A Conformal Hyperbolic Formulation of the Einstein Equations

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We propose a re-formulation of the Einstein evolution equations that cleanly separates the con-
formal degrees of freedom and the non-conformal degrees of freedom with the latter satisfying a first
order strongly hyperbolic system. The conformal degrees of freedom are taken to be determined by
the choice of slicing and the initial data, and are regarded as given functions (along with the lapse
and the shift) in the hyperbolic part of the evolution.

We find that there is a two parameter family of hyperbolic systems for the non-conformal degrees
of freedom for a given set of trace free variables. The two parameters are uniquely fixed if we
require the system to be “consistently trace-free”, i.e., the time derivatives of the trace free variables
remains trace-free to the principal part, even in the presence of constraint violations due to numerical
truncation error. We show that by forming linear combinations of the trace free variables a conformal
hyperbolic system with only physical characteristic speeds can also be constructed.

a. INTRODUCTION. With the advent of large
amounts of observational data from high-energy astron-
omy and gravitational wave astronomy, general relativis-
tic astrophysics — astrophysics involving gravitational
fields so strong and dynamical that the full Einstein field
equations are required for its accurate description — is
emerging as an exciting research area. This calls for an
understanding of the Einstein theory in its non-linear and
dynamical regime, in order to study the physics of gen-
eral relativistic events in a realistic astrophysical envi-
ronment. This in turn calls for solving the full set of Einstein
equations numerically. However, the complicated set of
partial differential equations present major difficulties in
all of these three tightly coupled areas: the understand-
ing of its mathematical structure, the derivation of its
physical consequences, and its numerical solution. The
difficulties have attracted a lot of recent effort, includ-
ing two “Grand Challenge” [1,2] efforts on the numerical
studies of black holes and neutron stars, respectively.

One major obstacle in solving the Einstein equations
numerically is that we lack a complete understanding of
the mathematical structure of the Einstein equations.
The difficulties in numerically integrating the Einstein
equations in a stable fashion have motivated intense ef-
fort in rewriting the Einstein equations into a form that
is explicitly well-posed [3–18] (for an excellent overview
see [19]). The main idea has been re-writing the six
space-space components of the Einstein equations into
a first order hyperbolic system. These space-space parts
of the Einstein equations are dynamical evolution equa-
tions, while the time-time and space-time parts of the
Einstein equations are (elliptic) constraint equations.

The central question we raise in this communication
is: In order to enable an accurate and stable numerical
integration of the full set of the Einstein equations, what
part of the system should be taken to form a hyperbolic
system?

Our question is motivated by two observations. First,
there are many recent proposals on re-formulating the six
space-space parts of the Einstein equations into a first
order hyperbolic system [8–18]. Three of the hyperbolic
formulations have been coded up for numerical treatment
(to the best of our knowledge), namely, the York et. al.
formulation [9,14,16] (see, e.g., [20,21]), the Bona-Masso
formulation [15] (see e.g., [17]), and Friedrich’s formulation
[3,4,22–26] (which is rather different from the first
two formulations in its use of a global conformal trans-
formation of the four-metric to compactify hyperboloidal
slices). However, in all these cases, the numerical integra-
tion of the first order hyperbolic system consisting of the
six space-space components of the Einstein equations so
far have not lead to a substantial improvement over those
using the traditional ADM [27] evolution equations. This
is despite the original hope that the well-posedness of the
hyperbolic formulations leads to an immediate numerical
advantage.

The second observation is that there have been vari-
ous attempts in re-writing the traditional ADM form of
the evolution equations by separating out the conformal
degree of freedom, beginning with Nakamura et. al. [28]
(see references cited therein). Lately this has received
much attention with [29] reporting that a variant of the
approach leads to highly stable numerical evolutions. A
detailed study of the approach using gravitational wave
systems carried out by our group [30] confirmed that the
approach has advantages over the standard ADM for-
mulation. We find that the approach yields results with accuracy comparable to that obtained by the standard ADM formulation with the K-driving technique [31] for weak to medium waves, and has better stability properties especially in the case of strong fields that needs high resolution with ADM [32] (see also [33]).

These two observations motivated us to study the possibility of a formulation that separates out the conformal degree of freedom in the 6 evolution equations, while requiring the remaining 5 equations governing the non-conformal degrees of freedom to form a first order hyperbolic system.

A re-cap of the various components of the Einstein equations is in order for a clearer discussion of our approach. In the standard ADM 3+1 formulation, the Einstein equations are broken into (a) the Hamiltonian constraint equation (the time-time part),

\[ H = (3)R + K^2 - K_{ij}K^{ij} - 16\pi p_{ADM} = 0, \]  

(b) the 3 momentum constraint equations (the time-space part)

\[ H^i = \nabla_j K^{ij} - \gamma^{ij} \nabla_j K - 8\pi j^i = 0, \]  

where \( p_{ADM}, j^i, S_{ij}, S = g^{ij}S_{ij} \) are the components of the stress energy tensor projected onto the 3-space, and (c) the 6 evolution equations (the space-space part) given as 12 first order equations

\[ \partial_t g_{ij} = -2\alpha K_{ij}, \]  \[ \partial_t K_{ij} = -\nabla_i \nabla_j \alpha + \alpha(R_{ij} + K K_{ij} - 2K_{im}K^{mj}) - 8\pi (S_{ij} - \frac{1}{2}g_{ij}S) - 4\pi p_{ADM}, g_{ij}), \]

where \( \nabla_i \) denotes a covariant derivative with respect to the 3-metric \( g_{ij} \), \( \partial_t \) stands for \( \partial_t - L_j \) with \( L_j \) being the Lie derivative with respect to \( \beta^i \), and \( R_{ij} \) is the Ricci curvature of the 3-metric. In the ADM formulation, Eqs. (3, 4) are used to evolve the 12 variables \( K_{ij}, g_{ij} \) forward in time for given lapse \( \alpha \) and shift vector \( \beta^i \). The constraint equations are automatically satisfied if \( \{K_{ij}, g_{ij}\} \) satisfy them on the initial time slice. However, in numerical evolutions the constraints will be violated due to truncation error. One major difficulty in numerical relativity is that the constraint violations often drive the development of instabilities, at least in the case of numerical evolution using the standard ADM equations (3, 4).

In the hyperbolic re-formulations of the evolution equations [9–12,14–17], one makes use of the constraint equations (1,2), and introduces additional variables (e.g., \( d_{ijk} = g_{ij,k} \) or its linear combinations) to cast Eqs. (3, 4) into a first order strongly hyperbolic system (often the symmetric hyperbolic subclass). (More variables would have to be introduced for formulations involving higher derivatives [9,11].) However, we note that hyperbolicity is often shown only under the assumption that some of the variables involved in the evolution equations, in particular the lapse \( \alpha \) and the shift \( \beta^i \), are considered as given functions of space and time. In actual numerical evolutions with no pre-determined choice of spacetime coordinates, \( \alpha \) and \( \beta^i \) have to be given in terms of the variables \( \{K_{ij}, g_{ij}, d_{ijk}\} \) (e.g., \( \alpha, \beta^i \) determined in a set of elliptic equations involving \( \{K_{ij}, g_{ij}, d_{ijk}\} \)). In the Bona-Masso formulation [15], the lapse can be part of the hyperbolic system for some choice of slicings (while the inclusion of the shift into the hyperbolic system severely restricts the class of applicable shifts). In [9,11], in addition to the lapse and the shift, the trace of the extrinsic curvature, \( K = g^{ij}K_{ij} \), is also regarded as a given function (\( K \) is used to specify the slicing, e.g., \( K = 0 \) for maximal slicing). The point we want to bring out here is that in all of the existing hyperbolic re-formulations of the Einstein evolution equations, part of the quantities \( \{K_{ij}, g_{ij}, d_{ijk}, \alpha, \beta^i\} \) are considered to be given, while others are evolved using hyperbolic equations.

In the following, we present a formulation in which the non-conformal degrees of freedom are separated out for hyperbolic evolution.

b. FORMULATION. For the evolution of the three-geometry, the conformal degree of freedom is represented by \( g \) (the determinant of the spatial 3-metric \( g_{ij} \)), its spatial derivative \( g_i \), and its time derivative \( K = -1/(2g_{ij})\partial_j g \). For the non-conformal degrees of freedom, we define

\[ \tilde{g}_{ij} = g_{ij}/g^{1/3}, \]  

\[ \tilde{A}_{ij} = (K_{ij} - \frac{1}{3}\tilde{g}_{ij}K)/g^{1/3}, \]  

\[ \tilde{D}^{ij}_{k} = \tilde{g}^{ij}_{,k}. \]  

\( \tilde{g}_{ij} \) has unit determinant, and \( \tilde{A}_{ij} \) is the rescaled trace-free part of \( K_{ij} \). All indices of tilde quantities are raised or lowered with the conformal 3-metric \( \tilde{g}_{ij} \). We note that \( \tilde{D}^{ij}_{k} \) is trace-free with respect to the indices \( (i,j) \). We take \( \{\tilde{g}^{ij}, \tilde{D}^{ij}_{k}, \tilde{A}^{ij}\} \), or their covariant component counterparts, to represent the non-conformal degrees of freedom.

In the following we develop a first order hyperbolic system for the non-conformal degrees of freedom, under the simplifying assumption that the 5 conformal degrees of freedom \( \{g, g_i, K\} \) and the gauge choice functions \( \{\alpha, \beta^i\} \) can be regarded as given functions of space and time. Note that these variables cannot be specified independently of each other. A concrete example is that of maximal slicing, \( K = 0 \), and vanishing shift, \( \beta^i = 0 \), in which case both \( g \) and \( g_i \) are part of the initial data (time independent), and are therefore truly given functions in the numerical evolution. In other cases, with \( K \) given to specify the slicing, it involves a non-trivial time integration to determine \( g \) (from the definition of \( K \) in terms of the time derivative of \( g \)).
We now discuss hyperbolicity of the evolution of the non-conformal variables, \( \{ \tilde{g}_{ij}, \tilde{D}^{ij} k, \tilde{A}^{ij} \} \), by examining the principal part of the evolution equations, which is the part that decides about strong hyperbolicity of the system [34]. To obtain the principal part we drop all terms that can be expressed by (1) the variables \( \{ \tilde{g}^{ij}, \tilde{A}^{ij} \} \) themselves, and (2) spacetime functions that are regarded as given, i.e. \( \{ \alpha, \beta, g, g_{i}, K \} \) and their space and time derivatives. We have

\begin{align}
\partial_t \tilde{g}_{ij} &\approx 0, \\
\partial_t \tilde{D}^{ij} k &\approx 2 \alpha \tilde{A}^{ij} k, \\
\partial_t \tilde{A}^{ij} &\approx \alpha g^{1/3} (R^{ij} - \frac{1}{3} \tilde{g}^{ij} R),
\end{align}

where \( \approx \) represents “equal up to principal part”, and where for the evolution equation of \( \tilde{D}^{ij} k \) we have used that spatial derivatives \( \partial_i \) and the time derivative \( \partial_t \) commute.

To evaluate \( R^{ij} \) and \( R \) in (10), we use

\begin{align}
R^{ij} &\approx g^{-2/3} \tilde{R}^{ij} \\
&\approx \frac{1}{2} g^{-2/3} (\tilde{g}^{kl} \tilde{D}^{ijkl} - \tilde{g}^{ij} \tilde{D}^{ijkl} - \tilde{g}^{jl} \tilde{D}^{ikl} - \tilde{g}^{il} \tilde{D}^{jkl}),
\end{align}

where the relation

\begin{equation}
g_{kl} \tilde{g}^{kl} = -g_{ij}/g \approx 0,
\end{equation}

and the spatial derivatives of it have been used. We obtain

\begin{align}
\partial_t \tilde{A}^{ij} &\approx \frac{1}{2} \alpha g^{-1/3} (\tilde{g}^{kl} \tilde{D}^{ijkl} - \tilde{g}^{ij} \tilde{D}^{ijkl} - \tilde{g}^{jl} \tilde{D}^{ikl} - \tilde{g}^{il} \tilde{D}^{jkl}) \\
&\quad + \frac{2}{3} \tilde{g}^{ij} \tilde{D}^{ijkl},
\end{align}

To make the non-conformal system strongly hyperbolic, one can add a combination of the momentum constraint (2) is \( H^j \approx g^{-1/3} \tilde{A}^{ij} \). We obtain

\begin{align}
\partial_t \tilde{D}^{ij} k &\approx 2 \alpha \tilde{A}^{ij} k - 2 \alpha g^{1/3} (\tilde{g}_{k} H^{j} + \tilde{g}_{j} H^{i}) \\
&\approx 2 \alpha (\tilde{A}^{ij} k - \tilde{g}_{k} \tilde{A}^{ij} - \tilde{g}_{j} \tilde{A}^{ik} l).
\end{align}

An energy norm can be constructed for the system:

\begin{equation}
E = \int \tilde{g}^{ij} \tilde{g}_{ij} + \tilde{A}^{ij} \tilde{A}_{ij} + \frac{1}{4} g^{-1/3} \tilde{D}^{ij} k \tilde{D}^{ijkl}.
\end{equation}

It is straightforward to demonstrate using (8), (14), and (16) that \( \partial_t E \) is a total derivative up to terms that can be expressed by the variables \( \{ \tilde{g}^{ij}, \tilde{A}^{ij}, \tilde{D}^{ij} k \} \) themselves. One can also show directly that the characteristic metric of the system (8), (14), and (16) has a complete set of eigenvectors with real eigen values. The system is similar to but not contained in the one parameter family of the hyperbolic systems constructed in [10].

Next we go one step beyond hyperbolicity. We make the following observations:

(i) Since \( \tilde{A}^{ij} \) and \( \tilde{D}^{ij} k \) are trace-free, one can add a term \( \epsilon_1 \alpha g^{-1/3} \tilde{g}^{ij} H \) to (14), and a term \( \epsilon_2 \alpha g^{ij} H_k \) to (16) without affecting the hyperbolicity. We have therefore a two parameter family of hyperbolic evolution equations (without making a variable change).

(ii) With these two terms added respectively to (14) and (16), the trace of the principle parts of the RHS’s of the equations are \( 3 \epsilon_1 \alpha \tilde{D}^{k} k, \) (proportional to the principal part of the Hamiltonian constraint), and \( \alpha (3 \epsilon_2 - 4) \tilde{A}^{ij} \) (proportional to the principal part of the momentum constraint), respectively. On the other hand, the LHS of the equations, \( \partial_t \tilde{A}^{ij} \) and \( \partial_t \tilde{D}^{ij} k \) are trace-free to the principal order. This means that truncation error in the numerical evolution which leads to a violation of the constraints will drive \( \tilde{A}^{ij} \) and \( \tilde{D}^{ij} k \) to evolve away from being trace-free, even up to the principal order.

(iii) We therefore propose to fix the freedom in the parameters \( \epsilon_1 \) and \( \epsilon_2 \) by requiring the system to be “consistently trace-free”, i.e., \( \epsilon_1 = 0 \) and \( \epsilon_2 = 4/3 \), so that the equations are trace-free to principal order consistently. Hence (14) for \( \tilde{A}^{ij} \) is left unchanged, but

\begin{align}
\partial_t \tilde{D}^{ij} k &\approx 2 \alpha \tilde{A}^{ij} k - 2 \alpha g^{ij} (\tilde{g}_{k} H^{j} + \tilde{g}_{j} H^{i}) + \frac{4}{3} \alpha \tilde{g}^{ij} H_k \\
&\approx 2 \alpha (\tilde{A}^{ij} k - \tilde{g}_{k} \tilde{A}^{ij} - \tilde{g}_{j} \tilde{A}^{ik} l + \frac{2}{3} \tilde{g}^{ij} \tilde{g}_{km} \tilde{A}^{ml} n).
\end{align}

The system \( \{8\}, \{14\}, \{19\} \) forms a strongly hyperbolic system with the same energy norm (17).

(iv)The remaining freedom in constructing conformal-hyperbolic systems that are “consistently trace-free” is through forming linear combinations of the variables. There are clearly infinite choices. Here we show for example a linear combination that leads to a system with only physical characteristic speeds, a property advocated by York et. al., see e.g., [14]. (14) can be written as

\begin{equation}
\partial_t \tilde{U}^{ij} k \approx \alpha g^{-1/3} \tilde{g}^{ij} \partial_t \tilde{U}^{ij} k \approx \alpha g^{ij} \partial_t \tilde{U}^{ij} k,
\end{equation}

where

\begin{equation}
\tilde{U}^{ij} k = \frac{1}{2} (\tilde{D}^{ij} k - \tilde{g}_{k} \tilde{D}^{ij} l - \tilde{g}_{l} \tilde{D}^{ij} k + \frac{2}{3} \tilde{g}^{ij} \tilde{g}_{km} \tilde{D}^{ml} n).
\end{equation}

We can take \( \tilde{U}^{ij} k \) to be our basic non-conformal variables (note \( \tilde{g}_{ij} \tilde{U}^{ij} k = 0 \). Taking the time derivative of \( \tilde{U}^{ij} k \) and commuting time and space derivatives leads to

\begin{equation}
\partial_t \tilde{U}^{ij} k \approx \alpha (\tilde{A}^{ij} k - \tilde{g}_{k} \tilde{A}^{ij} - \tilde{g}_{j} \tilde{A}^{ik} l + \frac{2}{3} \tilde{g}^{ij} \tilde{g}_{km} \tilde{A}^{ml} n).
\end{equation}

To make the system strongly hyperbolic, we follow the step leading to (15) and add the combination of momentum constraints \( \alpha g^{ij} (\tilde{g}_{k} H^{j} + \tilde{g}_{j} H^{i}) - 2 \alpha g^{ij} H_k/3 \) to (22) to arrive at
\[ \partial_t \tilde{U}^{ij}_{k} \approx \alpha \tilde{A}^{ij}_{k}. \]  

(23)  

(20) and (23) form a conformal hyperbolic system for \( \{ \tilde{U}^{ij}_{k}, \tilde{A}^{ij} \} \) with only physical characteristic speeds. The system can be symmetrized by contracting (23) with \( g^{kl} \).

c. **DISCUSSION AND CONCLUSION.** We raise the question of what part of the variables in the Einstein theory should be evolved in a hyperbolic fashion in numerical relativity. We propose a re-formulation of the Einstein evolution equations that cleanly separate the conformal degrees of freedom \( \{ g, g_{,k}, K \} \) and the non-conformal degrees of freedom \( \{ \tilde{g}^{ij}, \tilde{D}^{ij}_{k}, \tilde{A}^{ij} \} \) (or their linear combinations), with the latter satisfying a first order strongly hyperbolic system. The conformal degrees of freedom are taken to be determined by the choice of slicings and the initial data, and are regarded as given functions in the hyperbolic part of the evolution equations, along with the lapse and the shift.

We find a two parameter family of non-conformal hyperbolic system for \( \{ \tilde{g}^{ij}, \tilde{D}^{ij}_{k}, \tilde{A}^{ij} \} \). The two parameters are uniquely fixed if we require the system to be “consistently trace-free”, i.e., the time derivative of the trace-free variables \( \{ \tilde{g}^{ij}, \tilde{D}^{ij}_{k}, \tilde{A}^{ij} \} \) remains trace-free to principal part, even in the presence of constraint violations caused by numerical truncation error. We also show that certain linear combinations of the \( \tilde{D}^{ij}_{k} \) lead to a conformal hyperbolic system with physical characteristic speed.

This formulation merges two recent trends in rewriting the Einstein evolution equations for numerical relativity: first order hyperbolicity and the separating out of the conformal degrees of freedom. We believe it will lead to many interesting investigations: Given the coordinate conditions, e.g., maximal slicing and an appropriate shift condition, can the combined elliptic hyperbolic system be shown to be well-posed analytically \([9,35]\)? When posted as initial boundary value problem, what are the suitable boundary conditions for stability in numerical evolutions? How will the constraints propagate under this system of conformal-hyperbolic equations? One particularly interesting issue that will be reported on in a follow up paper is the stability of this formulation in numerical evolution, and how the stability is related to the slicing conditions \( (K) \) one chooses.

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