invariant univocally fixes \( V_i'(0) \) that turns out to coincide with the first derivative evaluated in \( z = 0 \) of the projective transformation which maps the points \( \infty, 0, 1 \) respectively in \( z_{i-1}, z_i, z_{i+1} \), corresponding to the so-called Lovelace choice [15]. But we would like here to stress that the Lovelace choice does not reproduce the value of \( V_i''(0)/(V_i'(0))^2 \), compatible with our requirement of projective invariance.

It is possible to show that the value of \( V_i'(0) \) so found is the one that ensures that the string Green function at tree level reduces to a particle Green function in the field theory limit [16].
The above prescription yields, differently from the other results present in literature [17], an expression of the off-shell three-gluon string amplitude at tree level which reproduces the corresponding one in field theory in the most natural gauge, i.e. the usual covariant gauge, and also coincides with the one obtained by the background field method that is the technique in which results relative to higher orders are obtained in literature [2] [17].

Our starting point is the $N$-string 0-loop vertex for $N$ massless closed bosonic strings:

$$V_{N;0}^c = C_0 \Omega \int [dm^c]^0_N \exp \left\{ \frac{1}{2} \sum_{i=1}^{N} \sqrt{\alpha'} p_i \cdot \left[ \sqrt{\frac{\alpha'}{2}} p_i + \alpha_1^{(i)} \partial_z + \tilde{\alpha}_1^{(i)} \partial_{\bar{z}} \right] \ln |V_1'(z)|^2 |_{z=0} \right\}$$

$$\times \exp \left\{ \sum_{i,j=1 \atop i \neq j}^{N} \left[ \sqrt{\frac{\alpha'}{2}} p_i + \alpha_1^{(i)} V_1'(0) \partial_{z_i} + \tilde{\alpha}_1^{(i)} \tilde{V}_1'(0) \partial_{\bar{z}_i} \right] \cdot \left[ \sqrt{\frac{\alpha'}{2}} p_j + \alpha_1^{(j)} V_1'(0) \partial_{z_j} + \tilde{\alpha}_1^{(j)} \tilde{V}_1'(0) \partial_{\bar{z}_j} \right] \right\}$$

$$\times \ln |z_i - z_j|$$

where $|\Omega|$ is given by the direct product of the vacuum states of the Fock spaces relative to the oscillators $\alpha^{(i)}$ and $\tilde{\alpha}^{(i)}$, while the measure on the moduli space for a Riemann surface of genus $g = 0$ is:

$$[dm^c]^0_N \equiv \prod_{i=1}^{N} d^2 z_i |z_A - z_B|^2 |z_A - z_C|^2 |z_B - z_C|^2$$

$$\overline{\prod_{i=1}^{N} |V_i'(0)|^2 d^2 z_A d^2 z_B d^2 z_C}.$$

Furthermore, the normalization constant $C_0$ is given by $4\pi^3/\alpha' k^2$, where $k$ is the gravitational coupling constant. In the following our convention will be to use a superscript only for closed string objects and not for open string ones; it will be denoted by $c$. Through the introduction of the tree-level two-point Green function for closed strings, defined as:

$$G^c(z_i, z_j) = \ln \left| \frac{z_i - z_j}{\sqrt{V_i'(0) V_j'(0)}} \right|,$$

the operator (1) can be rewritten as follows:

$$V_{N;0}^c = C_0 \Omega \int [dm^c]^0_N \exp \left\{ \frac{1}{2} \sum_{i=1}^{N} \sqrt{\alpha'} p_i \cdot \left[ \sqrt{\frac{\alpha'}{2}} p_i + \alpha_1^{(i)} \partial_z + \tilde{\alpha}_1^{(i)} \partial_{\bar{z}} \right] \ln |V_1'(z)|^2 |_{z=0} \right\}$$

$$\times \exp \left\{ \sum_{i,j=1 \atop i \neq j}^{N} \left[ \sqrt{\frac{\alpha'}{2}} p_i + \alpha_1^{(i)} V_1'(0) \partial_{z_i} + \tilde{\alpha}_1^{(i)} \tilde{V}_1'(0) \partial_{\bar{z}_i} \right] \cdot \left[ \sqrt{\frac{\alpha'}{2}} p_j + \alpha_1^{(j)} V_1'(0) \partial_{z_j} + \tilde{\alpha}_1^{(j)} \tilde{V}_1'(0) \partial_{\bar{z}_j} \right] \right\}$$

$$\times \ln |z_i - z_j|$$
\[
\times \exp \left\{ \frac{N}{2} \sum_{i,j=1}^{N} \left[ \sqrt{\frac{\alpha'}{2}} p_i + \alpha^{(i)}_1 V'_i(0) \partial z_i + \alpha^{(i)}_1 V'_i(0) \partial z_i \right] \cdot \left[ \sqrt{\frac{\alpha'}{2}} p_j + \alpha^{(j)}_1 V'_j(0) \partial z_j + \alpha^{(j)}_1 V'_j(0) \partial z_j \right] \right. \\
\left. \times \left[ \frac{1}{2} \ln |V'_i(0) V'_j(0)| + G^c(z_i, z_j) \right] \right\}. \tag{4}
\]

In order to get off-shell scattering amplitudes among \( N \) massless external states from the previous operator, we have to saturate it on them after relaxing their on-shell conditions. Massless states are defined by:

\[
|\varepsilon, p > = \frac{k}{\pi} \varepsilon \lambda p \alpha^\lambda_1 \alpha^\rho_{-1} |0, p > \text{ in the closed string case}, \tag{5}
\]
\[
|\varepsilon, p > = g_d \sqrt{2\alpha} \varepsilon \lambda p \alpha^\lambda_1 |0, p > \text{ in the open string case}, \tag{6}
\]

with \( g_d \) being the gauge coupling constant of the target space Yang-Mills theory. The on-shell conditions we are going to release are the following:

\[
p^2 = 0 \quad \varepsilon \cdot p = 0. \tag{7}
\]

At tree-level, the \( N \)-string Vertex does not contain any operator that mixes the left and right sectors of the closed bosonic string. Moreover in the three-string case, in which there is not an integration over the Koba-Nielsen variables, the amplitude can be simply factorized in two open string amplitudes [18]. Hence, taking one sector of the operator given in eq. (4), and writing it properly for \( N = 3 \) in the case of open bosonic strings, we get:

\[
V^a_{3,0} = C^\text{open}_0 < \Omega | \int [dm]_3^0 \exp \left\{ \frac{1}{2} \sum_{i=1}^{3} \sqrt{2\alpha'} p_i \cdot \left[ \sqrt{2\alpha'} p_i + \alpha^{(i)}_1 V'_i(0) \partial z_i \right] \ln V'_i(z) \right| z=0 \right\} \\
\times \exp \left\{ \sum_{i,j=1}^{3} \left[ \sqrt{2\alpha'} p_i + \alpha^{(i)}_1 V'_i(0) \partial z_i \right] \cdot \left[ \sqrt{2\alpha'} p_j + \alpha^{(j)}_1 V'_j(0) \partial z_j \right] \\
\times \left[ \frac{1}{2} \ln (V'_i(0) V'_j(0)) + G(z_i, z_j) \right] \right\}. \tag{8}
\]

with \( C^\text{open}_0 = 1/g^2_d \alpha'^2 \) and where \( G(z_i, z_j) \) is the open string two-point function at tree level, which is related to the corresponding closed string one by:

\[
G^a(z_i, z_j) = \frac{1}{2} \ln \frac{z_i - z_j}{\sqrt{V'_i(0)V'_j(0)}} + \frac{1}{2} \ln \frac{\bar{z}_i - \bar{z}_j}{\sqrt{\bar{V}'_i(0)\bar{V}'_j(0)}} \equiv \frac{1}{2} [G(z_i, z_j) + G(\bar{z}_i, \bar{z}_j)]. \tag{9}
\]
We would like here to observe that in the three-string case the integration is absolutely fictitious. Indeed, through the identification \( z_A = z_1, z_B = z_2, z_C = z_3 \), the measure \([dm]_3^0\) merely reduces to:

\[
[dm]_3^0 \equiv \frac{(z_A - z_B)(z_A - z_C)(z_B - z_C)}{\prod_{i=1}^{3}(V'_i(0))}.
\]  

At the moment we do not want to perform any explicit choice of the local holomorphic coordinate \( V'_i(z) \), our purpose being to get some constraints on that function by requiring the projective invariance of the entire amplitude. Indeed, by saturating the operator (8) on three states as the ones defined in (6), without specifying any choice of the local holomorphic function, we get:

\[
A_0^3 = 4g_d (2\alpha')^{-\frac{1}{2}} (z_A - z_B)^{1 + 2\alpha'}p_1p_2(z_A - z_C)^{1 + 2\alpha'}p_1p_3(z_B - z_C)^{1 + 2\alpha'}p_2p_3 \sum_{e=1}^{3} \alpha'p_i^2 \ln V'_i(0)
\times \varepsilon^{(1)}_{\lambda} \varepsilon^{(2)}_{\mu} \varepsilon^{(3)}_{\nu} \left\{ (2\alpha')^{\frac{3}{2}} \left[ \left( p_1^\lambda V_1''(0) \frac{2}{V_1'(0)^2} + p_2^\lambda (z_A - z_B) + p_3^\lambda (z_A - z_C) \right) \right. \\
\times \left( \frac{p_2^\mu V_2''(0)}{2 V_2'(0)^2} + \frac{p_1^\mu}{(z_B - z_A)} + \frac{p_3^\mu}{(z_B - z_C)} \right) \right. \\
\times \left. \left( \frac{p_3^\nu V_3''(0)}{2 V_3'(0)^2} + \frac{p_1^\nu}{(z_C - z_A)} + \frac{p_2^\nu}{(z_C - z_B)} \right) \right] \\
+ (2\alpha')^{\frac{3}{2}} \left[ \frac{\eta^{\lambda\mu}}{(z_A - z_B)^2} \left( \frac{p_2^\mu V_2''(0)}{2 V_2'(0)^2} + \frac{p_1^\mu}{(z_B - z_A)} + \frac{p_3^\mu}{(z_B - z_C)} \right) \right. \\
\left. \frac{\eta^{\lambda\nu}}{(z_A - z_C)^2} \left( \frac{p_3^\nu V_3''(0)}{2 V_3'(0)^2} + \frac{p_1^\nu}{(z_C - z_A)} + \frac{p_2^\nu}{(z_C - z_B)} \right) \right] \right\}. 
\]  

In the low-energy limit the only contribution that survives is:

\[
A_0^3(\alpha' \rightarrow 0) = 4g_d \varepsilon^{(1)}_{\lambda} \varepsilon^{(2)}_{\mu} \varepsilon^{(3)}_{\nu} \left\{ \frac{\eta^{\lambda\mu}}{(z_A - z_B)^2} \left( \frac{p_2^\mu V_2''(0)}{2 V_2'(0)^2} + \frac{p_1^\mu}{(z_B - z_A)} + \frac{p_3^\mu}{(z_B - z_C)} \right) \right. \\
\left. + \frac{\eta^{\lambda\nu}}{(z_A - z_C)^2} \left( \frac{p_3^\nu V_3''(0)}{2 V_3'(0)^2} + \frac{p_1^\nu}{(z_C - z_A)} + \frac{p_2^\nu}{(z_C - z_B)} \right) \right\}.
\]
Asking for the previous amplitude to be projective invariant, we get the following constraints for $V_0(0) = \left( V_0(0) \right)^2$:

$$\frac{V''(0)}{(V'(0))^2} = \frac{2}{(z_i - z_{i+1}) (z_i - z_{i-1})} \frac{(z_i - z_{i+1}) - \ell (z_{i+1} - z_{i-1})}{(z_i - z_{i+1}) (z_i - z_{i-1})}$$

where $\ell$ is a free parameter. Then there exist a family of functions of the punctures parametrized by $\ell$. In order to choose a value for that parameter, we ask for our low-energy amplitude to reproduce a gauge independent result such as the one relative to the three-photon scattering amplitude which is identically zero. For extracting this amplitude we sum the expression (12) over all the anticyclic permutations of the indices $(1, 2, 3)$ getting:

$$A_0^3(\text{photons}) = 4g_d \varepsilon^{(1)}_\lambda \varepsilon^{(2)}_\mu \varepsilon^{(3)}_\nu (2\ell + 1)\left\{ \eta^\nu_\lambda (p'_1 + p'_2) + \eta^\mu_\nu (p'_3 + p'_2) + \eta^\nu_\lambda (p'_1 + p'_3) \right\} = 0$$

which is satisfied only for

$$\ell = -\frac{1}{2}.$$ 

With this value of the parameter $\ell$ the eq. (13) becomes

$$\frac{V''(0)}{(V'(0))^2} = \frac{1}{(z_i - z_{i+1})} + \frac{1}{(z_i - z_{i-1})}.$$ 

It is easy to check that the Lovelace function does not satisfy this latter equation. By substituting (16) into (12) we get for the low-energy limit of the amplitude in exam the following expression:

$$A_0^3(\alpha' \to 0) = 2g_d \varepsilon^{(1)}_\lambda \varepsilon^{(2)}_\mu \varepsilon^{(3)}_\nu \left\{ \eta^\nu_\lambda (p'_1 - p'_2) + \eta^\mu_\nu (p'_3 - p'_2) + \eta^\nu_\lambda (p'_1 - p'_3) \right\}.$$ 

The choice (16) makes the entire amplitude (11) to become:

$$A_0^3 = g_d (z_A - z_B) (z_A - z_C) (z_B - z_C) \left( \sum_{i=1}^{3} V'_i(0)^{2\alpha' p_i \mu} V''_i(0)^{2\alpha' p_i \nu} \right) \left\{ (2\alpha') \left[ \frac{1}{2} (p''_2 - p''_3) (p''_3 - p''_1) (p'_1 - p'_2) \right] + 2 \left[ \eta^\nu_\lambda (p'_1 - p'_2) + \eta^\mu_\nu (p'_3 - p'_1) + \eta^\mu_\nu (p'_2 - p'_3) \right] \right\}.$$
If this latter amplitude has to be projective invariant, then $V_i(z)$ must satisfy the following equation:

$$(z_A - z_B)^2\alpha' p_1 p_2 (z_A - z_C)^2\alpha' p_1 p_3 (z_B - z_C)^2\alpha' p_2 p_3 \prod_{i=1}^{3} [V_i'(0)]^{\alpha_i p_i^2} = \text{const.} \quad (19)$$

From (16) we know that $V_i'(0)$ must be a function of all the punctures, then we can conjecture it be written as follows:

$$V_i'(0) = (z_i - z_{i+1})^a (z_i - z_{i-1})^b (z_{i+1} - z_{i-1})^c \quad (20)$$

where $a, b, c$, are unknown exponents to be determined by resolving the eq. (19). Indeed, by inserting (20) into (19) we get the following system equations for $a, b, c$:

$$\begin{cases}
p_1^2 a + p_2^2 b + p_3^2 c = -2p_1 \cdot p_2 \\
p_1^2 a + p_2^2 b + p_3^2 c = -2p_1 \cdot p_3 \\
p_2^2 a + p_2^2 b + p_3^2 c = -2p_3 \cdot p_2.
\end{cases} \quad (21)$$

This system admits a unique solution that, using the momentum conservation, is:

$$\begin{cases}
a = 1 \\
b = 1 \\
c = -1.
\end{cases} \quad (22)$$

Plugging the solution obtained into the expression (20) yields:

$$V_i'(0) = \frac{(z_i - z_{i+1})(z_i - z_{i-1})}{(z_{i+1} - z_{i-1})}. \quad (23)$$

It implies that all the tree Green functions that are present in the three-string case, given by eq. (9), vanish. The eq. (23) corresponds to the first derivative of the Lovelace function $V_i(z)$ evaluated at $z = 0$ [15].

By requiring, then, the projective invariance for the low-energy limit of the amplitude (12) and asking for this amplitude to reproduce the three-photon scattering amplitude, we could fix the ratio $V_i''(0)/(V_i'(0))^2$; at the same time, by requiring the projective invariance for the entire scattering amplitude (12) we get a specific prescription for the function.
$V^i_0(0)$. If the local holomorphic coordinate system $V_i(z)$ is chosen to be a projective transformation satisfying by construction the condition $V_i(0) = z_i$, we completely fix the function $V_i(z)$ by fixing $V^i_0(0)$ and $V^0_0(0) = (V^i_0(0))^2$.

Introducing the choices (23) and (16) in (18) gives the following projective invariant three-open string off-shell amplitude:

$$A^3_0 = g_d \varepsilon^{(1)}_{\lambda} \varepsilon^{(2)}_{\mu} \varepsilon^{(3)}_{\nu} \left\{ (2\alpha') \left[ \frac{1}{2} (p^\lambda_3 - p^\lambda_2) (p^\mu_1 - p^\mu_3) (p^\nu_2 - p^\nu_1) \right] + 2 \left[ \eta^{\lambda\mu} (p^\nu_2 - p^\nu_1) + \eta^{\lambda\nu} (p^\mu_1 - p^\mu_3) + \eta^{\mu\nu} (p^\lambda_3 - p^\lambda_2) \right] \right\}. \quad (24)$$

Let us now come back to the analysis of the field theory limit of the open string amplitude. We are interested in the evaluation of the three-gluon scattering amplitude. As it is well-known, at tree level we need to sum the expression in (17) over all the anticyclic permutations of the indices $(1, 2, 3)$ after having multiplied it by the Chan-Paton factor

$$\text{Tr} (\lambda_{a_1} \lambda_{a_2} \lambda_{a_3}). \quad (25)$$

In this object $\lambda$'s are the generators of the $SU(N)$ gauge group in the fundamental representation. Through the normalization conditions

$$\text{Tr} (\lambda_a \lambda_b) = \frac{1}{2} \delta_{ab} \quad (26)$$

the Chan-Paton factor can be rewritten as:

$$\text{Tr} (\lambda_{a_1} \lambda_{a_2} \lambda_{a_3}) = \frac{1}{4} (f^{a_1 a_2 a_3} + d^{a_1 a_2 a_3}), \quad (27)$$

where $f$ is an antisymmetric tensor of the internal indices $a_i$ and $d$ a symmetric one. An explicit evaluation of the three-gluon amplitude yields:

$$A^3_0(\text{gluons}) = g_d \varepsilon^{(1)}_{\lambda} \varepsilon^{(2)}_{\mu} \varepsilon^{(3)}_{\nu} f^{abc} \left\{ \eta^{\lambda\mu} (p^\nu_2 - p^\nu_1) + \eta^{\lambda\nu} (p^\mu_1 - p^\mu_3) + \eta^{\mu\nu} (p^\lambda_3 - p^\lambda_2) \right\}. \quad (28)$$

This expression perfectly coincides with the one of the three-gluon scattering amplitude obtained through a covariant quantization of a non abelian gauge theory with gauge group $SU(N)$ or by using the background field method.
In computing a string amplitude in our formalism, the choice of the function $V_i(z)$ seems to be strictly connected with the gauge choice in the field theory limit of those amplitudes. Since the request of consistency, like that we have here made, together with the conjecture (20), completely fixes $V_i(z)$, it turns out that the only way in which the open string can reproduce the field theory gauge-dependent result of the three-gluon scattering amplitude at tree-level is in the usual covariant gauge [2].

Let us analyze the closed string case, by extending to it the result in (17). We find that the scattering amplitude involving three external off-shell massless states admits the following low-energy limit:

$$A_{0c}^3 (\alpha' \to 0) = k \varepsilon_\mu^{(1)} \varepsilon_\nu^{(2)} \varepsilon_\tau^{(3)} \left\{ \left[ \eta^{\lambda \mu} (p_1^\nu - p_2^\nu) + \eta^{\lambda \nu} (p_1^\mu - p_2^\mu) + \eta^{\lambda \tau} (p_1^\sigma - p_2^\sigma) \right] \times \left[ \eta^{\rho \sigma} (p_1^1 - p_2^1) + \eta^{\rho \tau} (p_1^2 - p_2^2) + \eta^{\tau \tau} (p_1^3 - p_2^3) \right] \right\}. \quad (29)$$

After symmetrizing this expression to eliminate the contribution of the antisymmetric tensor, we get the scattering amplitude of three gravitons mixed with dilatons:

$$A_{0c}^{3 \text{gravitons}}_{\text{dilatons}} = k \varepsilon_\mu^{(1)} \varepsilon_\nu^{(2)} \varepsilon_\tau^{(3)} \left\{ I^{\lambda \rho \mu \sigma} (p_2^\nu - p_1^\nu) (p_2^\tau - p_1^\tau) + I^{\lambda \rho \nu \tau} (p_1^\mu - p_3^\mu) (p_1^\sigma - p_3^\sigma) \right.$$ 
$$+ I^{\lambda \mu \sigma \tau} (p_3^3 - p_2^3) (p_3^\rho - p_2^\rho) + \frac{1}{2} \left[ I^{\lambda \rho \nu \tau} (p_2^\nu - p_1^\nu) (p_1^\sigma - p_3^\sigma) \right.$$ 
$$+ I^{\lambda \sigma \nu \tau} (p_2^\nu - p_1^\nu) (p_2^\mu - p_1^\mu) + I^{\lambda \mu \nu \sigma} (p_1^\rho - p_3^\rho) (p_2^\tau - p_1^\tau) + I^{\lambda \rho \mu \sigma} (p_1^\nu - p_3^\nu) (p_2^\tau - p_1^\tau)$$ 
$$+ I^{\lambda \tau \mu \sigma} (p_2^\nu - p_1^\nu) (p_3^\rho - p_2^\rho) + I^{\rho \mu \nu \sigma} (p_2^\nu - p_1^\nu) (p_3^\rho - p_2^\rho) + I^{\rho \lambda \mu \sigma} (p_1^\nu - p_3^\nu) (p_3^\rho - p_2^\rho)$$ 
$$+ I^{\rho \lambda \nu \sigma} (p_3^\rho - p_2^\rho) (p_3^\nu - p_2^\nu) + I^{\rho \sigma \nu \tau} (p_1^\mu - p_3^\mu) (p_3^\nu - p_2^\nu) + I^{\rho \mu \nu \tau} (p_1^\sigma - p_3^\sigma) (p_3^\nu - p_2^\nu)$$ 
$$+ I^{\rho \sigma \nu \tau} (p_1^\mu - p_3^\mu) (p_1^\nu - p_3^\nu) + I^{\mu \nu \rho \sigma} (p_1^\rho - p_3^\rho) (p_1^\sigma - p_3^\sigma) \left\} \right\} \quad (30)$$

with

$$I_{\alpha \beta \gamma \delta} = \frac{1}{2} \left( \eta_{\alpha \gamma} \eta_{\beta \delta} + \eta_{\alpha \delta} \eta_{\beta \gamma} \right). \quad (31)$$

Although the De Donder gauge is commonly believed to be the preferred gauge choice in quantum gravity [19], there are some arguments [20] which suggest that it is not the one chosen by the bosonic closed string.
We have investigated this point, following ref. [3], by computing the expansion of the Einstein-Hilbert Lagrangian coupled to a dilaton field with the De Donder gauge fixing term, up to the third order in the fields. On the complete Lagrangian we have performed the following transformations [3]:

\begin{align}
    h_{\mu\nu} &= \tilde{h}_{\mu\nu} + \frac{\eta_{\mu\nu}}{\sqrt{D-2}} \tilde{\phi} \\
    \phi &= \sqrt{\frac{D-2}{2}} \tilde{\phi} + \frac{1}{\sqrt{2}} \tilde{h}_\mu \tilde{h}^\mu.
\end{align}

These latter have been determined by requiring that the propagator obtained from the transformed Lagrangian were proportional to the unit tensor given in (31), that is the expected tensor structure for the string propagator [3]. Then the computation of the three-point function from the Lagrangian so obtained has not reproduced the result shown in (30). This means that, within this comprehension of decoupling gravitons and dilatons, and in agreement with the arguments of ref. [20], the closed string does not choose, in the low-energy limit, the De Donder gauge, which, naively, might be considered as the most natural counterpart of the usual covariant gauge in the open string case.

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