Towards the Standard Model spectrum from elliptic Calabi-Yau

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Abstract

We show that it is possible to construct supersymmetric three-generation models of Standard Model gauge group in the framework of non-simply-connected elliptically fibered Calabi-Yau, without section but with a bi-section. The fibrations on a cover Calabi-Yau, where the model has 6 generations of SU(5) and the bundle is given via the spectral cover description, use a different description of the elliptic fibre which leads to more than one global section. We present two examples of a possible cover Calabi-Yau with a free involution: one is a fibre product of rational elliptic surfaces dP9; another example is an elliptic fibration over a Hirzebruch surface. There we give the necessary amount of chiral matter by turning on in the bundles a further parameter, related to singularities of the fibration and the branching of the spectral cover.

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1 Introduction

Efforts to get a (supersymmetric) phenomenological spectrum From the $E_8 \times E_8$ heterotic string on a Calabi-Yau $Z$ started with embedding the spin connection in the gauge connection which gave an unbroken $E_6$ (times a hidden $E_8$ which couples only gravitationally). This procedure was combined in a second step with a further breaking of the gauge group by turning on Wilson lines using a non-trivial $\pi_1(Z)$. The simplest constructions of Calabi-Yau spaces, as hypersurfaces in toric varieties will always have a trivial fundamental group. However one can still produce $\pi = G$ by dividing $Z$ by a freely acting group $G$, provided such an operation on $Z$ exists. This then led at the same time to a reduction of the often high number of generations (being equal to $\chi(Z)/2$) by the order of $G$.

This approach was generalised [1] to the case of embedding instead of the tangent bundle an $SU(n)$ bundle for $n = 4$ or 5, leading to unbroken $SO(10)$ resp. $SU(5)$ of even greater phenomenological interest then $E_6$. This subject was revived when, as a consequence of the investigation of heterotic/$F$-theory duality, the bundle construction was made much more explicit for the case of elliptically fibered Calabi-Yau $\pi : Z \rightarrow B$ [2]. This extended ansatz showed among other things to have a much greater flexibility in providing one with three-generation models (of the corresponding unbroken GUT group) [3].

There remains to go to the Standard Model gauge group in this framework\(^4\). For the following bases one can show that the elliptic fibrations, we will discuss, are smooth: the Hirzebruch surfaces $F_m$, $m = 0, 1, 2$, the del Pezzo surfaces $dP_k$ with $k = 0, \ldots, 6$, the rational elliptic surface $dP_9$ and 10 additional examples from the list of classified [4] toric varieties, which correspond to two dimensional reflexive polyhedra. However none\(^5\) of them has non-trivial $\pi_1(Z)$.

The purpose of this note is to show that the elliptic framework is nevertheless capable of providing one with a three generation model of Standard Model gauge group by giving just a typical example how to proceed. Namely\(^6\) we will look for an elliptically fibered Calabi-Yau $Z$ which has besides the usually assumed section of its elliptic fibration a

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\(^{4}\)Three generation models with Standard model gauge group were constructed in orbifold and free fermion models before but these are, in contrast to our models, defined only at isolated points in moduli space.

\(^{5}\)The Enriques surface as base $B$ has $\pi_1(B) = \mathbb{Z}_2$ leading to a non-trivial $\pi_1(Z)$ but, as pointed out by E. Witten, does not lead to a three generation model (cf. [17]).

\(^{6}\)following a suggestion of E. Witten
second section (a construction which we will call $B$ model in contrast to the usual form, here called $A$ model); this will lead to a free involution on $Z$ which after modding it out provides you with a *smooth* Calabi-Yau where one now has the possibility of turning on Wilson lines which break $SU(5)$ to the Standard Model gauge group. Thereby we will have achieved our goal when we specify on $Z$ an $SU(5)$ bundle that leads to 6 generations and fulfills some further requirements of the spectral cover construction. A variant of this procedure would be to consider a fibration with two $\mathbb{Z}_2$ operations ($D$ model) and to break $SO(10)$ to the Standard model gauge group times $U(1)_{B-L}$; we will comment on this possibility\(^7\) too.

In *section 2* the spaces along with their cohomological data are introduced. In *section 3* the spectral cover construction of bundles is recalled and the necessary adjustments for our base Calabi-Yau as well as some consistency conditions are spelled out; finally, there is given an example of a 6 generation model (‘above’) for $B = F_1$. In *section 4* the question of modding these spaces/bundles is treated; for $F_0$, $F_2$ and a type of $dP_9$ (with a $B$-type elliptic fibre) we succeed in writing down a free involution. In *section 5* we point to a method to get a new class of bundles, which though is not capable of producing chiral matter. Finally in *section 6* we turn on a further possible parameter (an option already existent in the $A$ model), related to singularities of the fibration and the branching of the spectral cover, to generate chiral matter; here we give for $B = F_0$ or $F_2$ for the first time moddable 6 generation models.

\section{The spaces}

\subsection{Change of fibre type}

As it will be our goal to mod not just the Calabi-Yau spaces but also the geometric data describing the bundle (and this transformation of bundle data into geometric data uses in an essential way the elliptic fibration structure) we will search only for operations which preserve the fibration structure, i.e. $\mathbf{L} \cdot \pi = \pi \cdot \mathbf{L}$ where the $\mathbf{L}$ is an action on the base. This has the consequence that our fibrations will have $n$ sections for $n = |G|$.

To have this extra structure, which allows for free $\mathbb{Z}_n$ actions on the elliptically fibered Calabi-Yau we will use a different elliptic curve than the usually taken $\mathbb{P}_{1,2,3}(6)$, which we refer to as the $A$-fibre. We will have to extend the construction of [2] to the new fibre

\[\text{\footnotesize [5] which we do not get in our approach.}\]
types, labeled by $B, C, D$

<table>
<thead>
<tr>
<th>$(y, x, z) \in E$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $P_{3,2,1}(6)$ $y^2 + x^3 + z^6 + sxz^4 = 0$</td>
</tr>
<tr>
<td>B $P_{2,1,1}(4)$ $y^2 + x^4 + z^4 + sx^2z^2 = 0$</td>
</tr>
<tr>
<td>C $P_{1,1,1}(3)$ $x^3 + y^3 + z^3 + sxyz = 0$</td>
</tr>
<tr>
<td>D $P^3(2, 2)$ $x^2 + y^2 + szw = 0$, $z^2 + w^2 + sxy = 0$</td>
</tr>
</tbody>
</table>

**Tab. 1 The $A, B, C, D$ fibre types**

These different possibilities were considered before, mainly in the context of $F$-theory [6-9]. In particular [10] contains an in depth study of these fibrations over a one-dimensional base (i.e. $P^1$), which becomes useful here.

We will be interested in a global version of these descriptions over a complex surface $B$ so that our Calabi-Yau $Z$ can be described by a generalized Weierstrass equation in a $P^2$ bundle $W$ over $B$ (note that in the case of the $B$-model the fibre $P^2$ is actually a weighted $P_{2,1,1}$ with $y$ being a section of $O(2)$). In the following table we indicate the power $i$ of the line bundle $\mathcal{L} = K^{-1}_B$ over $B$ so that the given variables $x, y, z$ resp. coefficient functions $a, b, c, d, e$ are sections of $\mathcal{L}^i$.

<table>
<thead>
<tr>
<th>$Z$</th>
<th>$z$</th>
<th>$x$</th>
<th>$y$</th>
<th>$w$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
<th>$s$</th>
<th>$\text{def}(K3)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $zy^2 + x^3 + axz^2 + bz^3 = 0$</td>
<td>0</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>1</td>
<td>18</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>B $y^2 + x^4 + ax^2z^2 + bxz^3 + cz^4 = 0$</td>
<td>0</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>2</td>
<td>17</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C $x^3 + y^3 + axyz + bxz^2 + cyz^2 + dz^3 = 0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>3</td>
<td>3</td>
<td>16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D $x^2 + y^2 + azw + dz^2 = 0$, $w^2 + xy + bxz + cyz + ez^2 = 0$</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
<td>4</td>
<td>15</td>
<td></td>
</tr>
</tbody>
</table>

**Tab. 2 Generic fibrations with $A, B, C, D$ fibre types**

Here we indicated also the number of cohomological inequivalent sections $s$. It is obtained by counting the solutions of the equation at the locus of the section $z = 0$, taking into account the equivalence relations in the (weighted) projective spaces. E.g. for the $B$-fiber we get at $z = 0$ the equation $y^2 = x^4$, which has naively 8 solutions which lie however only in two equivalence classes in $P(2, 1, 1)$: $[y, x] = [1, 1]$ and $[-1, 1]$ etc.. Note that always one of the $s$ sections is the zero section, while the others have rank 1 in the Mordell-Weil group, i.e. they generate infinitely many sections. For special points in the moduli space we can bring them to torsion points of finite order $s$, see below.

Furthermore we indicated the number of complex structure deformations $d$ if we consider an elliptically fibered $K3$, with $P^1$ base. Then $a, b, c, \ldots$ are polynomials of order $2i$ in the $P^1$ variables. The number of the parameters in the polynomials minus the 3 parameters of the $SL(2, C)$ reparametrisations of the $P^1$, minus one for the overall scaling.
of all variables gives the number of independent complex structure deformations. E.g. for the $B$-fibre $K_3$ we have $d = 5 + 7 + 9 - 3 - 1 = 17$ etc. The Picard-group of the $K_3$ is generated by the $s$-sections plus the base, i.e. $\rho = s + 1$ and in particular $\rho + d = 20$.

The construction of free $\mathbb{Z}_n$ actions on $Z$ will be closely connected with shift symmetries, which are free at least on the generic fibre. The existence of the shift symmetry implies that the $n$ sections are at order $n$ points ($\frac{1}{n}$-periods), which are fixed under the monodromy; this in turn means specialization of the complex parameters$^8$. The following table shows for each of the models two consecutive specializations and their ensuing monodromy enhancements [10].

<table>
<thead>
<tr>
<th>specialization</th>
<th>$B$</th>
<th>$C$</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = 0$</td>
<td>$c = h^4$</td>
<td>$b = c = 0$</td>
<td>$d = h^3$</td>
</tr>
<tr>
<td>symmetry</td>
<td>$(y, x, z)$</td>
<td>$(y, x, z)$</td>
<td>$(y, x, z)$</td>
</tr>
<tr>
<td>$\downarrow$ &amp; $\downarrow$ &amp; $\downarrow$ &amp; $\downarrow$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$(-y, -x, z)$</td>
<td>$(y, zh, xh^{-1})$</td>
<td>$(y, x, z)$</td>
<td>$(x, zh, yh^{-1})$</td>
</tr>
<tr>
<td>Mordell-Weil</td>
<td>$Z_2$</td>
<td>$Z_2 \times Z_2$</td>
<td>$Z_3$</td>
</tr>
<tr>
<td>Monodromy</td>
<td>$\Gamma_1(2)$</td>
<td>$\Gamma(2)$</td>
<td>$\Gamma_1(3)$</td>
</tr>
<tr>
<td>def($K3$)</td>
<td>10</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>sing.</td>
<td>$(A_1)^8$</td>
<td>$(A_1)^{12}$</td>
<td>$(A_2)^6$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>specialization</th>
<th>$D$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$b = c = d = 0$</td>
<td>$&amp; a = h^2, e = g^4$</td>
</tr>
<tr>
<td>symmetry</td>
<td>$(y, x, z, w)$</td>
</tr>
<tr>
<td>$\downarrow$ &amp; $\downarrow$</td>
<td></td>
</tr>
<tr>
<td>$(\mu_1^{-1}y, \mu_4x, z, \mu_2^2w)$</td>
<td>$(g^2hz, hw, g^{-1}x, gy)$</td>
</tr>
<tr>
<td>Mordell-Weil</td>
<td>$Z_4$</td>
</tr>
<tr>
<td>Monodromy</td>
<td>$\Gamma_1(4)$</td>
</tr>
<tr>
<td>def($K3$)</td>
<td>4</td>
</tr>
<tr>
<td>sing</td>
<td>$(A_1)^2 \times (A_3)^4$</td>
</tr>
</tbody>
</table>

Tab. 3 Properties of specializations of the $B, C, D$-type fibres

The possible degenerations of the associated Weierstrass forms employing the Kodaira classification tables of singular fibres are also analysed [10]. I.e. the group is $\Gamma_1(k)$ ($\Gamma(k)$) for $k = 9 - j$ ‘the number’ of the model related to $E_j$ (i.e. 1, 2, 3, 4 for the $A, B, C, D$ model; cf. the Euler number later). The relation to the $E_j$ classification becomes apparent, if we blow up (down) a $(-1)$ curve in the base. In this case we create (or shrink) an $E_j$ $j = 8, \ldots 5$ del Pezzo surface [13, 8], which can be geometrically described by $P_{3,2,1,1}(6)$,

$^8$The spaces then turn out to be singular; so this is only a rudimentary version of what we are actually going to do.
$P_{2,1,1,1}(4)$, $P^3(3)$ and $P^1(2,2)$. The local singularities which have the same structure, were analysed in [14]. We assume that elliptic fibrations related to $E_j$ for $j < 5$ with $9 - j$ sections exist, but it is known that the associated singularities mentioned above can not be described as complete intersections. The further specializations in the table, as well as forms of the A-fibre, which have additional sections, are singular and it is not known whether an extremal transition leads to new fibre types.

2.2 Our procedure

Our main goal will be to give a free $\mathbb{Z}_n$ action on $Z$. One does not find free operations on the base $B$. Therefore the operation over the fix-locus in the base must be free in the fibre. But this means that it is a shift by an $n$-torsion point. To force the existence of this shift we have to use actually a fibration where such a shift exists globally (even if only on a sublocus of moduli space). So clearly the first idea in choosing the $A, B, C, D$ models is to use the fibre shift and to enhance it to an operation in the total Calabi-Yau by combining it with an operation in the base. This will be necessary for two reasons: first at the singular fibers the pure fibre operation ceases to be free, and secondly to use just the fibre operation would us restrict to the first specialisation locus given in the table which is singular\(^9\). For example in the $B$ model the involution $(y, x) \rightarrow (-y, -x)$ in the coordinates (which for $b = 0$ is identical to the shift provided by the second section, which is then an involution) does not necessarily restrict us to the $b = 0$ locus when combined with a base operation. In other words the involution $(y, x) \rightarrow (-y, -x)$ does not exist on the fibre per se, but can exist, when combined with a base involution, on a subspace (not as restrictive as the locus $b = 0$; actually only some monomials within the coefficient functions $a, b, c$ are then forbidden) of the moduli space where the generic member is still smooth. Note that even away from $b = 0$ the coordinate involution maps still the two sections on another.

In general the full action, being given by these actions on the fibre accompanied by suitable group actions on the basis, have, besides the requirement of not restricting us to a non-smooth model 'above' by the condition of their definedness, also then to fulfill the further conditions to provide free group actions, which leave the holomorphic three-form invariant, as will be discussed further in section 4.

\(^9\)The unspecialised, generic form of the $B, C, D$ fibrations are smooth for the bases $F_m$ for $m = 0, 1, 2$ and $dP_k$ for $k = 0, 1, 2, 9$ and then ten additional toric polyhedra described in [4].
2.3 Transitions

Finally we discuss the relation between the $A, B, C, D$ smooth Weierstrass Calabi-Yau three folds. If we force a second section for the $A$-type Weierstrass fibration $Z$ it takes the general form $y^2 = 4(x + f)(x^2 - fx - g)$, where $f, g$ are sections of $\mathcal{L}^2, \mathcal{L}^4$. From

$$g_2 = 16 \cdot 2^{2/3}(f^2 + g), \quad g_3 = 64fg, \quad \Delta = (g - 2f^2)^2(f^2 + 4g)$$

we see using Kodaira's classifications of singular fibers that it acquires an $A_1$ fiber over a divisor $B'$ in the class $[B_{A_1}'] = -4K_B = 4c_1$ and the calculational framework [2] needs modifications. The smooth $A$-fibre Weierstrass model had Euler number $f_Zc_3(Z) = -60f_Bc_1^2(B)$ and if we resolve the singularities over the divisor $B'$ we get an extremal transition\(^\text{10}\) From $Z$ to a model $\hat{Z}$, whose change in the Euler number \([9]\) is given by $\delta = \text{cox}(G)\text{rank}(G)f_{B'}c_1(B')$. By the adjunction formula we calculate $\delta = 2f_{B'}c_1(B') = 24f_Bc_1^2(B), \text{ which yields } f_Zc_3(\hat{Z}) = -36f_Bc_1^2(B)$. This model is up to birational transformations the elliptic fibration with $B$-fibre type over the same base, whose Euler number depends on the base cohomology in precisely this way \([9]\). Similarly forcing three sections produces an $A_2$ singularity over $[B'] = -3K_B [10]$, and the extremal transition leads with $\delta = 6f_{B'}c_1(B') = 36f_Bc_1^2(B)$ to $f_Zc_3(\hat{Z}) = -24f_Bc_1^2(B)$, the $C$-fibre type. Finally forcing four sections leads by a slightly more complicated transition to the $D$-fibration.

2.4 The cohomology of the spaces

The different fibration structures lead in the following way to the cohomological data of $Z$ (we give, as an example, a consideration of the $B$-model, cf. the $A$-model [2]; unspecified Chern classes will always refer to the base $B$).

As noted $z, x, y$ can be thought of as homogeneous coordinates on $W$ respectively globally as sections of line bundles $\mathcal{O}(1), \mathcal{O}(1) \otimes \mathcal{L}$ and $\mathcal{O}(1)^2 \otimes \mathcal{L}^2$ whose first Chern classes are given by $r, r + c_1, 2r + 2c_1$ with $c_1(\mathcal{O}(1)) = r$. The cohomology ring of $W$ is generated by $r$ with the relation $r(r+c_1)(2r+2c_1) = 0$ expressing the fact that $z, x, y$ have no common zeros. Since the $B$-model is defined by the vanishing of a section of $\mathcal{O}(1)^4 \otimes \mathcal{L}^4$, which is a line bundle over $W$ with first Chern class $4(r + c_1)$, the restriction from $W$ to $Z$ is effected by multiplying by this Chern class, so that $c(W) = (1 + 4r + 4c_1)c(Z)$. One can then simplify $r(r + c_1)(2r + 2c_1) = 0$ to $r(r + c_1) = 0$ in the cohomology ring of $Z$.

\(^{10}\)Take type IIA on the same Calabi-Yau space, then we get physical interpretation of the transition. Unhiggsing of a $SU(2)$ with $g(B')$ hypermultiplets in the adjoint $[11,12]$.\)
and finds for the total Chern class of $Z$ (B-model)

$$c(Z) = c(B) \frac{(1 + r)(1 + r + c_1)(1 + 2r + 2c_1)}{1 + 4r + 4c_1}$$

(2.1)

Using $r^2 = -rc_1$ and taking into account that the divisor $(z = 0)$ of the section $z$ of the line bundle $O(1)$ of class $r$ shows that $r = \sigma_1 + \sigma_2$ (as $z = 0$ implies $y^2 = x^4$, leading to $(x, y) = (i, 1)$ and $(i, -1)$) we find\(^\text{11}\)

$$c_2(Z) = c_2 + 6(\sigma_1 + \sigma_2)c_1 + 5c_1^2$$

$$c_3(Z) = -36c_1^2$$

(2.2)

The computations for the $C$ and $D$ model can be done the same way and one finds for the cohomological data of $Z$ (if one defines the sum of the sections $\Sigma = \sum_{i=1}^{k} \sigma_i$, so that one has $r = \Sigma$ for the $B, C, D$ models in contrast to the $r = 3\sigma$ of the $A$ model (note that the 'simplification relation' is $r(r + c_1) = 0$ in the $B, C, D$ models in contrast to the $r(r + 3c_1) = 0$ in the $A$ model, the only case where a section occurs with a multiplicity (namely 3) in $r$); so in all cases the adjunction relations $\sigma_i^2 = -\sigma_i c_1$ are satisfied, as they should)

<table>
<thead>
<tr>
<th>$G$</th>
<th>$c_3(Z)$</th>
<th>$c_2(Z)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A $E_8$</td>
<td>$-60c_1^2$</td>
<td>$c_2 + 12\sigma c_1 + 11c_1^2$</td>
</tr>
<tr>
<td>B $E_7$</td>
<td>$-36c_1^2$</td>
<td>$c_2 + 6\Sigma c_1 + 5c_1^2$</td>
</tr>
<tr>
<td>C $E_6$</td>
<td>$-24c_1^2$</td>
<td>$c_2 + 4\Sigma c_1 + 3c_1^2$</td>
</tr>
<tr>
<td>D $D_5$</td>
<td>$-16c_1^2$</td>
<td>$c_2 + 3\Sigma c_1 + 2c_1^2$</td>
</tr>
</tbody>
</table>

**Tab. 4** Chern classes of the $A, B, C, D$ models

What concerns the volume form, e.g. for the $B$-fibre we have $-18 \int_Z r c_1^2 = -18 \int_Z c_1^2 \Sigma = -36 \int_B c_1^2$.

So one sees that in general (with $k$ the 'number' of the model (cf. the monodromy groups above) and $h$ is the Coxeter number of $G$)

$$c_2(Z) = c_2 + \frac{12}{k} \Sigma c_1 + (\frac{12}{k} - 1)c_1^2$$

$$c_3(Z) = -2hc_1^2$$

(2.3)

\(^{11}\)for the Euler number we give the cohomology class which has to be integrated over the base
2.5 Examples

2.5.1 Over Hirzebruch bases

Let us consider as an example the $B$-type fibration over $B = F_0$. As one reads off from the table of weight given above, $a, b, c$ are sections of line bundles corresponding to polynomials of bi-degree $(4,4)$, $(6,6)$ and $(8,8)$ over $F_0 = \mathbb{P}^1 \times \mathbb{P}^1$, which yields $h_{2,1} = 5^2 + 7^2 + 9^2 - 3 - 3 - 1 = 148$, where the subtraction comes from the two SL$(2, \mathbb{C})$ automorphism on the two $\mathbb{P}^1$ and the rescaling. Of the $(1,1)$-forms two come from base and two from the two sections. Over $F_2$ our Calabi-Yau is the $\mathbb{P}_{1,1,2,4,8}(16)$ just as in the $A$ fibre case one had the $\mathbb{P}_{1,1,2,8,12}(24)$.

2.5.2 Over rational elliptic bases

Let us consider as an example, which becomes important later in section (4.2), the case $B = dP_9$. Let us first recall that we denote by $dP_9$ the rational elliptic surface being given by the projective plane $\mathbb{P}^2_{x,y,z}$ blown up in the nine intersection points of two cubics. The later will be denoted by $C, C'$. This representation $sC(x, y, z) + tC'(x, y, z) = 0$ (with $s, t$ coordinates on $\mathbb{P}^1$) of $dP_9 = [\mathbb{P}^2_{x,y,z}]$ shows at the same time the elliptic fibration, with $C$-type elliptic fibre. This may also be called $\frac{1}{2}K3$ for the $K3 = [\begin{array}{c} p^2 \\ p^1 \end{array}, \begin{array}{c} 3 \\ 1 \end{array}]$: it has $h^{2,0} = 0$ and $h^{1,1} = 10$ and 8 complex deformations. These can be counted either as 8 points (the ninth is determined as they have to add up to zero in the group law) on $\mathbb{P}^2$, leading to $8 \times 2 - 8 = 8$ parameters, where the each point contributes 2 degrees of freedom and the rescaling comes from PGL$(3, \mathbb{C})$. Alternatively one can count the monomials as $10 \cdot 2 - 8 - 3 - 1 = 8$. From this surface one can build the Calabi-Yau space $[\begin{array}{c} p^2 \\ p^1 \end{array}, \begin{array}{c} 3 \\ 1 \end{array}]$. Note that our earlier constructed Calabi-Yau spaces have such a fibre product structure in case the base is a $dP_9$: the fact that the coefficient functions $f$ and $g$ are sections of a line bundle $\mathcal{L}$ of class $c_1(dP_9) = f$ (here $f$ denotes the elliptic fibre of $dP_9$) means that they are linear in the base $\mathbb{P}^1$ and independent of the fibre direction (so having polar divisor $f$). Note further that these Calabi-Yau spaces $Z$ when restricted to the base $\mathbb{P}^1$ (of the horizontal $dP_9$) have again the structure of a (vertical) $dP_9$, with correspondingly changed elliptic fibre type. This is the well known representation $dP_9 = \frac{1}{2}K3$: if $B$ would be one-dimensional one would get a $K3$ for $c_1(\mathcal{L}) = c_1(\mathbb{P}^1) = 2$ which would double the degrees of the coefficient functions. So changing From $K3$ to $\frac{1}{2}K3$ means in the $B$ model to consider $y^2 + x^4 + a_2(s, t)x^2z^2 + b_3(s, t)xz^3 + c_4(s, t)z^4 = 0$.

Note that although the $K3$ in the $A, B, C$ model have because of the added sections a decreasing number of deformations, one has nevertheless by going to the rational elliptic
surfaces, i.e. the corresponding $\frac{1}{2}K3$, always the 8 deformations as one easily convinces oneself in dividing the mentioned degrees by two (for example $3 + 4 + 5 - 3 - 1 = 8$ in the $B$ model); i.e. one always has for them the same characteristics $h^{1,1} = 10 = \rho$ and 8 complex deformations, the difference between the $A, B, C$ models and further the mentioned $C'$ model $\left[ \begin{array}{c} 3 \\ 1 \\ 1 \end{array} \right]$, whose ambient space is in contrast to our $C$ model a constant $\mathbb{P}^2$-fibration over the base, being given by the type of the elliptic fibre in which different group actions are manifest.

Note that the Euler number vanishes for a $dP_9$ base in our examples anyway (because of $c_1^2 = f^2 = 0$) as it should for double elliptic fibrations (the two sets of 12 points in the base $P^1$ where both the elliptic fibres of the two $dP_9$ fibre factors degenerate are generically disjoint).

\section{The bundles}

\subsection{The spectral cover description}

Let us recall the idea of the spectral cover description of an $SU(n)$ bundle $V$: one considers the bundle first over an elliptic fibre and then pastes together these descriptions using global data in the base $B$. Now over an elliptic fibre $E$ an $SU(n)$ bundle $V$ over $Z$ (assumed to be fibrewise semistable) decomposes as a direct sum of line bundles of degree zero; this is described as a set of $n$ points which sums to zero. If you now let this vary over the base $B$ this will give you a hypersurface $C \subset Z$ which is a ramified $n$-fold cover of $B$. If one denotes the cohomology class in $Z$ of the base surface $B$ (embedded by the zero-section $\sigma_1$) by $\sigma \in H^2(Z)$ one finds that the globalization datum suitable to encode the information about $V$ is given by a class $\eta \in H^{1,1}(B)$ with\footnote{\label{footnote} $C$ is given as a locus $w = 0$ with $w$ a section of $O(\sigma)^n \otimes \mathcal{M}$ with a line bundle $\mathcal{M}$ of class $\eta$.} \begin{equation}
C = n\sigma + \eta \tag{3.4}
\end{equation}

The idea is then to trade in the $SU(n)$ bundle $V$ over $Z$, which is in a sense essentially a datum over $B$, for a line bundle $L$ over the $n$-fold (ramified) cover $C$ of $B$: one has

\begin{equation}
V = p_\ast(p_\ast^c L \otimes \mathcal{P}) \tag{3.5}
\end{equation}

with $p : Z \times_B C \to Z$ and $p_\ast : Z \times_B C \to C$ the projections and $\mathcal{P}$ the global version of the Poincare line bundle over $E \times E$ (actually one uses a symmetrized version of this),
i.e. the universal bundle which realizes the second $E$ in the product as the moduli space of degree zero line bundles over the first factor.

A second parameter in the description of $V$ is given by a half-integral number $\lambda$ which occurs because one gets From the condition $c_1(V) = \pi_*(c_1(L) + \frac{c_1(C)-c_1}{2}) = 0$ that with $\gamma \in \ker \pi_* : H^{1,1}(C) \to H^{1,1}(B)$ one has

$$c_1(L) = -\frac{1}{2}(c_1(C) - \pi_*c_1) + \gamma$$

where $\gamma$ is being given by ($\lambda \in \frac{1}{2}\mathbb{Z}$)

$$\gamma = \lambda(n\sigma - \eta + nc_1)$$

as $n\sigma |_C - \eta + nc_1$ is the only generally given class which projects to zero. As we are in a different set-up than [2] one has actually a further possibility which is described in section 5.

### 3.2 The generation number

Considering the chiral matter content one finds then as the number of net generations [3]

$$\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1)$$

Note that because of the decomposition

$$248 = (3, 27) \oplus (\bar{3}, 27) \oplus (1, 78) \oplus (8, 1)$$

$$= (4, 16) \oplus (\bar{4}, 16) \oplus (6, 10) \oplus (1, 45) \oplus (15, 1)$$

$$= (5, 10) \oplus (10, \bar{5}) \oplus (\bar{5}, 10) \oplus (\bar{10}, 5) \oplus (1, 24) \oplus (24, 1)$$

one has, in order to get all the relevant fermions, in the case of an $SU(5)$ bundle - unlike the case of an $SU(3)$ or $SU(4)$ bundle - to consider also the $\Lambda^2V = 10$ to get the $\bar{5}$ part of the fermions $10 \oplus \bar{5}$; but the 10 and the $\bar{5}$ will come in the same number of families by anomaly considerations.

### 3.3 The cohomology of the bundles

Note that in order to apply the formalism of [2], to compute the generation number of $V$ and to make the checks on the effectiveness of the five-brane class below we have
to take into account that we are now working in Calabi-Yau spaces with a different representation of the elliptic fibre $E$. This changes the cohomological data of $Z$ itself (as described above) but could also change the expression for $c_2(V)$ and $c_3(V)$ if you look at Grothendieck-Riemann-Roch

$$
\pi_*(e^{c_1(L)}Td(C)) = ch(V)Td(B)
$$

which shows an influence in $c_2(V)$ of the change of $c_2(Z)$ relative to the $A$ model: as $c_2(Z)$ occurs in the Chern classes $c_1(C) = -(n\sigma_1 + \eta)$ and $c_2(C) = c_2(Z) + c_1^2(C)$ of the spectral cover (note that the whole spectral cover construction always uses only the zero section $\sigma_1$) one gets the following. The new term to consider is the push-down (from $C$ to $B$) of $\Sigma$ (coming from $c_2(Z)$ in $c_2(C)$); the complete term there is a $\frac{12}{k} \Sigma c_1$ inside $c_2(Z)$, times $\frac{1}{12}$ because of $Td(C)$; this can also be understood as the push-down from $Z$ to $B$ of $C\Sigma$ which is $(n\sigma_1 + \eta)\Sigma_{i=1}^k \sigma_i = -(nc_1 + \eta)\sigma_1 + \eta\Sigma_{i=2}^k \sigma_i = -nc_1\sigma_1 + \eta\Sigma_{i=1}^k \sigma_i$; this is pushed down to $-nc_1 + k\eta$, so the whole term inside $c_2(Z)$ goes to $\frac{1}{12}(-n\frac{12}{k}c_1 + 12\eta)c_1$.

The second summand here is especially important as it gives\(^\text{13}\) the first term (which, as we see, remains unchanged compared to the $A$ model as $k$ cancels out) in the final result for $c_2(V) = \eta\sigma_1 + \omega$ (where $\omega \in H^4(B)$, pulled back to $Z$.

Now let us give the second Chern class for $V$ (one has $\pi_*(\gamma^2) = -\lambda^2 n(\eta (\eta - nc_1))$):

\[
c_2(V) = \frac{12k}{12} \eta\sigma_1 - \frac{n^3 - (2(a - b) - 1)n}{24} c_1^2 - \frac{n}{8} (\eta (\eta - nc_1) - \frac{1}{2} \pi_*(\gamma^2))
\]

\[
= \eta \sigma_1 - \frac{n^3 - n}{24} c_1^2 + \frac{1}{2} (\lambda^2 - \frac{1}{4} n \eta (\eta - nc_1))
\](3.11)
i.e. there is no correction as $a - b = 1$ in $c_2(Z) = c_2 + a \Sigma c_1 + bc_1^2$. Note that $c_3(V)$ is not changed too.

### 3.4 The parabolic approach

Of course this is no accident that the bundle does not see the changed fibre. One has a second approach to describe bundles and compute their Chern classes [2],[18]. In the parabolic approach one starts with an unstable bundle on a single elliptic curve $E$. For this one fixes a point $p$ on $E$ with the associated rank 1 line bundle $O(p) = W_1$. Rank

\(^\text{13}\)By doing the GRR for $\pi: C \rightarrow B$ we actually compute only $c_2(V)|_{\sigma_1}$; so the term '$-\eta c_1$' in the expression for the restriction comes because of $\sigma_1|_{\sigma_1} = -c_1|_{\sigma_1}$ (from adjunction) from a term $\eta \sigma_1$ in the total expression for $c_2(V)$. This, and the further statement that in the cases with changed elliptic fibre the element which restricts to $\sigma_1$ does not involve the other $\sigma_i$ (say in the combination $\Sigma$), can be checked from the corresponding GRR formula for $p: Z \times_B C \rightarrow Z$ (this is described in [3]); there one sees that the corresponding term comes from the Poincare bundle, which involves only $\sigma_1$.  

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$k$ line bundles $W_k$ are then inductively constructed via the unique non-split extension $0 \to \mathcal{O} \to W_{k+1} \to W_k \to 0$. If one writes the dual of $W_k$ as $W_k^*$ then the unique (up to translations on $E$) minimal unstable bundle with trivial determinant on $E$ is given by $V = W_k \oplus W_{n-k}^*$. This can be deformed by an element of $H^1(E, W_k^* \otimes W_{n-k}^*)$ to a stable bundle $V'$ which fits then into the exact sequence $0 \to W_{n-k}^* \to V' \to W_k \to 0$.

To get a global version of this construction one replaces the $W_k$ by their global versions, i.e. replace $\mathcal{O}(p)$ by $\mathcal{O}(\sigma_1)$. The global versions of $W_k$ are inductively constructed by an exact sequence $0 \to L_{n-k} \to W_k \to W_k^{*-1} \to 0$. Using the fact that one can globally twist by additional data coming from the base, i.e. line bundles $\mathcal{M}$ and $\mathcal{M}'$ on $B$, one finds the unstable $SU(n)$ bundle

$$V = W_k \otimes \mathcal{M} \oplus W_{n-k}^* \otimes \mathcal{M}'$$

(3.12)

with $W_k = \bigoplus_{a=0}^{k-1} L^a$, $W_{n-k}^* = \bigoplus_{b=0}^{n-k-1} L^{-b}$ and $\mathcal{M}, \mathcal{M}'$ are constrained so that $V$ has trivial determinant. Note that the specification of the twisting line bundle $\mathcal{M}$ corresponds to the specification of $\eta$ in the spectral cover approach (cf. [2], [18]). Since the topology of $V$ is invariant under deformations one computes Chern classes simply of $V$ using

$$c(V) = \prod_{a=0}^{k-1} (1 + c_1(L^a) + c_1(\mathcal{M})) \prod_{b=0}^{n-k-1} (1 + c_1(L^{-b}) + c_1(\mathcal{M}'))$$

(3.13)

which is independent of the fibre type $(A, B, C, D)$.

### 3.5 Bundle moduli and index theorems

Let us also point out that the $\mathbb{Z}_2$ equivariant index $I = n_e - n_o$ of [2], counting the bundle moduli which are even respectively odd under the $\tau$-involution, does not change under exchanging the elliptic fibre of $Z$. Recall that $I$ can be interpreted as giving essentially the holomorphic Euler characteristic of the spectral surface [20] which is

$$1 + h^{2,0}(C) - h^{1,0}(C) = \frac{c_2(C) + c_2^2(C)}{12} |_C = \frac{c_2(Z)C + 2C^3}{12} = n + \frac{n^3 - n}{6} c_1^2 + \frac{n}{2} \eta(\eta - nc_1) + \frac{12}{12} \eta c_1$$

(3.14)

Now identifying the number of local complex deformations $h^{2,0}(C)$ of $C$ with $n_e$ respectively the dimension $h^{1,0}(C)$ of Jac$(C) := Pic_0(C)$ with $n_o$, one finds

$$I = n - 1 + \frac{n^3 - n}{6} c_1^2 + \frac{n}{2} \eta(\eta - nc_1) + \eta c_1.$$

(3.15)
3.6 Restrictions on the bundle parameters

After these introductory remarks we proceed to the actual construction of the examples. This will be done in two steps: first we specify the vector bundle (‘above’) over $Z$ and then we make explicit the involution. For this let us recapitulate what are the requirements: we are searching an $SU(5)$ resp. $SO(10)$ bundle $V$ with 6 resp. 12 net-generations on a Calabi-Yau admitting a freely acting $Z_2$ which respects the holomorphic three-form. The conditions concerning the Calabi-Yau $Z$ will be treated in the next section. The conditions concerning the bundle $V$ amount to the following.

3.6.1 Restriction on $\lambda$

From
\[ c_1(L) = n\left(\frac{1}{2} + \lambda\right)\sigma + \left(\frac{1}{2} - \lambda\right)\eta + \left(\frac{1}{2} + n\lambda\right)c_1 \] (3.16)

one sees that the easiest (and the only general) way to fulfill the requirement of integrality\(^{14}\) for $c_1(L)$ is to require $\lambda$ to be strictly half-integral for $n$ odd resp. to be strictly integral and $\eta \equiv c_1 \mod 2$ for $n$ even. More exotic possibilities involve $\lambda = \frac{1}{2n}$ for $n$ odd and $\eta \equiv 0 \mod n$ [18] or for example $\lambda = \frac{1}{4}$ for $n = 4$ and $\eta = 2c_1 \mod 4$.

Combined with the searched for generation number obvious restrictions result on the possible $\lambda$ (for example $\lambda = \pm 1/2, \pm 3/2$ where one has then to construct on the Calabi-Yau ‘above’ (before the modding) a model with $\eta(\eta - 5c_1) = \pm 12$ or $\pm 4$ in the $SU(5)$ case).

3.6.2 The upper bound on $\eta$

The essential restrictions on $V$ come from bounds on the $\eta$ class. The upper bound comes from the anomaly cancellation condition $c_2(Z) = c_2(V_1) + c_2(V_2) + W$ (we have here $V_2 = 0$) giving the effectiveness restriction $c_2(V) \leq c_2(Z)$ on the five-brane class $W = W_B + a_f F$; clearly here a second effectiveness condition is emerging for $a_f$: it has to be (integral and) non-negative\(^{15}\). Now remember that in the final result for $c_2(V) = \eta\sigma_1 + \omega$ (where $\omega \in H^4(B)$, pulled back to $Z$) the number $k$ of the model ($1, 2, 3, 4$ for $A, B, C, D$) cancelled out leaving the $A$ model result unchanged. On the other hand note that we have a corresponding decomposition $c_2(Z) = \frac{12}{k}\Sigma c_1 + (c_2 + (\frac{12}{k} - 1)c_1^2)$. This is the term

\(^{14}\)Note that these conditions assure the integrality of $c_2(V)$ too.

\(^{15}\)in the case of a Hirzebruch surface $F_k$ with $k \geq 3$ it is sufficient to have non-negativity of $a_f - W_B c_1$ (cf. [17])

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responsible for the upper bound \( \eta \sigma_1 \leq \frac{12}{k} \Sigma c_1 \), i.e. \( \eta \leq \frac{12}{k} c_1 \) (which is thus much sharper than in the \( A \) model) from the effectiveness restriction on the five-brane class.

### 3.6.3 The lower bound on \( \eta \)

This is a bound on 'how much instanton number has to be turned on to generate/fill out a certain \( SU(n) \) bundle', or speaking in terms of the unbroken gauge group \( G \) (the commutator of \( SU(n) \) in \( E_8 \)) a condition 'to have no greater unbroken gauge group than a certain \( G' \).

Let us recall the situation in six dimensions. There the easiest duality set-up is given by the duality of the heterotic string on \( K3 \) with instanton numbers \((12 - m, 12 + m)\) (and no five-branes) with \( F \)-theory on the Hirzebruch surface \( F_m \) [13]. The gauge group there is described by the singularities of the fibration and a perturbative heterotic gauge group corresponds to a certain degeneration over the zero-section \( C_0 \) (of self-intersection \(-m\)): for example to get an \( SU(3) \) one needs a certain \( A_2 \) degeneration over \( C_0 \) available first for \( m = 3 \); in general this means that the discriminant divisor \( \Delta = 12c_1(F_m) \) has a component \( \delta(G)C_0 \) where \( \delta(G) \) is the vanishing order of the discriminant (equivalently the Euler number of the affine resolution tree of the singularity), giving also the relation \( m \leq \frac{24}{12 - \delta(G)} \) for the realization over a \( F_m \) to have no singularity worse than \( G \). The last relation follows (cf. [21]) from the fact that after taking the \( C_0 \) component with its full multiplicity \( \delta(G) \) out of \( \Delta \) the resulting \( \Delta' = \Delta - \delta(G)C_0 \) has transversal intersection with \( C_0 \) and so \( \Delta' \cdot C_0 \geq 0 \), leading with \( c_1(F_m) = 2C_0 + (2 + m)f \) to the mentioned result.

So the instanton number \( 12 - m \) to give a \( G \) gauge group has to be \( 12 - m \geq 12 - \frac{24}{12 - \delta(G)} = (6 - \frac{12}{12 - \delta(G)})c_1(B_1) \) with \( B_1 \) the common \( P^1 \) base of the heterotic \( K3 \) resp. the \( F_m \). From this it was conjectured in [21] (see also [19]) that a similar effectiveness bound could in four dimensions look like the generalizations of both sides of the six-dimensional bound, i.e. in view of the fact that the \((12 - m, 12 + m)\) structure generalizes in four dimensions to \( \eta_1 = 6c_1 - t \), \( \eta_2 = 6c_1 + t \) (for this cf. the anomaly cancellation condition \( c_2(V_1) + c_2(V_2) + a_f F = c_2(Z) \) and its component \( \eta_1 \sigma + \eta_2 \sigma = 12c_1 \sigma \) concerning the classes not pull-backed from \( H^4(B) \) for the case of an \( A \) model with \( W_B = 0 \))

\[
\eta_1 \geq (6 - \frac{12}{12 - \delta(G)})c_1 \quad \text{(3.17)}
\]

So finally one has to fulfill the lower bound \( \eta \geq \frac{30}{7} c_1 \) resp. \( \eta \geq \frac{18}{5} c_1 \) in the case of an \( SU(5) \) resp. \( SO(10) \) unbroken gauge group (the coefficients come from the expression \( 6 - \frac{12}{12 - \delta} \) in [21] with \( \delta \) the vanishing order 5 resp. 7 of the discriminant). As our breaking
scheme involves one resp. two $\mathbb{Z}_2$'s (and so the $B$ resp. $D$ model) for unbroken $SU(5)$ resp. $SO(10)$ we see that, in view of the upper bound $\eta \leq 3c_1$ in the $D$ model case, we are left with the possibility of breaking $SU(5)$ via the $B$ model.

3.7 Restricting attention to $V_1$

When giving in the following subsection our examples we will restrict attention to the visible sector, i.e. to the '1' sector in the bundle embedding $(V_1, V_2) \in E_8 \times E_8$, and will neglect the hidden '2' sector concerning $V_2$. This is justified up to the occurrence of the compensating five-brane class in

\[ c_2(V_1) + c_2(V_2) + W = c_2(Z) \]  

(3.18)

which connects the two sectors. As we will consider a $B$ type model of $Z$ we will have $c_2(Z) = 6c_1\sigma_1 + 6c_1\sigma_2$. Now the decomposition (suitable pull-backs understood) $(\oplus_i \sigma_i H^2(B)) \oplus H^4(B)$ leads for the parts not coming from $H^4(B)$ to

\[ \eta_1\sigma_1 + \eta_2\sigma_j + W_B = 6c_1\sigma_1 + 6c_1\sigma_2 \]  

(3.19)

where $j = 1$ or 2 according to whether we choose to build $V_2$ using the same section $\sigma_1$ as for $V_1$ or the other one, i.e. $\sigma_2$ (the spectral cover construction uses in its essential parts just the possible globalization of a point $p$ on an elliptic fiber curve, relative to which degree zero line bundles are transmuted into divisors $Q - p$ for another point $Q$; there is no geometric distinction which forces one of the sections to be called the zero-section with respect to the group law).

So we have actually two possible strategies for saturating on the LHS of the last equation the $6c_1\sigma_2$ of the RHS: either we put it in a "$W_B^{(2)}$" or we choose $j = 2$ and $\eta_2 = 6c_1$. So let us now also introduce the decomposition $W_B = W_B^{(1)} + W_B^{(2)}$ by which we mean $W_B^{(2)} = 6c_1\sigma_2$ for $j = 1$ (actually we will then not even turn on at all a second bundle, i.e. $\eta_2 = 0, \lambda_2 = 0$, as the lower bound leaves no space for this) respectively $W_B^{(2)} = 0$ for $j = 2$ in which latter case also $\eta_2 = 6c_1$ is understood. We do so just for the sake of simplicity; of course one can easily elaborate a more general scheme for $V_2$ lying between the extreme cases $j = 1$ (i.e. no bundle constructed over $\sigma_2$ and further even $\eta_2 = 0, \lambda_2 = 0$) together with a maximal $W_B^{(2)} = 6c_1\sigma_2$ on the one hand or $j = 2$ and a maximal $\eta_2 = 6c_1$ and no $W_B^{(2)}$. This would repeat just the treatment of the "1" sector apart from the fact that now both these sectors are not disentangled because there is still the $H^4(B)$ part of the anomaly equation giving the number of five-branes wrapping
the elliptic fibre

\[-\frac{n_3 - n_1}{24}c_1^2 + \frac{1}{2}(\lambda_2^2 - \frac{1}{4})n_1\eta_1(\eta_1 - n_1c_1) +\]

\[-\frac{n_2^2 - n_2}{24}c_1^2 + \frac{1}{2}(\lambda_2^2 - \frac{1}{4})n_2\eta_2(\eta_2 - n_2c_1) + a_f F = c_2 + 5c_1^2 \] (3.20)

Decompose now \(a_f = a_f^{(1)} + a_f^{(2)}\). Our procedure below will be to solve for \(a_f^{(1)}\) (which then has to be integral and non-negative\(^{16}\)) in

\[-\frac{n_3 - n_1}{24}c_1^2 + \frac{1}{2}(\lambda_2^2 - \frac{1}{4})n_1\eta_1(\eta_1 - n_1c_1) + a_f^{(1)}F = c_2 + 5c_1^2 \] (3.21)

thereby saturating the base part of \(c_2(Z)\). The other \(a_f^{(2)}\) (in the case where a second bundle is turned on)

\[a_f^{(2)} = \frac{n_3^2 - n_2}{24}c_1^2 - \frac{1}{2}(\lambda_2^2 - \frac{1}{4})n_2\eta_2(\eta_2 - n_2c_1) \] (3.22)

will take care of itself: the non-negativity and integrality of \(a_f^{(2)}\) are easily accomplished by choosing \(n_2 = 3\) or 5 and \(\eta_2 = 6c_1, \lambda_2 = 1/2\).

So the upshot is we have now reduced the discussion to the '(1)’ sector and shall omit below the (1) index, it being understood that a \(W_B^{(2)} = 6c_1\sigma_2\) (and \(\eta_2 = 0, \lambda_2 = 0\)) or an \(\eta_2 = 6c_1\) runs in the second sector.

3.8 Examples

So one has to find an \(\eta\) in the strip \(\frac{20}{7}c_1 \leq \eta \leq 6c_1\) with \(\eta(\eta - 5c_1) = \pm 12\) (for \(\lambda = \pm \frac{3}{2}\)) resp. \(\eta(\eta - 5c_1) = \pm 4\) (for \(\lambda = \pm \frac{1}{2}\)) with \(a_f\) non-negative (we will be considering only \(F_0, F_1, F_2\) and the \(dP_k\)'s) where \(a_f = c_2 + 10c_1^2\) for \(\lambda = \pm \frac{1}{2}\) resp. \(a_f = c_2 + 10c_1^2 \mp 20\) for \(\lambda = \pm \frac{3}{2}\) (so this only excludes for \(\lambda = +\frac{3}{2}\) the \(dP_3\); of course one can work in that case simply still with \(\lambda = -\frac{3}{2}\) getting \(-6\) net-generations).

Now let us specify the \(SU(5)\) bundle over \(Z\). Recall that the Hirzebruch surfaces \(F_m\) are \(P^1_{(2)}\) fibrations over \(P^1_{(1)}\) possessing a section of self-intersection \(-m\) and have \(c_1(F_m) = (2, 2 + m)\) in a basis (for the effective cone) \((b, f)\) with \(b^2 = -n\) and \(f\) the fiber. The data which specify our bundle over \(F_1\) are now given by

\[\lambda = +\frac{3}{2}, \eta = (11, 15)\]

\(^{16}\)Of course one could imagine a more general procedure which demands these conditions only for the total \(a_f\) as only this has an absolute meaning.
Note that besides leading to a net-number of 6 generations these data fulfill the series of further conditions mentioned above with \( W = (1, 3) + 64F \). This is (in the given set-up) the only solution on a Hirzebruch surface \( F_r, r = 0, 1, 2 \). Over the del Pezzo surfaces \( dP_k \) there are many more possibilities (here of course the example over \( F_1 \) reoccurs; we are using the basis \((l, (E_i)_{i=1...k})\) where for example \( c_1 = (3, -1^k) \); here one has to keep in mind that the classes \( l - E_i \) and \( l - E_i - E_j \) when \( n \geq 2 \), \( 2l - \sum_{j=1}^5 E_{ij} \) when \( n \geq 5 \), \( 3l - 2E_i - \sum_{j=1}^5 E_{ij} \) when \( n \geq 7 \) and \( 4l - 2 \sum_{j=1}^3 E_{ij} - \sum_{m=1}^5 E_{im} \) when \( n \geq 8 \) are effective when verifying bounds on \( \eta \)). Note that the five-brane class \( W = W_B + a_f F \) is given by \( W_B = 6c_1 - \eta \) and \( a_f = 93 - 9k \) for \( \lambda = \pm \frac{1}{2} \) resp. \( a_f = 93 - 9k \pm 20 \) for \( \lambda = \pm \frac{3}{2} \).

Besides a long list of 6 generation models which can be given for the del Pezzo surfaces up to \( dP_8 \) (which we will not list as we will have no suitable involution on \( Z \)), on \( dP_9 \) (\( C' \)-type; we will give in the next section a fixpoint free involution for \( dP_9^B \) of \( B \)-type), one sees that one has no relevant examples, as the bounds \( \frac{3}{4} c_1 \leq \eta \leq 6c_1 \), together with the effectivity requirement on \( \eta \), force \( \eta \) to be proportional to \( c_1 \) (and so the model leads to no generations, because of \( c_1^2 = 0 \)) as one convinces oneself in the equivalent basis given by a section, the elliptic fibre (being equal to \( c_1 \)) and the \( E_8 \) lattice.

### 4 Free \( Z_n \) symmetries

#### 4.1 Modding the spaces

Here we construct elliptically fibered Calabi-Yau threefolds \( Z \) with sections, which admit an action \( \iota \) of an discrete group \( G \), such that the following requirements are met:

- a.) \( Z \) is smooth,

- b.) The action is free, i.e. the fixpoint set is empty,

- c.) The action leaves the holomorphic \((3,0)\)-form invariant.

We call \( Z' = Z/G \), let \( pr \) denote the projection. Note that \( Z' \) is again an elliptic fibration (of \( B \) type elliptic fibre) over a base \( B' \). To get a \( Z_n \) action we will start with the elliptic fibration with \( n \) sections and use the algebraic \( Z_n \) action from sect. 2, together with an in general non-free order \( n \) action on the base. More precisely what happens is this (let us put \( n = 2 \), the case of our main example). As we consider an involution \( \iota \) compatible with the fibration we have an involution \( \underline{\iota} \) already defined on \( B \) alone with \( \underline{\iota} \cdot \pi = \pi \cdot \iota \) (note that \( B' = B/G \)). The group action \( \iota \) maps both sections onto each other, their
image downstairs in $Z'$ will be an irreducible surface $\sigma$ (still isomorphic to $B$). Is this $\sigma$ a section in $Z'$? Now, from $pr^*\sigma \cdot pr^*f = 2 \sigma \cdot f$ where $f$ denotes the fibre downstairs (lying over a generic point $b'$ in the base $B'$ of $Z'$), one learns that it is actually only a bi-section because the left hand side is evaluated as $4 = 2 + 2$ as each of the two sections $\sigma_i, i = 1, 2$, in the preimage of $\sigma$ intersects each of the two fibers in $Z$ 'above' (lying over the preimages of $b'$ in $B$) twice. So in general after the modding $n$ sections become one $n$-section. One losses therefore at least $n - 1$ independent divisor classes in $H^{1,1}$.

In the following we will describe more closely the modding process: first in section (4.1) for a Hirzebruch basis; here we find free involutions for $F_0$ and $F_2$, so they will not lead to 3 generations on $Z'$ as a corresponding 6 generation model on $Z$ was found in the last section only over $F_1$; then in section (4.2) we find a free involution for the case $B_2 = dP^B_9$ of $B$-type elliptic fibre.

### 4.1.1 Toric descriptions

In the following we will describe Calabi-Yau spaces $Z$ defined as hypersurfaces in toric ambient spaces, with the three fiber types $A, B, C$. As we mentioned in the introduction, the 15 2-d toric reflexive polyhedra $\Delta_B$ from [4] lead to smooth fibrations for all fibre types. The four dimensional polyhedra, which define the Calabi-Yau threefold following [15], are given by the convex hull of the points $\{[0,0,-1,0], [0,0,0,-1], [\nu^B, \nu^{(i)}]\}$, where the $\nu^B$ runs over the points in $\Delta_B$ and $\nu^{(1)} = (2,3), \nu^{(2)} = (1,2)$ and $\nu^{(3)} = (1,1)$ for the $A, B, C$ fiber types. By the formulas derived in section 2.4, we can express the cohomological information of $Z$ entirely in terms of the base cohomology.

\[
\begin{align*}
  c_3(Z) &= -2hc_1(B)^2 \\
  c_2(Z)J_E &= kc_2(B) + k\left(\frac{12}{k} - 1\right)c_1(B)^2, \quad c_2J_i = 12kc_1(B)J_i \\
  J_E^3 &= kc_1^3(B), \quad J_E^2J_i^B = k c_1(B)J_i^B, \quad J_EJ_iJ_k = kJ_iJ_k,
\end{align*}
\]

where the lefthand side is integrated over $Z$ and the righthand side over $B$. $J_E$ is as cohomology element supported on the elliptic fibre, its dual homology element is the base, while the $J_i$ are as cohomology elements supported on curves in $B$, while the homology element is the dual curve in $B$ and the fiber over it.

In order to see whether requirements a.)-c.) are met we give a coordinate description of $Z$, which turns out to be a straightforward generalization of the notion of a homogeneous polynomial in a projective space. We shall proceed in an example and consider as bases
The polyhedron $\Delta_B$ contains the points $[1, 0], [0, 1], [-1, 0], [-1, -n], [0, 0]$ and hence $\Delta$ is given by the convex hull of points $\nu_i \ i = 0, \ldots, 7$ in $\mathbb{R}^4$ equipped with the standard $\mathbb{Z}$-basis for a lattice $\Lambda$

<table>
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<th></th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>0</th>
<th>-4</th>
<th>x₀</th>
</tr>
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<td>0</td>
<td>0</td>
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<td>s = x₁</td>
<td></td>
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<tr>
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<td>0</td>
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<td>t = x₂</td>
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</tr>
<tr>
<td>[1,0,1,2]</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>u = x₃</td>
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</tr>
<tr>
<td>[-1,0,1,2]</td>
<td>-n</td>
<td>1</td>
<td>0</td>
<td>v = x₄</td>
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<td></td>
</tr>
<tr>
<td>[0,0,-1,0]</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>x = x₅</td>
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</tr>
<tr>
<td>[0,0,0,-1]</td>
<td>0</td>
<td>0</td>
<td>2</td>
<td>y = x₆</td>
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<td></td>
</tr>
<tr>
<td>[0,0,1,2]</td>
<td>n-2</td>
<td>-2</td>
<td>1</td>
<td>z = x₇</td>
<td></td>
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</tr>
</tbody>
</table>

where we also indicated as columns the generators of linear relations among the $\nu_i$ $l^{(i)}$ spanning the Mori cone, which is dual to the Kähler cone. We can now write $Z$ as hypersurface representation in coordinates $(s, t, v, u, x, y, z)$, one for each $\nu_i$, as

$$x₀y² = x₀y \sum_{l=0}^{2} xₗz^{2-l} \sum_{k=k_{min}^1}^{4-2l-k} v^k u^{4-2l-k} g_{l,n} + x₀ \sum_{l=0}^{4} xₗz^{1-l} \sum_{k=k_{min}^2}^{8-2l-k} v^k u^{8-2l-k} f_{l,n}.$$  \hspace{1cm} (4.26)

Here $d_{l,n} = (2-n)(2-l) + nk$ and $d_{l,n} = (4-l)(2-n) + nk$ are the degrees of $f(s,t), g(s,t)$ in $s, t$ and the lower bounds on $k$ is such that this degree does not negative\(^{17}\) $k_{min}^1 = \lceil \frac{(2-n)(l-2)}{n} \rceil$ and $k_{min}^2 = \lceil \frac{(2-n)(l-4)}{n} \rceil$. The form [4.26] is manifestly invariant under the $\mathbb{C}^*$-actions $x_i \rightarrow \lambda_\mu^{(i)} x_i$, $\lambda_\mu \in \mathbb{C}^*$, $\mu = 1, 2, 3$ and therefore welldefined in the coordinate ring\(^{18}\) of the $x_i$ $\mathcal{R} = \{ C[x₁, \ldots, x₇] - SRI \}/C^*$ where the Stanley-Reissner ideal $SRI$ is generated by $x₀ = 0, x₁ = x₂ = 0, x₃ = x₄ = 0, x₅ = x₆ = x₇ = 0$. We may set $x₀ = 1$ in the following. Note in particular that for $n > 2$ one has $k_{min}^i > 0$ and hence [4.26] becomes singular, with singularities that can be directly read off e.g. an $A₃$ near $v = 0$ for $n = 3$ e.t.c. In general, the failure of transversality of the constraint has by Bertinis theorem only to be checked on the base locus, i.e. for the constraint $p = 0$ restricted to hyperplanes defined by $\{ x_i = 0 \}$, which also have to avoid of course the $SRI$. Still this is a formidable task in general, but for 3-d Calabi-Yau manifolds it is equivalent to reflexivity of $\Delta$ [15]. This fact has been used to check that the fibrations over bases from [4] are indeed smooth and will be used to show that the member of the family admitting the $G$-action is smooth. Using the description of the cohomology by [15] we get $h_{1,1} = 4(1)$ in all cases and $h_{2,1} = 148(0)$ for $F₀$ and $F₁$ and $148(1)$ for $F₂$.\hspace{1cm}

\(^{17}\)[x] means the next integer greater then x.

\(^{18}\)Known as Batyrev-Cox coordinate ring.
Alternatively we can fix most of the automorphism group of the toric ambient space by writing the equation (4.26) into a more specialized form

\[ y^2 = 4x^4 + x^2z^2a(s, t, u, v) + xz^3b(s, t, u, v)/3 + z^4c(s, t, u, v)/4, \]  

(4.27)

where the weight constraints on \(a, b, c\) are as in (4.26). The part of the automorphism group, which is not fixed by this choice is the automorphism group of the base and an overall scaling. Then the discriminant of (4.27) is sufficiently simple to be analysed directly.\(^{19}\)

\[ \Delta = (9a^4c + 36ab^2c - 72a^2c^2 + 144c^3 - a^3b^2 - 3b^4). \]  

(4.28)

Starting from (4.27) we may count the \(h_{2,1}\) forms by enumerating perturbations modulo the remaining automorphism. E.g. for \(F_0\) \(a, b, c\) are sections of line bundles corresponding to polynomials of bi-degree (4, 4), (6, 6) and (8, 8) over \(F_0 = \mathbb{P}^1 \times \mathbb{P}^1\), which yields \(h_{2,1} = 5^2 + 7^2 + 1^2 = 148\) (cf. section (2.5.1)).

In order that the holomorphic three-form is not projected out the total action of the \(\mathbb{Z}_n\) on the cotangent space of \(Z\) must be a subgroup of \(SU(3)\). The condition is easily checked on the explicit expression of the holomorphic three-form \(\Omega \sim \prod_{i \neq k, l, m, p} \frac{x_kdx_ldx_mx}{\partial x_p}\) from which we see that phase actions \(x_i \rightarrow \mu^{wi}x_i\) with \(\mu^n = 1\) and \(\sum w_i = 0 \mod n\) leave \(\Omega\) invariant. Hence \(\mathbb{Z}_2\) acting on the Calabi-Yau \(Z\) by

\[ (s, t, u, v, z, x, y) \rightarrow (-s, t, u, -v, -x, -y, z) \]  

(4.29)

is an action with property c). To find the fix point set we must take into account the \(SRI\) and the \(C^*\) actions. If \(n\) is even then \((0, t, u, 0, 0, 0, z)\) is a fixed stratum in the ambient space, the \(SRI\) enforces \(z \neq 0, t \neq 0\) and \(u \neq 0\) and (4.26) restricted to the stratum is empty \((z^4u^8t^4(2-n) = 0\) in contradiction with the \(SRI\) on \(Z\). Similarly the other fixed strata on which the action (4.29) can be undone by the \(C^*\)-actions have no intersection with \(p = 0\). On the other hand if \(n\) is odd then by the \(C^*\) actions \((s, 0, u, v, x, 0, z)\) turn out to be the fixed stratum, which intersects \(Z\) in particular for \(n = 1\) in a \(\chi_{fix} = -16\) curve.

Hence to have a.) we shall restrict ourselves to the fix point free actions on \(F_0\) and \(F_2\) and consider transversality of the specialized polynomial which admits the \(\mathbb{Z}_2\) involution. We may check transversality by showing that the polyhedron, which corresponds to the invariant monomials is again reflexive. The invariant polyhedron \(\Delta'\) is constructed as in [16] by considering a sublattice \(\Lambda'\) of index \(\frac{1}{2}\) in \(\Lambda\). In the new \(\mathbb{Z}\)-basis the points

\[^{19}\]g_2 = \frac{1}{12}(a^2 + 12c)\) and \(g_3 = \frac{1}{10}(36ac - 6b^2 - a^3)\)
transform to $\nu'_i = M_{i,j} \nu_j$ with

$$M = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 2
\end{pmatrix}$$

(4.30)

By constructing $\Delta'(F_0)$, $\Delta'(F_2)$ we see that they are indeed reflexive polyhedra and the 
cohomology of the associated Calabi-Yau space is $\chi = -144$, $h_{1,1} = 3$ and $h_{2,1} = 75$, 
with one non-generic complex structure modulus for the $\Delta'(F_2)$ case\(^{20}\). Again in the $F_0$ case it is simpler to count in (4.27) the invariant monomials in $a, b, c$, which yields $13 + 24 + 41$. Note that each $\text{SL}(2, \mathbb{C})$ is broken by (4.29) to the diagonal rescaling. Hence

$h_{2,1} = 78 - 3 = 75$. The change in $(1, 1)$ forms from 4 to 3 is due to the fact that the 2 sections become a 2-section, as explained above. Note also that (4.28) remains generic since enough terms in $(a, b, c)$ survive the projection, hence $Z$ and so $Z'$ is smooth. I.e. for these cases all requirements a.)-c.) are fulfilled.

Similarly we can obtain a free $\mathbb{Z}_3$ action on the $C$-fiber over $\mathbb{P}^2$. If we introduce 
coordinates $(s, t, r)$ for the base and $(x, y, z)$ for the fibre as before, then the action

$$(r, s, t, z, x, y) \rightarrow (\mu_3 r, \mu_2 s, t, \mu_3 x, \mu_3 y, z)$$

(4.31)

is a free $\mathbb{Z}_3$ which changes the Hodge numbers from $\chi = -216$, $h_{2,1} = 112$, $h_{1,1} = 4(2)$ to

$\chi = -72$, $h_{2,1} = 38$ and $h_{1,1} = 2$.

An afterthought

Let us remark about some related points in constructing free involutions. In two dimensions analogue there is a famous construction which is somewhat analogous, the free Enriques involution (killing the holomorphic two-fro m) on (certain) $K3$. This gives as quotient again a surface elliptically fibered over $\mathbb{P}^1$ of the same Hodge numbers as $dP_9$ but has generically 10 deformations and two multiple fibers (so in particular does not have a section; if one performs logarithmic transformations on them one connects to the $dP_9$ surface), whose difference constitutes the canonical class which is two-torsion (after all, it vanishes 'above'), a second marked difference to $dP_9$.

Note that if we take in our construction of $B$ fibered Calabi-Yau (which are also $K3$ fibrations) just the $K3$ fibre part (i.e. neglect the overall base $\mathbb{P}^1$ and the part of the involution operating there) this leads to a $K3$ cover of an Enriques (the well-defindness

\(^{20}\)In accordance with the fix point formula $\chi^{\text{new}} = (\chi - \chi_{\text{fix}})/N + N\chi_{\text{fix}}$, we find for $\Delta'(F_1)$: $\chi = -168$, $h_{1,1} = 4$ and $h_{2,1} = 88(9)$, i.e. nine non-generic complex structure moduli.
of the operation restricts one then in the deformation space of the $K3$ where the $B$ type fibration is kept manifest to a nine-dimensional space).

Note further that if one would take just the Calabi-Yau $(T^2 \times K3_{Enr})/\mathbb{Z}_2$ and considered it as $K3$ fibered over the first factor\(^{21}\) then one gets a Calabi-Yau with a fundamental group 'including' $\mathbb{Z}_2$ but this model has no section (although the generic $K3$ fibre has a section, over four points in the base one has the Enriques as fibre which has no section but only a bi-section); as one would here have 'above' an $N = 2$ situation one has there no chiral matter.

4.1.2 An example over $dP_9$

Let us consider now the $B$ model fibration over $dP_9$, i.e. the Calabi-Yau space $Z = dP_9^B \times_{P^1} dP_9$ (the first fibre factor indicates the vertical $dP_9$, the second one the horizontal base). We will choose for the base the $dP_9$ too, in contrast to the choice of $C'$ model we did in the generation search in section 3.8. We want to supplement the involution $(y, x) \to (-y, -x)$ in the fibre by a second involution coming from the base. For this we 'combine' (but see next footnote) the action $(s, t) \to (-s, t)$ in the base $P^1$ with the hyperelliptic involution in the fibre of the horizontal base $dP_9$.

So let $(x, y, z)$ be the coordinates of the fiber of the vertical $dP_9$, $(s, t)$ the coordinates of the common base $P^1$ and $(x', y', z')$ the coordinates of the fiber of the horizontal $P^1$. Let us recall the coordinate form

$$y^2 + x^4 + a_2(s, t)x^2z^2 + b_3(s, t)xz^3 + c_4(s, t)z^4 = 0$$  \hspace{1cm} (4.32)

of the vertical $dP_9^B$, with $\mathcal{R} = (C[x, y, z, s, t] - \mathcal{S}\mathcal{R}\mathcal{I})/(C^*)^2$ where $\mathcal{S}\mathcal{R}\mathcal{I}$ is generated by $x = y = z = 0, s = t = 0$ and $l^{(1)} = (1, 2, 1, 0, 0), l^{(2)} = (0, 0, -1, 1, 1)$, shows that monomials within $a, b, c$ are invariant under the action; i.e. we are not forced to a singular locus like $b = 0$.

This is easily enhanced to a fibre product structure, which is given by the following complete intersection (we choose $B = dP_9^B$)

$$y^2 + x^4 + a_2(s, t)x^2z^2 + b_3(s, t)xz^3 + c_4(s, t)z^4 = 0$$

$$y'^2 + x'^4 + \tilde{a}_2(s, t)x'^2z'^2 + \tilde{b}_3(s, t)x'z'^3 + \tilde{c}_4(s, t)z'^4 = 0$$  \hspace{1cm} (4.33)

The $C^*$-actions are specified by

$$l^{(1)} = (1, 2, 1, 0, 0, 0, 0)$$

\(^{21}\)if one fibres over the second factor [17] one encounters difficulties with the effectiveness restrictions as the Enriques base then has a torsion canonical class
\[
I^{(2)} = (0, 0, -1, 1, 1, 0, 0, -1) \\
I^{(3)} = (0, 0, 0, 0, 1, 2, 1) 
\tag{4.34}
\]

Then we consider the action
\[
(x, y, z, s, t, x', y', z') \to (-x, -y, z, -s, t, x', -y', z'), 
\tag{4.35}
\]
i.e. as before we supplement the involution in the fibre by a second involution acting on the base \(dP_9\). Clearly this operation leaves the holomorphic \((3, 0)\) form invariant as \((-1)^4 = +1\).

![Diagram](image)

**Figure 1:** The fibre product \(Z = dP^\text{vert}_9 \times_{P^1} dP^\text{hor}_9\), showing the fixpoints of the action restricted to the factors.

Let us look for possible fix loci by searching strata in the coordinates \((x, y, z, s, t, x', y', x')\). We start by distinguishing between two cases, which are necessarily to have fixed strata, namely either \(s \neq 0\) and \(t = 0\) or \(s = 0\) and \(t \neq 0\). In the first case the largest possible fixed strata are either \((x, 0, z, s, 0, 0, 0, z')\) or \((x, 0, z, s, 0, x', 0, 0)\), which both do not intersect \((4.33)\) as the second equation forces either \(z' = 0\) or \(s = 0\) or \(x' = 0\) all in the \(SRI\) and therefore excluded. In the second case the largest possible strata are either \((x, 0, 0, 0, t, x', 0, z')\) or \((0, 0, z, 0, t, x', 0, z')\), which for similar reasons as above do not intersect with the first equation in \((4.33)\). If we look for the first case only in the vertical fibre then we see here 4 fixpoints over one tip\(^{22}\) of \(P^1\) at \(t = 0\). Likewise for the second case, if we look only in the horizontal fibre, we see here 4 fixpoints over the

\(^{22}\)Note here a special property of our construction of \(dP_9\) in a nontrivial (weighted) \(P^2\) bundle as opposed to the usual \(\begin{bmatrix} P^2 & 3 \\ \cdot & 1 \end{bmatrix}\): In that case one would really have over both fixpoints in the base an hyperelliptic involution with 4 fixed points. In that sense our involution is not just a naive combination of the \(P^1\) involution and the hyperelliptic involution.
other tip of $\mathbb{P}^1$ $s = 0$. The fact that, on the other hand, at $t = 0$ ($s = 0$) in the $\mathbb{P}^1$ the action in the horizontal (vertical) fibre is the shift $(y', x') \rightarrow (-y', -x')$ ($(y, x) \rightarrow (-y, -x)$) is a more conceptual way to see that the total action is actually free.

The number of deformations which is $h^{2,1}(Z) = 2(3 + 4 + 5) - 3 - 1 - 1 = 19$ on the covering drops by the modding to $h^{2,1}(Z') = 2(2 + 2 + 3) - 1 - 1 - 1 = 11$. Likewise by the Lefschetz fixpoint formula one finds for the number of surviving $(1, 1)$-forms $n_{1,1}^+ = 11$ as $0 = 2 + 2(n_{1,1}^+ - n_{1,1}^-) - [2 + 2(n_{2,1}^+ - n_{2,1}^-)] = 2 + 2(n_{1,1}^+ - 19) - [2 + 2(11 - 8)]$.

What concerns the smoothness of $Z$ note that for the individual factors one can easily check using eq. (4.28) that the invariant monomials, just counted, are generic enough to keep the space smooth. So, as the Euler number remains zero, we mod from a $(19, 19)$ Calabi-Yau $Z$ to an $(11, 11)$ Calabi-Yau $Z'$. Note that again our conditions a.)-c.) are fulfilled.

### 4.2 Modding the bundles

In the 'old' case of using the tangent bundle, obviously the bundle was carried along when just modding out the space: one knows then that the bundle 'above' on the Calabi-Yau $Z$ is a pullback of the tangent bundle 'downstairs' of the modded Calabi-Yau $Z'$ and it follows, in the case of modding by an involution $\iota$ (the case of interest here for us), that the Euler number relevant for the net generation number (= Euler/2) is accordingly reduced, i.e. $\int_Z T(Z) = 2 \int_{Z'} T(Z')$. To 'mod' the bundle in the more general case one has just to assume that one really starts from a bundle on $Z'$ where one later breaks the gauge group by turning on Wilson lines using $\pi_1(Z') = \mathbb{Z}_2$. On the other hand to compute the generation number one has to work 'upstairs' as $Z'$ does not have a section but only a bi-section (left over from the two sections of $Z$ as discussed above) and so one cannot use the spectral cover method, as developed so far, directly on $Z'$: remember that the crucial translation from bundle data to geometrical data occurred when a bundle (decomposing fibrewise into a sum of line bundles) is described fibrewise by a collection of points, for which a choice of reference point $p$ (the "zero" in the group law) has to be made (to represent a degree zero line bundle by the divisor $Q_i - p$); one then needed, for this collection of points to fit together to the branched $n$-fold covering $C$ of $B$, the existence of a global reference point, i.e. a section. So we will actually lift back the bundle $V'$ we have downstairs on $Z'$ (whose generation number we want to compute as 3) to a bundle $V$ on $Z$ which should then, because of $\int_Z V = 2 \int_{Z'} V'$, have 6 generations (and is then 'moddable' by construction).

Now, how can one be sure when one constructed a bundle above with 6 generations
that it actually was such a pull-back from \( Z' \)? Clearly for this \( V \) should be \( \iota \) invariant.

For this let us make the following consideration\(^{23}\). First, when one just has a bundle on \( Z \), where \( Z \) is a \( B \) type elliptically fibered Calabi-Yau, one can \( V \) translate into geometrical data in two ways: namely one can analyze \( V \) with respect to both global reference points (i.e. sections) giving the alternative descriptions \( C_i = n\sigma_i + \eta_i \) for \( i = 1 \) and 2. But as it is both times actually the same bundle all invariants associated with \( V \), such as the Chern classes, must coincide. This will be the case if both \( \eta \)’s and \( \lambda \)’s coincide (this is just a sufficient condition).

Now, secondly, let us assume that we are actually analyzing a bundle \( V \) on \( Z \) which is a pull-back from a bundle \( V' \) on \( Z' \). Again we can analyze \( V \) relative to \( \sigma_1 \). But now we have to make explicit the condition of \( \iota \)-invariance, i.e. (as the spectral cover of \( \iota^*V \) with respect to the \( \sigma_2 \) would be\(^{24}\) \( \iota C_1 \)) we have to move \( C_1 \) by \( \iota \) (of course, as the geometrical information encoded in a spectral cover as \( C_1 \) is defined only relative to the global reference point \( \sigma_1 \), one has for the correct interpretation of \( \iota(C_1) \) to read it relative to \( \iota(\sigma_1) = \sigma_2 \) and get a spectral cover \( \iota(C_1) \) over \( \sigma_2 \) of class \([\iota(C_1)] = n\sigma_2 + \iota(\eta_1)\); so, from what was said in the preconsideration and as we require now \( \iota^*V = V \), one finds the \( \iota \)-invariance condition fulfilled for \( \iota(\eta_1) = \eta_1 \) (and \( c_1(B_2) \) should be fix, too; similar considerations pertain to \( \gamma \); again we are presenting sufficient conditions, but note that in our case \( (B = F_0 \text{ or } F_2) \) the \( \iota \) operation induced on \( H^{1,1}(B) \) will be trivial anyway).

Now, in the two actual examples over \( F_0 \) and \( F_2 \), where we could mod the Calabi-Yau, one has a basis of \( H^{1,1}(B) \) consisting in the classes \( b \) and \( f \), the base (of negative self-intersection, say) and the fibre class. Now, in Dolbeault cohomology the relevant representatives of the top \((1,1)\) form of these two \( \mathbf{P}_1 \) are given in local coordinates by, say, \( dzd\bar{z} \); but these classes are invariant under the phase rotations (actually \((-1)\)’s) on the \( \mathbf{P}_1 \). So that condition does not represent a restriction.

Note that this could be a true restriction. For example, in case of a \( dP_9 \) base, the involution in the base (which can be non-free) could have, say, 8 or 4 fix points (in the case\(^{25}\) of \( dP_9^{C'} \) or \( dP_9^B \)) if we combine the \((s, t) \rightarrow (-s, t)\) operation in the base \( \mathbf{P}_1 \) with the hyperelliptic involution. By Lefschetz we see that, say, \( 8 = 2 + n_+ - n_- \) with \( n_\pm \) the \( \pm 1 \) eigenclasses of the induced \( \iota \) operation on \( H^{1,1}(B) \); from \( n_+ + n_- = 10 \) one gets that \( n_- = 2 \) resp. 4 classes for the 4 fixed point case are actually projected out. By

\(^{23}\)Note that we consider fibration compatible involutions, which are thus already defined on the base \( B \) alone and map a fiber over a base point to another fibre.

\(^{24}\)the transport by \( \iota \) of line bundles over an elliptic fibre, which is a map between \( \text{Pic}_0 \)'s, is, when the \( \text{Pic}_0 \)'s are again identified with the elliptic fibre, again the map \( \iota \)

\(^{25}\)Cf. the footnote 22 the case of \( dP_9^{C'} \).
contrast in our cases of Hirzebruch bases, where our involution not only respects the elliptic fibration of the Calabi-Yau but also the internal $\mathbb{P}_1$ fibration of the base, our operation there again 'factorizes' into operations (like $(s, t) \mapsto (-s, t)$) of both the $\mathbb{P}_1$'s, so that from the two individual fixed points on each of them one gets four overall fixed points in $B$ and so, of course obvious anyway, from Lefschetz in that case that $n_\gamma = 0$, i.e. no real restriction results.

5 Genuine $B$ type bundles

Up to now the influence of the choice of a $B$ type elliptic fibered base Calabi-Yau $Z$ had a rather restricted impact on the general set-up. Essentially the influence of this alteration was restricted to the change in $c_2(Z)$ which had its consequences for the upper bound of $\eta$ and a possibility to build the 'other' bundle 'over' $\sigma_2$. But there is an even more interesting twist in the story which we have not mentioned before as we wanted to keep the different steps of complication of the building of bundles disentangled as far as possible. The new freedom we are now speaking about stems from the fact that we have more divisors in the game and so a greater chance to build up a line bundle on the spectral cover. Remember that in the push-forward construction of $V$ from $L$ the class of $L$ was constrained by the requirement $c_1(V) = 0$; this determined $c_1(L)$ up to a class in $\ker \pi_* : H^{1,1}(C) \to H^{1,1}(B)$ killed by the push-forward\footnote{we neglect here the continuous moduli from $H^{1,0}(C)$}. In the $A$ model one has two obvious sources of divisors, consisting of either pull-back’s from $B$ or the section (restricted to $C$); then the combination in $\ker \pi_*$ was $\gamma = \lambda(n\sigma - \eta + nc_1)$. Now, with a second section, a new option arises:

$$\delta = \mu(\sigma_1 + c_1 - \sigma_2)$$

(5.36)

One has $\gamma \cdot \delta = 0$, $\pi_*\delta^2 = -2\mu^2\eta c_1$ and finds that the general combination of $\gamma + \delta$ can be used in building up $L$ with $\mu$ integral and $\lambda$ restrictions unchanged. The influence in cohomological data is (here a major computation is to be carried out which for the $A$ fibre is outlined in the appendix of [3])

$$c_2(V) = \eta\sigma_1 - \eta\delta - \frac{n^3 - n}{24} c_1^2 + \frac{1}{2}(\lambda^2 - \frac{1}{4})n\eta(\eta - nc_1) + \mu^2\eta c_1$$

$$\frac{1}{2}c_3(V) = \lambda\eta(\eta - nc_1) - \frac{3}{2}\mu\eta c_1(\sigma_1 - \sigma_2) = \lambda\eta(\eta - nc_1)$$

(5.37)

where the generation number is unchanged because $\int_Z c_3(V) = 0$ as both $\sigma_i$ are sections leading after integration over the fibre to an integral over $B$ times $(1 - 1)$. Of course that
δ doesn’t contribute is not an accident: the matter computation, described in [3] too, shows that the matter is localized on the curve \( A = C \cdot \sigma_1 \subset \sigma_1(B) \); as \( V|_{\sigma_1(B)} = \pi_*L \) the computation of \( h^0(Z, V) - h^1(Z, V) \) is reduced to \( h^1(A, L|_A \otimes K^{1/2}_A) - h^0(A, L|_A \otimes K^{1/2}_A) = -c_1(L|_A) = -\deg \gamma|_A = -\gamma \cdot \sigma_1 = \lambda \eta \sigma_1 = \lambda \eta (\eta - nc_1) \) where in the last equality the intersection number was evaluated in \( B \cong \sigma_1(B) \) instead of \( C \). But the corresponding consideration for \( \delta \) shows that it doesn’t contribute as \( \delta \cdot \sigma_1 = 0 \).

So one must still search for 6 generations with the pure \( \gamma \) formula, i.e. this generalization is not capable of generating chiral matter by itself, and allows one (within the mentioned \( \eta \)-bounds) not, even when combined with the ordinary \( \gamma \) class, to find 6 generation models over a Hirzebruch surface where one can ‘mod’.

6 A further bundle parameter

As the two requirements in our search, 6 net generations and a moddable Calabi-Yau, have lead us so far to mutually exclusive examples (of bases \( F_1 \) respectively \( F_0, F_2 \) and \( dP_B^9 \)), we have still to broaden our arsenal of constructions. For this we will put the burden on the generation number search and will use a second method to get chiral matter besides turning on \( \gamma \).

Recall that in section (3.1) we explained the idea of the spectral cover construction to trade in the \( SU(n) \) bundle \( V \) over \( Z \) for a line bundle \( L \) over a \( n \)-fold branched covering \( C \) of \( B \): one had

\[
V = p_*(p^{*}_C L \otimes \mathcal{P}) \tag{6.38}
\]

with \( p : Z \times_B C \rightarrow Z \) and \( \mathcal{P} \) the (global) Poincare line bundle over \( Z \times Z \) (restricted to \( Z \times_B C \)). We remarked there that the trade occurs ‘essentially’ over \( B \), i.e. on a surface level, which referred to the fact that we got the line bundle \( \mathcal{L} \) over \( Z \times_B C \), which is needed for the construction \( V = p_*(\mathcal{L} \otimes \mathcal{P}) \), as a pull-back From a line bundle \( L \) over \( C \).

Actually the construction is naturally slightly generalized leading to a dependence on a further parameter\(^{27} \cite{3} \) when one takes into account the full capability of constructing line bundles \( \mathcal{L} \) over the three-fold \( Z \times_B C \): \( Z \times_B C \) will have a set \( S \) of isolated singularities (generically ordinary double points) when in the base the discriminant referring to the \( Z \)-direction meets the branch locus referring to the \( C \) direction. I.e. they are lying over points in the base \( B \) where the branch divisor \( r = K_C - \pi^*K_B \) of \( \pi : C \rightarrow B \), resp. its image in \( B \pi_*r = n(2\eta - (n-1)c_1) \), meets the discriminant \( 12c_1 \) (cf. the case of an elliptic

\(^{27}\text{a possibility already mentioned in } \cite{22} \)
$K3$ over $\mathbb{P}^1$ of $\pi : Z \to B$. Their number is (as detected in $B$) $|S| = 12c_1n(2\eta - (n-1)c_1)$.

One has a resolution $\nu : Y \to Z \times_B C$ of the isolated singularities with $E$ the exceptional divisor. Each of its $|S|$ components is a divisor $D = \mathbb{P}^1 \times \mathbb{P}^1$ of triple self-intersection $D^3 = (-1, -1)^2 = +2$.

The resolution $Y \to Z \times_B C$ leads to the possibility to formulate the whole construction on $Y$ and twist there by the line bundle corresponding to a multiple $l \in \mathbb{Z}$ of the exceptional divisor $E$. So one arrives at the description $V = p_\ast \nu_\ast \mathcal{L}$ where $\mathcal{L} = \nu_\ast p_\ast \mathcal{L} \otimes \nu_\ast \mathcal{P} \otimes \mathcal{O}_Y(lE)$. Including this twist one gets for the generations in total

$$
\frac{1}{2}c_3(V) = \lambda \eta(\eta - nc_1) + \frac{l(l-1)(2l-1)}{6}|S|
$$

Note that the additional contribution coming from $c_1(\mathcal{O}_Y(lE))$ is even componentwise integral as for example (for $l \geq 0$) $\frac{l(l-1)(2l-1)}{6} = \sum_{i=1}^{l} i^2$.

Now what concerns the application in our search for 6 generation models over $F_0$ or $F_2$, fate wants it that numerically it is still not possible to reach the number $\pm 6$, essentially because $|S| = 12c_1n(2\eta - (n-1)c_1) = 120c_1(\eta - 2c_1)$ turns out to be a somewhat unpleasant large number which is unsuitable to be tuned to $\pm 6$ even with the combined effort of the $\lambda \eta(\eta - 5c_1)$ term ($\lambda \in \frac{1}{2} + \mathbb{Z}$).

But there is actually an easy way out. The exceptional divisor $E$ decomposes into $|S|$ components and we are not forced to buy its effect in the total collection. I.e., we do not have just a second discrete parameter $l$ besides $\lambda$, but this $l$ is actually a vector $(l_i)_{i \in S}$ of $|S|$ components of integral numbers! The only thing we could be forced to is to take an $G$-orbit (where $G$ is the freely acting group by which finally we want to mod out; its operates on the set $S$ of singularities) to secure 'moddability'.

But as in our case $G$ is just $\mathbb{Z}_2$, generated by the involution $\iota$, this forces us at most to turn on two individual $l_i$ in parallel. When we take their common value to be 2 (so that $\frac{l(l-1)(2l-1)}{6} = 1$) their contribution in the generation number is only $1 \cdot 2 = 2$. But now it is easy, by turning on enough $l_i$’s, to reach 6.

G.C. thanks E. Witten for pointing out the possibility of a construction similar to the one considered here.
References


