Moments of event observable distributions and many-body correlations

M. Belkacem\textsuperscript{1}, Z. Aouissat\textsuperscript{2}, M. Bleicher\textsuperscript{1}, H. Stöcker\textsuperscript{1} and W. Greiner\textsuperscript{1}

\textsuperscript{1} Institut für Theoretische Physik, J. W. Goethe-Universität, D-60054 Frankfurt am Main, Germany

\textsuperscript{2} Institut für Kernphysik, Technische Hochschule Darmstadt, Schloßgartenstraße 9, D-64289 Darmstadt, Germany

(March 8, 1999)

Abstract

We investigate event-by-event fluctuations for ensembles with non-fixed multiplicity. Moments of event observable distributions, like total energy distribution, total transverse momentum distribution, etc, are shown to be related to the multi-body correlations present in the system. These moments reduce in the absence of any correlations to the moments of particle inclusive momentum distribution. As a consequence, a zero value for the recently introduced Φ-variable is shown to indicate the vanishing of two-body correlations from one part, and of correlations between multiplicity and momentum distributions from the other part. It is often misunderstood as a measure of the degree of equilibration in the system.
One of the main goals of relativistic heavy-ion collisions is the study of hadronic matter under extreme conditions of temperature and density. This offers the unique opportunity to investigate the possible phase transition of hadronic matter to quark-gluon plasma. As is well known from statistical mechanics [1,2], large fluctuations may occur at such a phase transition. These fluctuations are maximum at the critical point where large portions of the system become strongly correlated. Moreover, the investigation of these fluctuations is now possible with the advent of large acceptance detectors which allow for the first time an event-by-event analysis of the data. Already, event-by-event fluctuations of the transverse momentum distributions have been proposed to provide information about the heat capacity [3–5], or about a possible equilibration of the system [6–10]. On the experimental side, first preliminary result of the NA49 collaboration seem to indicate the absence of any non-statistical fluctuations in the mean transverse momentum distribution for Pb-Pb collisions at 160 AGeV [11].

In this letter, we propose a general method to investigate the presence of many-body correlations and of non-statistical fluctuations in momentum distributions of multiparticle events. The method applies to all p-p, p-A, or A-A collisions. The Φ-value introduced recently by Gazdzicki and Mrowczynski [6] appears to be one of the moments proposed to give evidence for the presence or no of these fluctuations.

We consider the global observable defined for each event by:

\[ Z = \sum_{i=1}^{N} y(\vec{p}_i) \]  

where \( N \) indicates the multiplicity of the event considered and \( y(\vec{p}_i) \) is any function which depends of the momentum of particle \( i \) in the event. This quantity could be e.g. the energy, the transverse momentum, etc. We are interested by the fluctuations of this global observable from event to event and in particular by its moments

\[ \langle Z^k \rangle = \frac{1}{M} \sum_{j=1}^{M} \left[ \sum_{i=1}^{N_j} y(\vec{p}_i) \right]^k \]  

where \( M \) is the total number of events and \( N_j \) indicates the multiplicity of event \( j \).

Let us now consider the N-body distribution function \( f_N(N, \vec{p}_1, \cdots, \vec{p}_N) \) which gives the probability for a system of \( N \) particles that particle 1 has momentum \( \vec{p}_1 \), particle 2 momentum \( \vec{p}_2 \), and so on. Because we want to describe systems with different multiplicity, the distribution function \( f_N(N, \vec{p}_1, \cdots, \vec{p}_N) \) may also depend on the multiplicity \( N \). The distribution function \( f_N(N, \vec{p}_1, \cdots, \vec{p}_N) \) is defined such that

\[ \int d\vec{p}_1 \cdots d\vec{p}_N \ f_N(N, \vec{p}_1, \cdots, \vec{p}_N) = P(N) \]  

where \( P(N) \) is the probability of finding the system with exactly \( N \) particles, regardless of their momenta, and

\[ \sum_{N=0}^{\infty} P(N) = 1 \]
The reduced s-body distribution functions \((s < N)\) for a system of indistinguishable particles is given by [1,2]:

\[
f_s(N, \vec{p}_1, \cdots, \vec{p}_s) = \frac{N!}{(N-s)!} \int dp_{s+1} \cdots dp_N f_N(N, \vec{p}_1, \cdots, \vec{p}_N)
\]  

From the above definitions, we have:

\[
\int dp_1 \cdots dp_s f_s(N, \vec{p}_1, \cdots, \vec{p}_s) = \frac{N!}{(N-s)!} P(N)
\]  

After these definitions, it appears then that the moments of the event variable \(Z\) are defined as:

\[
\langle Z^k \rangle = \sum_{N=0}^{\infty} \int dp_1 \cdots dp_N \left[ \sum_{i=1}^{N} y(\vec{p}_i) \right]^k f_N(N, \vec{p}_1, \cdots, \vec{p}_N)
\]  

and in particular,

\[
\langle Z \rangle = \sum_{N=0}^{\infty} \int dp_1 y(\vec{p}) f_1(N, \vec{p}_1);
\]  

\[
\langle Z^2 \rangle = \sum_{N=0}^{\infty} \left[ \int dp_1 y^2(\vec{p}_1) f_1(N, \vec{p}_1) + \int dp_1 dp_2 y(\vec{p}_1)y(\vec{p}_2) f_2(N, \vec{p}_1, \vec{p}_2) \right];
\]  

\[
\langle Z^3 \rangle = \sum_{N=0}^{\infty} \left[ \int dp_1 y^3(\vec{p}_1) f_1(N, \vec{p}_1) + \int dp_1 dp_2 y^2(\vec{p}_1)y(\vec{p}_2) f_2(N, \vec{p}_1, \vec{p}_2)
\right.
\]

\[
+ \int dp_1 dp_2 dp_3 y(\vec{p}_1)y(\vec{p}_2)y(\vec{p}_3) f_3(N, \vec{p}_1, \vec{p}_2, \vec{p}_3) \right].
\]

and so on for the higher moments. We see then that the fluctuations of the event observable \(Z\) are related to the higher n-body correlations; the second moment is related to 2-body correlations, the third moment to 2- and 3-body correlations and so on. Note also that these moments are related to the possible correlation of the multiplicity of particles to the s-body momentum distributions \(f_s(N, \vec{p}_1, \cdots, \vec{p}_s)\). This result is similar to that obtained by Bialas and Koch [12].

Let us now answer the following question: how do the above defined moments of the event variable \(Z\) reduce in the absence of any correlation? If no correlations are present in the system, the s-body distribution functions, consistent with Eqs.(3-6), read:

---

1What makes the s-body momentum distributions (possibly) depend on the particle multiplicity? It appears that these are precisely the multi-body correlations. On the contrary, in the absence of correlations, the many-body distribution functions consist of a product of one-body distribution functions. In this case, every particle in the system does not feel the presence of the other particles. Then its distribution function should not depend on whether there is only one particle or many of them, hence it should not depend on the multiplicity of particles in the system.
\[ f_s(N, \vec{p}_1, \ldots, \vec{p}_s) = \frac{N!}{(N - s)!} P(N) \tilde{f}_1(\vec{p}_1) \cdots \tilde{f}_1(\vec{p}_s) \]  
with
\[ \int d\vec{p} \tilde{f}_1(\vec{p}) = 1 \]

Eqs.(8-10) then reduce to
\[ \langle Z \rangle = \langle N \rangle \int d\vec{p} y(\vec{p}) \tilde{f}_1(\vec{p}) \]  
\[ \langle Z^2 \rangle = \langle N \rangle \int d\vec{p} y^2(\vec{p}) \tilde{f}_1(\vec{p}) + \left( \langle N^2 \rangle - \langle N \rangle \right) \left[ \int d\vec{p} y(\vec{p}) \tilde{f}_1(\vec{p}) \right]^2 \]  
\[ \langle Z^3 \rangle = \langle N \rangle \int d\vec{p} y^3(\vec{p}) \tilde{f}_1(\vec{p}) + \left( \langle N^2 \rangle - \langle N \rangle \right) \left[ \int d\vec{p} y^2(\vec{p}) \tilde{f}_1(\vec{p}) \right] \left[ \int d\vec{p} y(\vec{p}) \tilde{f}_1(\vec{p}) \right] \\
+ \left( \langle N^3 \rangle - 3\langle N^2 \rangle + 2\langle N \rangle \right) \left[ \int d\vec{p} y(\vec{p}) \tilde{f}_1(\vec{p}) \right]^3 \]

where
\[ \langle N^k \rangle = \sum_{N=0}^{\infty} N^k P(N) \]  

1. Note that in the absence of correlations, the distribution function \( \tilde{f}_1(\vec{p}) \) coincides with the inclusive particle distribution function. Indeed, the normalized inclusive particle distribution function is defined\(^2\) as:
\[ f^{incl}(\vec{p}) = \frac{1}{\langle N \rangle} \sum_{N=0}^{\infty} f_1(N, \vec{p}) \]  

which reduces in the absence of correlations to
\[ f^{incl}(\vec{p}) = \frac{1}{\langle N \rangle} \sum_{N=0}^{\infty} N P(N) \tilde{f}_1(\vec{p}) = \tilde{f}_1(\vec{p}) \]  

\(^2\)The inclusive particle distribution function is defined such as the average value of a given particle observable \( O \) is given by: \( \langle O \rangle = \frac{1}{N_{tot}} \sum_{i=1}^{N_{tot}} O(\vec{p}_i) = \frac{1}{N_{tot}} \sum_{i=1}^{M} \sum_{j=1}^{N_i} O(p^r_{i,j}) \) where \( M \) is the number of events and \( p^r_{i,j} \) is the momentum of particle \( j \) in event \( i \). \( N_{tot} \) is the total number of particles in all events; it is given by \( N_{tot} = \sum_{i=1}^{M} N_i = M\langle N \rangle \) with \( \langle N \rangle = \frac{1}{M} \sum_{i=1}^{M} N_i \). We obtain then \( \langle O \rangle = \frac{M}{N_{tot}} \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{N_i} O(p^r_{i,j}) = \frac{1}{\langle N \rangle} \sum_{N=0}^{\infty} \int d\vec{p} O(\vec{p}) f_1(N, \vec{p}_i) \) (see Eq.(8).
2. Note also that by a judicious choice of the particle variable \( y(\vec{p}) \) in such a way that in the absence of correlations, the mean value of \( y(\vec{p}) \) vanishes,

\[
\int d\vec{p} \ y(\vec{p}) \ \tilde{f}_1(\vec{p}) = 0 \tag{19}
\]

the moments of the event observable \( Z \) will be exactly proportional to the moments of the particle observable \( y(\vec{p}) \). A good choice of the particle variable \( y(\vec{p}) \) is:

\[
y(\vec{p}) = x(\vec{p}) - \bar{x} \tag{20}
\]

where \( x(\vec{p}) \) is any function of momentum \( \vec{p} \) (kinetic-, transverse-energy, transverse momentum, ... etc) and

\[
\bar{x} = \int d\vec{p} \ x(\vec{p}) \ f^{incl}(\vec{p}) \tag{21}
\]

Note that with this choice, and according to Eqs.(8,17), the average value of \( Z \) is always zero, even in the presence of strong correlations.

It appears then that, in the absence of any correlations in the system and by a judicious choice of the particle variable \( y(\vec{p}) \), the moments of the event variable \( Z \) are exactly proportional to the inclusive moments of the particle variable \( y(\vec{p}) \):

\[
\langle Z \rangle = \langle N \rangle \int d\vec{p} \ y(\vec{p}) \ f^{incl}(\vec{p}) \equiv 0 \tag{22}
\]

\[
\langle Z^2 \rangle = \langle N \rangle \int d\vec{p} \ y^2(\vec{p}) \ f^{incl}(\vec{p}) \tag{23}
\]

\[
\langle Z^3 \rangle = \langle N \rangle \int d\vec{p} \ y^3(\vec{p}) \ f^{incl}(\vec{p}) \tag{24}
\]

The proportionality factor is the average number of particles \( \langle N \rangle \). If multi-body correlations are present in the system, Eqs.(23,24) do not hold anymore and the difference of the lhs and the rhs of both equations will not vanish.

In conclusion, we have investigated the moments of event observable distributions for systems with non-fixed multiplicity. These moments are shown to be related to the higher many-body correlations. In the absence of any correlations, these moments reduce to the moments of inclusive particle momentum distribution. Moreover, we have shown that a non-zero value for the quantities

\[
\frac{\langle Z^k \rangle}{\langle N \rangle} - \bar{y}^k; \quad k > 1 \tag{25}
\]

with

\[
\bar{y}^k = \int d\vec{p} \ y^k(\vec{p}) \ f^{incl}(\vec{p}) \tag{26}
\]
indicates the presence of strong many-body correlations and non-statistical fluctuations in the system. A non-vanishing value of, e.g., $\frac{\langle Z^2 \rangle}{\langle N \rangle} - y^2$ indicates the presence of two-body correlations from one part and of possible correlations between the particle multiplicity and the momentum distributions from the other part. The absence of correlations between the particle multiplicity and the momentum distributions alone does not necessarily imply a zero value for this quantity [6]. The generalization of this method to different particle distributions like, e.g., hadronic type distributions [7] is straightforward. In this case, the n-particle momentum distribution functions $f_N(N, p_1, \cdots, p_N)$ would be replaced by the n-particle hadronic-type distribution functions $f_N(N, h_1, \cdots, h_N)$, where $h_i$ indicates the hadronic-type of particle $i$. The normalization of $f_N(N, h_1, \cdots, h_N)$ would then be:

$$\sum_{h_1=1}^{N_h} \cdots \sum_{h_N=1}^{N_h} f_N(N, h_1, \cdots, h_N) = P(N)$$

(27)

where $N_h$ indicates the number of all possible hadron types. The event observable in this case is defined as

$$Z = \sum_{i=1}^{N} y(h_i)$$

(28)

In this case, the vanishing of the quantities, Eq.(25) would indicate an uncorrelated hadron-type production in the system.

A remark here is in order: There is a common misinterpretation of the quantity $\Phi = \sqrt{\frac{\langle Z^2 \rangle}{\langle N \rangle} - y^2}$: it has been argued that a zero-value of this quantity would indicate an equilibration (thermal or chemical) of the system [7,10,11]. It appears from our study that such a zero value merely indicates the absence of 2-body correlations and of correlations of the particle multiplicity with momentum distributions. No specific form for the distribution functions was assumed. It is true that a possible equilibration of the system implies a completely uncorrelated system and would give a zero value for this quantity and all higher moments, but a zero value for $\Phi$ does not necessarily imply a complete equilibration of the system.

This work was supported by BMBF, GSI, DFG and Graduiertenkolleg "Schwerionenphysik". M. Bleicher was supported by the Josef Buchmann Foundation.
REFERENCES