Central charges and Hamiltonians for 0-brane dynamics

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We consider the general properties of central charges of 0-branes and associated duality invariants, in view of their double role, on the bulk and on the world volume (quantum mechanical) theory. A detailed study of the BPS condition for the mass spectrum arising from toroidal compactifications is given for 1/2, 1/4, and 1/8 BPS states in any dimension. As a by-product, we retrieve the U -duality invariant conditions on the charge (zero mode) spectrum and the orbit classification of BPS states preserving different fractions of supersymmetry. The BPS condition for 0-branes in theories with 16 supersymmetries in any dimension is also discussed.
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I. INTRODUCTION

In recent times, the role of the duality symmetries of a dynamical theory encompassing quantum gravity has received increasing attention in several contexts. Particular examples where the duality takes an important role, especially in connection with nonperturbative properties, is the anti-de Sitter (AdS)/conformal field theory (CFT) correspondence [1], related to the horizon geometry of p -branes and their world-volume conformal field theory description. Another example is the connection between M-theory compactified on a torus T^d [2–6] and $(d+1)$ Yang-Mills theory compactified on the dual torus $\tilde{T}^d$.

More closely related to the latter is the recent investigation of D-brane Born-Infeld actions and the role played by duality in explaining several properties of their Hamiltonian formulation and the corresponding energy spectrum of Bogomol'nyi-Prasad-Sommerfield (BPS) states [7]. In this framework it is believed that Born-Infeld non-Abelian gauge theories with nontrivial Ramond-Ramond (RR) backgrounds are naturally described by some generalization of gauge theories on noncommutative tori [8].

The framework of noncommutative geometry offers, for instance, a new interpretation of the T -duality group $O(d, d; \mathbf{Z})$ of quantum mechanical systems obtained by compactifying the Born-Infeld action of D-branes on T^d . The latter occurs in type II string theory compactified on T^d [9–12].

These quantum mechanical systems have also been shown [13], at least for $d \leq 4$, to exhibit the full extended U -duality symmetry¹ $E_{d+1(d+1)}$, rather than the smaller symmetry

$E_{d(d)}$ present in matrix gauge theory on T^d , where it appears as an extension of the geometrical symmetry $SL(d)$ [4,6].

In previous investigations, the central charge matrix Z for 0-branes played a role, not only as a central extension of supersymmetry algebra in theories with a nontrivial 0-brane background metric, but also as an effective potential of the geodesic action of a one-dimensional Lagrangian system derived from the bulk Einstein-Maxwell Lagrangian, in the presence of moduli fields $\{\phi\}$ and quantized charges q_A of 0-branes [14,15].

The critical points of this potential were seen to determine the Bekenstein-Hawking entropy formula as the extremization of the Weinhold potential [15],

$$W = \frac{1}{2} \text{Tr}(ZZ^\dagger), \quad (1)$$

or equivalently of the BPS mass $m_{BPS} = |Z_h|$ where $|Z_h|$ is the highest eigenvalue of $\sqrt{ZZ^\dagger}$ [17].

In the world-volume description of 0-branes, the very same function W appears as the Hamiltonian of the 0-brane quantum mechanics [7,13,16]:

$$H_\phi(\hat{q}) = \sqrt{\frac{1}{N} \text{Tr}[ZZ^\dagger(\phi, \hat{q})]}, \quad (2)$$

where the quantized charges are replaced by a set of Hamiltonian variables $\hat{q}$, which belong to the same duality multiplet as the quantized charges of the bulk supergravity theory in the presence of 0-brane sources.

The appearance of the central charge in the 0-brane action in arbitrary $D=4$ supergravity backgrounds has recently been shown to occur as a consequence of κ supersymmetry [18].

¹In this paper U duality will mean both classical and quantum U dualities.

In this framework the energy spectrum of the Hamiltonian (2) is given by the BPS mass formula of the effective supergravity theory [7]:

$$m_{BPS} = |Z_h(\phi, q_0)|, \quad (3)$$

where the Hamiltonian variables $\hat{q}$ are replaced by their zero-mode part q_0 which eventually coincide with the same duality multiplet of the quantized charges of the bulk theory, but now with the interpretation of “fluxes” and “momenta” of the world-volume Hamiltonian description [4,6].

These zero modes fill representations of the U -duality group $E_{d+1(d+1)}(\mathbf{Z})$ for systems with maximal supersymmetry and the BPS spectrum preserves some fraction of supersymmetry depending on the particular orbit of the charge vector state [19–21].

Note that the BPS energy $|Z_h(\phi, q_0)|$ is not the same as replacing in $\sqrt{ZZ^\dagger}(\phi, \hat{q})$ the zero mode q_0 of q , unless the states are 1/2 BPS [7], which, as we will see, can only occur if the charge duality multiplet satisfies some duality-invariant conditions. At the classical level, where the charges are continuous, this is equivalent to saying that the zero-mode part belongs to a particular orbit of the charge vector representation of the duality group G .

It is the aim of the present investigation to derive general formulas of the energy spectrum for any torus T^d , $d = 1, \dots, 6$, and provide a new derivation of the different BPS conditions in terms of the U -duality-invariant constraints, retrieving then the analysis of Ferrara and Maldacena [19] as well as the classification of Ferrara and Gunaydin [20] and Lu, Stelle, and Pope [21]. We deal also with the case of 0-branes in theories in any dimension with 16 supersymmetries. This is interesting because it is related to heterotic strings compactified on T^d or type II theories compactified on more general manifolds (such as K_3).

The paper is organized as follows:

In Sec. III we consider systems compactified on T^d , $d = 1, \dots, 4$, where only 1/2 and 1/4 BPS states occur.

In Sec. IV we consider the richer structure occurring for $d = 5, 6$ where a complete understanding of the world-volume theory is still missing.

In Sec. V the BPS conditions are derived for the case of theories with 16 supersymmetries in any dimension.

II. CENTRAL CHARGES AND GEOMETRICAL TOOLS OF COSET SPACES

In the present section we review the central charges for 0-branes in theories with maximal supersymmetry and the BPS conditions on the U multiplet of quantized charges which entail different orbits of the duality group which preserve different fractions of supersymmetry. The analysis for theories with 16 supersymmetries will be considered in the last section.

The supergravity theories describing these systems can be obtained in three different ways, by compactifying M-theory on T^{d+1} [$(d+1)$ -dimensional torus] or type IIA and type IIB string theories on T^d . We will consider here supergravity

theories compactified down up to $D=4$ space-time dimensions ($d=1, \dots, 6$).

Some of the results presented here overlap with previous analysis for $d=1, \dots, 4$, when only 1/2 or 1/4 BPS states are present [5,7,16]. The analysis of $d=5, 6$ is essentially novel although the BPS conditions for 1/2, 1/4 and 1/8 BPS states were previously discussed in the literature and the orbit classifications derived [19–21].

A. R Symmetry and U duality

The supersymmetry algebra of type II string theory compactified on T^d down to $10-d$ dimensions has an R -symmetry group and a continuous duality group which depends on d . The R symmetry is given below [22]:

R -symmetry group H

$$\begin{aligned} d=1 & \quad U(1), \\ d=2 & \quad SU(2) \times U(1), \\ d=3 & \quad USp(4) \approx O(5), \\ d=4 & \quad USp(4) \times USp(4) \approx O(5) \times O(5), \\ d=5 & \quad USp(8), \\ d=6 & \quad SU(8). \end{aligned} \quad (4)$$

The U -duality groups G are $E_{d+1(d+1)}$ [22], and the R -symmetry groups are their maximal compact subgroups $H \equiv R$. The quantum U -duality groups are $E_{d+1(d+1)}(\mathbf{Z})$ [23]. Because of the connection between M-theory and string theory, the groups $E_{d+1(d+1)}$ contain both

$$GL(d+1) \subset E_{d+1(d+1)}, \quad (5)$$

which is the classical isometry group of the moduli space of a T^{d+1} torus in M-theory, and

$$\begin{aligned} O(1,1) \times O(d,d) & \subset E_{d+1(d+1)} \quad (d \neq 6), \\ Sl(2) \times O(6,6) & \subset E_{7(7)} \quad \text{for } d=6, \end{aligned} \quad (6)$$

which is the S - T duality group of string theory [24,25,23].

In string theory the $O(d,d)$ group combines the geometric isometry of the T^d torus $GL(d)$ with the shift of the antisymmetric tensor $B_{ij} \rightarrow B_{ij} + N_{ij}$ while the $O(1,1)$ factor corresponds to the dilaton shift. The group $E_{d+1(d+1)}$ emerges from the combination of $Sl(d+1)$ with $O(d,d)$ and this operation gives rise to nonperturbative symmetries which combine the Neveu-Schwarz–Neveu-Schwarz (NS-NS) and Ramond-Ramond (RR) fields in supermultiplets.

The spinorial charges of the supersymmetry algebra transform in representations of the R -symmetry group and this implies that the central charges of interest to us have certain symmetry and reality properties.

In the case of Lorentz scalar central charges, appropriate to 0-branes, the classification goes as follows: the central charge matrix $Z(\phi, q)$ is in the same representation of the R symmetry as the vector fields A_μ of the corresponding theory. This gives the following result:

TABLE I. Duality multiplets.

d	Supergravity vector fields	Born-Infeld YM fluxes
IIA		
2	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$	$\int \text{Tr } F_{ij}, \int \text{Tr } P_i, \int \text{Tr } E_i, \text{rank}$
4	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$ $A_{\mu ijkl}^D$	$\int \text{Tr } F_{ij} F_{kl}, \int \text{Tr } P_i, \int \text{Tr } E_i, \int \text{Tr } F_{ij}$ rank
6	$Z_\mu, g_{\mu i}, b_{\mu i}, A_{\mu ij}$ $A_{\mu ijkl}^D, Z_{\mu ijklpq}^D$ $b_{\mu ijklp}^{NS}, g_{\mu i}^D$	$\int \text{Tr } F_{ij} F_{kl} F_{pq}, \int \text{Tr } P_i, \int \text{Tr } E_i, \int \text{Tr } F_{ij} F_{kl}$ $\int \text{Tr } F_{ij}, \text{rank}$
IIB		
1	$g_{\mu 1}, b_{\mu 1}, b_{\mu 1}^C$	$\int \text{Tr } P, \int \text{Tr } E, \text{rank}$
3	$g_{\mu i}, b_{\mu i}, b_{\mu i}^C, A_{\mu ijk}$	$\int \text{Tr } P_i, \int \text{Tr } E_i, \int \text{Tr } F_{ij}, \text{rank}$
5	$g_{\mu i}, b_{\mu i}, b_{\mu i}^C, A_{\mu ijk}$ $b_{\mu ijklp}^D$ $b_{\mu ijklp}^{NS}$	$\int \text{Tr } P_i, \int \text{Tr } E_i, \int \text{Tr } F_{ij} F_{kl}, \int \text{Tr } F_{ij}$ rank

Central charge representation of the R -symmetry

- $d=1$ **3** of $O(2)$ (real symmetric tensor),
 $d=2$ **3₁** of $SU(2) \times U(1)$ (complex triplet),
 $d=3$ **10** of $USp(4)$ (real antisymmetric tensor),
 $d=4$ **16** of $USp(4) \times USp(4)$
 [bispinor (4,4) of $O(5) \times O(5)$],
 $d=5$ **27** of $USp(8)$ (Ω -traceless symplectic
 antisymmetric tensor),
 $d=6$ **28** of $SU(8)$ (complex antisymmetric tensor). (7)

The previous results follow both from a dynamical reduction of the 11- or 10-dimensional supergravities with 32 supercharges or by an analysis of extended superalgebras in the appropriate dimensions [26,27].

Since in the original IIA theory there is only one D 0-brane (one scalar central charge) [26], all the charges in lower dimensions come by wrapping branes on the torus cycles, other than momenta and string windings. In the type IIB on T^d , 0-branes emerge as momenta, string windings, and D-branes wrapped on the torus cycles.

If we want to discuss quantum mechanical systems emerging from $d+1$ Born-Infeld Lagrangians compactified on T^d , we must consider IIA D-branes compactified on even dimensional tori and IIB D-branes compactified on odd dimensional tori.

The world-volume description of the central charges Z and quantized charges q is fairly well understood for the case of T^d with $d=1, \dots, 4$. The U -duality multiplets of the quantized charges q correspond to fluxes, momenta, instanton number, and rank of the gauge groups in the world-volume Yang-Mills (YM) theory. The moduli dependent central charge determines the Hamiltonian of the quantum mechanical system as well as the energy spectrum of the BPS states [7,6].

The main role played by duality is that the central charge vector extends the representation of the R -symmetry group to

a representation of the full duality group $E_{d+1(d+1)}$ acting on the vector field strength. The relevant extensions are as follows [22]:

0-brane representation of the U -duality group

- $d=1$ **2+1** of $E_2 = SL(2) \times O(1,1)$,
 $d=2$ **(3,2)** of $E_3 = SL(3) \times SL(2)$,
 $d=3$ **10** of $E_4 = SL(5)$,
 $d=4$ **16** of $E_5 = O(5,5)$,
 $d=5$ **27** of $E_{6(6)}$,
 $d=6$ **56** of $E_{7(7)}$. (8)

The moduli space of these theories is G/H . At the string level this space can be quotiented further by $E_{d+1}(\mathbf{Z})$ [23,24], something similar to the fundamental domain versus the half plane for the prototype $SL(2, \mathbf{R})/O(2)$.

In Table I we present the U -duality multiplets for 0-branes in the bulk and world-volume descriptions [7].

The gauge fields of type II supergravity theory, as well as the Yang-Mills world-volume fluxes, complete U -duality multiplets of $E_{d+1(d+1)}$ for $d=1, \dots, 4$. These multiplets are obtained by the $E_{d(d)}$ flux and momenta multiplets of matrix gauge theory on T^d and by adding an $E_{d(d)}$ singlet, the rank of the gauge group. For $d=5,6$, the Yang-Mills theory side misses some states corresponding to a NS five brane and Kaluza-Klein (KK) monopoles [4,6].

The coset spaces G/H ($G \equiv U, H \equiv R$, in our case) can be described by choosing a representative in each equivalence class. If ϕ denotes the coordinates of a point in G/H , then the coset representative will be given by an element $L(\phi) \in G$, in the equivalence class corresponding to ϕ ; that is, $L(\phi)$ is a local section in the principal bundle G over G/H with structure group H . Under the action of $g \in G$ on G/H we have $\phi \mapsto \phi_g$ and the coset representative $L(\phi)$ will be mapped to $gL(\phi)$ which is in the fiber over ϕ_g , so neces-

sarily $L(\phi_g) = gL(\phi)h(\phi)$. Taking a representation of the group G (which is also a representation of H), we obtain $L(\phi)$ as a matrix $L_a^\Lambda(\phi)$ where the indexes a and Λ run in principle over the same representation space, the different names being used to label the covariance properties of L . If the representation of G is reducible under H , one can project down the subspace where the representation of H is irreducible, so the index a will be understood as running on that subspace. We will use the representations appropriated to 0-brane multiplets.

The central charge is given by

$$Z_a(\phi, q) = (q^T L)_a = (q^T)_\Lambda L_a^\Lambda(\phi). \quad (9)$$

q is a vector transforming under the contravariant representation of G ,

$$q^g = (g^{-1})^T q, \quad (10)$$

so the central charge is U -duality covariant in the sense that under a transformation,

$$\phi \mapsto \phi_g, \quad (q^g)^T_\Lambda = (q^T)_\Sigma (g^{-1})^\Sigma_\Lambda, \quad (11)$$

and then $Z \mapsto Zh$. It then follows that any H -invariant function $I(Z)$ is also U -duality invariant in the sense that

$$I(\phi_g, q^g) = I(\phi, q) \quad \text{or} \quad I(Zh) = I(Z). \quad (12)$$

Among the duality-invariant combinations there are some which are ‘‘topological,’’ i.e., they do not depend on the moduli [28,30]. This happens when the H invariant is also invariant with respect to the right action of G . In fact, since $Z = q^T L$, with $L \in G$, it is obvious that if $I(Z)$ is G invariant for $Z \mapsto Zg$, then

$$I(Zg) = I(Z) = I(q). \quad (13)$$

Such objects exist for $d=5,6$ [17,31], but not for $d=1, \dots, 4$, with the implication that a Bekenstein-Hawking entropy formula for 0-branes exists only in four and five dimensions [32].

We can make a generalization of this analysis to obtain other moduli-invariant conditions. Let us consider now a covariant expression

$$E_\alpha(Z) = E_\alpha(\phi, q), \quad (14)$$

where the index α runs over the space of some representation T of G (and H). The covariance property means that under a right G transformation,

$$E_\alpha(Zg) = E_\beta(Z) T(g)^\beta_\alpha. \quad (15)$$

It follows that an equation of the form $E_\alpha(Z) = E_\alpha(\phi, q) = 0$ is moduli independent, so $E_\alpha(q) = 0$.

Now, assume that the representation T admits an H -invariant norm (which is positive since H is compact)

$$\|E_\alpha(Z)\|^2 = g^{\alpha\beta} E_\alpha E_\beta. \quad (16)$$

An equation like

$$\|E_\alpha(Z)\|^2 = 0 \quad (17)$$

is in principle moduli dependent since this expression is not G invariant. But the constraint $\|E_\alpha(Z)\| = 0$ implies that $E_\alpha(Z) = 0$, which is moduli independent, so $E_\alpha(q) = 0$.

The central charges satisfy some differential identities that are inherited from the coset representatives. To see this, let us consider the algebra valued Maurer-Cartan form in G , usually expressed for a matrix group as

$$\alpha_{MC} = g(x)^{-1} dg(x), \quad (18)$$

where x denotes coordinates on the group G . The components of the Maurer-Cartan form are left invariant forms, and one can take the pullback to G/H by the local section $L(\phi)$, giving a local, algebra-valued left invariant form on G/H :

$$\Omega(\phi) = L^{-1}(\phi) dL(\phi). \quad (19)$$

Consider now the Cartan decomposition of the Lie algebra of G as $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{k}$, where $\mathfrak{h}$ is the Lie algebra of H , the maximal compact subgroup of G , and $\mathfrak{k}$ can be identified with the tangent space at the identity of the coset. Since our coset spaces are symmetric spaces, the following properties are satisfied:

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}. \quad (20)$$

The second equation in (20) means that by the adjoint, $\mathfrak{h}$ acts on $\mathfrak{k}$ as a representation R of dimension equal to $\dim(G/H)$. We can write Ω according to this decomposition of the algebra:

$$\Omega = \omega^i T_i + P^\alpha T_\alpha, \quad (21)$$

where $\{T_i\}$ form a basis of $\mathfrak{h}$ and $\{T_\alpha\}$ form a basis of $\mathfrak{k}$. The projection of Ω on $\mathfrak{h}$, $\omega^i T_i$, is a G -invariant connection on the bundle with fiber $\mathfrak{k}$ and basis G/H , associated with the principal bundle $G(G/H)$ by the representation R of H . We call this bundle $E(G/H)$. The spin connection is expressed in an open set as

$$(\omega)^\alpha_\beta = \omega^i C^\alpha_{i\beta}. \quad (22)$$

The other components of Ω , P^α , provide us with the local expression of a homomorphism between $E(G/H)$ and the tangent bundle $T(G/H)$. By means of this homomorphism we can pull back the invariant Riemannian connection on G/H which coincides with ω . Finally, the invariant metric on G/H , induced by the Cartan-Killing metric on G , can be locally represented as

$$g_{\mu\nu} = \text{Tr}(T_\alpha T_\beta) P^\alpha_\mu P^\beta_\nu. \quad (23)$$

[The indices (μ, ν) are one-form indices on G/H .]

We want now to write Eq. (19) using a representation of G (and H) labeled as we explained above indistinctly by indices of the type Λ or a ; then we have

$$dL^\Lambda_a = L^\Lambda_b \omega^b_a + L^\Lambda_b P^b_a. \quad (24)$$

By defining as usual the covariant derivative with respect to H ,

$$\nabla_H L^\Lambda_a = dL^\Lambda_a - L^\Lambda_b \omega^b_a, \quad (25)$$

we obtain

$$\nabla_H L^\Lambda_a = L^\Lambda_b P^b_a. \quad (26)$$

TABLE II. Orbits of BPS states ($\propto$ means semidirect product).

Orbits	1/2 BPS	1/4 BPS	1/8 BPS
$d=1$	$Sl(2)$ or $\mathbf{R}$	$Sl(2) \times \mathbf{R}$	
$d=2$	$Sl(3) \times Sl(2)/Gl(2) \propto \mathbf{R}^3$	$Sl(3) \times Sl(2)/Sl(2) \propto \mathbf{R}^2$	
$d=3$	$Sl(5)/[Sl(3) \times Sl(2)] \propto \mathbf{R}^6$	$Sl(5)/O(2,3) \propto \mathbf{R}^4$	
$d=4$	$O(5,5)/Sl(5) \propto \mathbf{R}^{10}$	$O(5,5)/O(3,4) \propto \mathbf{R}^8$	
$d=5$	$E_{6(6)}/O(5,5) \propto \mathbf{R}^{16}$	$E_{6(6)}/O(4,5) \propto \mathbf{R}^{16}$	$E_{6(6)}/F_{4(4)}$
$d=6$	$E_{7(7)}/E_{6(6)} \propto \mathbf{R}^{27}$	$E_{7(7)}/[O(5,6) \propto \mathbf{R}^{32}] \times \mathbf{R}$	$E_{7(7)}/F_{4(4)} \propto \mathbf{R}^{26},$ $E_{7(7)}/E_{6(2)}$

Suppose now that the representation of H is reducible and we want to project onto an irreducible factor. Since ω_a^b is block diagonal, indices of type a in $\nabla_H L_a^\Lambda$ can be understood as running on the smaller representation, while P_a^b will have in general off-diagonal components, so we could denote it by $P_a^{b'}$, b' running on the large representation space, but still specifying the covariant properties under H . This happens when matter fields are present. In that case the H group is a direct product $H_R \times H_M$. H_R is the R -symmetry group and H_M is some matter flavor symmetry. We assume now (as will happen in all our examples) that the representation of G decomposes under H as $(\mathbf{1}, \mathbf{T}_M) + (\mathbf{T}_R, \mathbf{1})$, where $\mathbf{T}_M$ is a representation of H_M and $\mathbf{T}_R$ is a representation of H_R . Then, there is a basis where the generic index Λ splits into (a, I) , where a runs over the vector space representation of $\mathbf{T}_R$ and I runs over the representation space of $\mathbf{T}_M$. Then Eq. (26) decomposes as

$$\begin{aligned}\nabla_{H_R} L_a^\Lambda &= L_b^\Lambda P_a^b + L_I^\Lambda P_a^I, \\ \nabla_{H_M} L_I^\Lambda &= L_a^\Lambda P_I^a + L_J^\Lambda P_I^J,\end{aligned}\quad (27)$$

where

$$\begin{aligned}\nabla_{H_R} L_a^\Lambda &= dL_a^\Lambda - L_b^\Lambda \omega_a^b, \\ \nabla_{H_M} L_I^\Lambda &= dL_I^\Lambda - L_J^\Lambda \omega_I^J.\end{aligned}\quad (28)$$

The central charges $Z_a = (q^T L)_a$ and matter charges $Z_I = (q^T L)_I$ satisfy consequently the identities

$$\begin{aligned}\nabla_{H_R} Z_a &= Z_b P_a^b + Z_I P_a^I, \\ \nabla_{H_M} Z_I &= Z_a P_I^a + Z_J P_I^J.\end{aligned}\quad (29)$$

In the forthcoming sections we will see that these properties enter in the discussion of the BPS conditions and their duality invariant character.

B. Orbit classification of BPS states

In order to further study properties of central charges which will be useful in the following section, we would like to remind the orbit classification of 0-brane BPS configurations [20,21]. To do so we will state some results of matrix algebra that will be useful for our analysis. We consider matrices over the real, complex, and quaternion fields, M^r, M^c, M^q . For each of these matrices, the following polar decomposition holds:

$$\begin{aligned}M^r &= \sqrt{M^r M^{rT}} O, \\ M^c &= \sqrt{M^c M^{c\dagger}} U, \\ M^q &= \sqrt{M^q M^{q\dagger}} U_q,\end{aligned}\quad (30)$$

where the matrices $\sqrt{MM^\dagger}$ are Hermitian and O , U , and U_q are orthogonal, unitary, and quaternionic unitary (unitary symplectic), respectively.

From this decomposition it then follows that if M^r is symmetric, M^c Hermitian, and M^q symplectic Hermitian, they can be diagonalized by an appropriate transformation which is, respectively, orthogonal, unitary, and unitary symplectic. Instead, for general matrices we can bring them to a diagonal form in the following way:

$$M_D = U_1 M U_2^\dagger, \quad (31)$$

where U_1 and U_2 belong to $O(n)$, $U(n)$, or $USp(n)$ in each case.

It can also be shown that any antisymmetric matrix can be brought to a skew-diagonal form (normal form) by a transformation [33]

$$M_{SD} = U M U^T, \quad (32)$$

where, as before, U belongs to the appropriate group.

Since the central charge vector is a two-tensor representation of H , we can always apply one of the above results.

From the structure of the R -symmetry group listed above and the representation properties from Eq. (7) we will see that it follows that with an R rotation we can always diagonalize (or skew diagonalize) the matrix Z . For $d=1, \dots, 4$ there will be only two eigenvalues, three eigenvalues for $d=5$, and four eigenvalues for $d=6$.

The richer structure occurring for $d=5,6$ is the why also 1/8 BPS states occur, instead of the two possibilities of $d=1, \dots, 4$.

In Table II we list the orbits of the representations in Eq. (1) corresponding to the 0-brane BPS configurations [20,21].

These orbits correspond, for $d=1, \dots, 4$, to the possibility of having two coinciding eigenvalues (1/2 BPS) or not (1/4 BPS) for the central charge matrix. For $d=5$ and 6, 1/2 BPS correspond to three and four coinciding eigenvalues, respectively, 1/4 BPS orbits correspond to two equal eigenvalues and two pairs of equal eigenvalues, respectively, and 1/8 BPS orbits correspond to all different eigenvalues. In $d=6$

there are there two kinds of 1/8 BPS orbits depending whether the quartic invariant vanishes or not (lightlike or timelike orbit) [19,20].

In the following section we will see that, in spite of the fact that such statements look moduli dependent, they are actually moduli independent and therefore U -duality invariant, as expected from physical considerations.

III. BPS SPECTRUM FOR 0-BRANES IN TYPE II STRING THEORY COMPACTIFIED ON T^d , $d=1, \dots, 4$

In the present section we consider the BPS spectrum and the central charge matrix for 0-branes in the cases when only 1/2 and 1/4 BPS states exist. These are the cases where the central charge has only two eigenvalues, as happens for $d=1, \dots, 4$.

Let us consider the relevant anticommutators, containing the scalar central charge:

$$\begin{aligned} d=1 \quad \{Q_i, Q_j\} &= Z_{ij} \quad (i, j=1, 2), \quad Z_{ij} \text{ real symmetric,} \\ d=2 \quad \{Q_A, Q_B\} &= Z_I \sigma_{AB}^I = Z_{AB} \quad (A, B=1, 2), \\ &\quad Z^I \text{ complex, } Z_{AB} \text{ symmetric,} \\ d=3 \quad \{Q_a, Q_b\} &= Z_{IJ} (\gamma^{IJ})_{ab} = Z_{ab} \quad (a, b=1, \dots, 4), \\ &\quad Z_{ab} \text{ symmetric and symplectic,} \\ d=4 \quad \{Q_a, Q_{b'}\} &= Z_{ab'} \quad (a, b'=1, \dots, 4), \\ &\quad Z_{ab'} \text{ symplectic,} \end{aligned} \quad (33)$$

where σ^I are the Pauli matrices and $\gamma^{IJ} = 1/2[\gamma^I, \gamma^J]$ [γ^I are the O(5) gamma matrices]. We say that Z_{ab} is a symplectic (or quaternionic) matrix if

$$\bar{Z} = -\Omega Z \Omega, \quad (34)$$

where Ω is the bilinear form invariant under $USp(4)$, which satisfies

$$\Omega = \bar{\Omega} = -\Omega^T = -\Omega^{-1}. \quad (35)$$

The indices a, b in the gamma matrices are raised and lowered with Ω .

Cases $d=1, 2$. In these cases the charge matrix Z is 2×2 and has two independent eigenvalues. The 1/2 BPS conditions correspond to these eigenvalues equal in magnitude. We consider separately both cases.

For $d=1$, Z is real and symmetric. We can decompose it as

$$Z_{ij} = Y \delta_{ij} + Z^\alpha (T_\alpha)_{ij}, \quad (36)$$

where $\alpha=1, 2$, $T_1 = \sigma_1$ and $T_2 = \sigma_3$ (the Pauli matrices). The characteristic equation for Z is

$$\lambda^2 - \text{Tr } Z \lambda + \det Z = 0, \quad (37)$$

where

$$\text{Tr } Z = 2Y, \quad (38)$$

$$\det Z = Y^2 - Z^\alpha Z_\alpha. \quad (39)$$

It then follows that the two solutions of Eq. (37) satisfy $|\lambda_1| = |\lambda_2|$ if either

$$\text{Tr } Z = 0 \quad (40)$$

or

$$\det Z = 1/4(\text{Tr } Z)^2, \quad (41)$$

then implying

$$YZ^\alpha Z_\alpha = 0. \quad (42)$$

It is obvious that condition (42) is only O(2) or R invariant, but the unique solution $YZ_\alpha = 0$, is an $SL(2) \times O(1,1)$ or U invariant. This is the first example of a condition of the type (17). So we retrieve the result of Ref. [19] for $d=1, 1/2$ BPS states:

$$q=0 \quad \text{or} \quad q_\alpha=0. \quad (43)$$

In the $d=2$ case the Hermiticity condition is lacking, but still the matrix $Z^I \sigma_I$ can be diagonalized with real eigenvalues using Eq. (31), where the difference between U_1 and U_2 is simply a phase. We just use a transformation of the R -symmetry group $U(2)$, with $SU(2)$ acting on the σ matrices and $U(1)$ acting as a phase on Z^I .

In fact, note that

$$Z_I Z^I = (A_I + iB_I)(A^I + iB^I) = A_I A^I - B_I B^I + 2iA_I B^I. \quad (44)$$

Therefore, with a $U(1)$ transformation we bring $A_I B^I$ to zero, which means that $\vec{A}$ and $\vec{B}$ are orthogonal vectors, so by an orthogonal transformation we can bring them to coincide with the axes, and only two real numbers (related to the two eigenvalues) are left.

We proceed by diagonalizing the Hermitian matrix $ZZ^\dagger$. The square root of the eigenvalues will be the eigenvalues of Z , which we denote by λ_1, λ_2 . (In this way we include both cases, when the eigenvalues are equal and when they are opposite in sign). We have

$$\text{Tr } ZZ^\dagger = \lambda_1^2 + \lambda_2^2, \quad \text{Tr } (ZZ^\dagger)^2 = \lambda_1^4 + \lambda_2^4, \quad \det ZZ^\dagger = \lambda_1^2 \lambda_2^2, \quad (45)$$

where λ_i are the eigenvalues obtained as in Eq. (31). From the characteristic equation for $ZZ^\dagger$ we have

$$\lambda_{1,2}^2 = \frac{1}{2} [\text{Tr } ZZ^\dagger \pm \sqrt{(\text{Tr } ZZ^\dagger)^2 - 4 \det ZZ^\dagger}]. \quad (46)$$

Using now the properties of the σ matrices,

$$ZZ^\dagger = Z^I \sigma_I \bar{Z}^J \sigma_J = Z^I \bar{Z}_I \mathbf{1} + i \epsilon_{IJK} Z^I \bar{Z}^J \sigma^K. \quad (47)$$

We denote $\hat{Z}_K = i \epsilon_{IJK} Z^I \bar{Z}^J$. Then we have

$$\text{Tr } ZZ^\dagger = 2Z^I \bar{Z}_I, \quad \text{Tr } (ZZ^\dagger)^2 = 2[(Z^I \bar{Z}_I)^2 + \hat{Z}_I \hat{Z}^I]. \quad (48)$$

Hence, the discriminant in Eq. (46) is given by

$$\text{Tr } (ZZ^\dagger)^2 - \frac{1}{2} (\text{Tr } ZZ^\dagger)^2 = 2\hat{Z}^I \hat{Z}_I. \quad (49)$$

We set $Z_1^I = A^I$, $Z_2^I = B^I$. The 1/2 BPS condition $\hat{Z}_I \hat{Z}^I = 0$ can be written as

$$\|\epsilon^{\alpha\beta} Z_\alpha^I Z_\beta^J \epsilon_{KIJ}\| = 0 \Rightarrow \epsilon^{\alpha\beta} Z_\alpha^I Z_\beta^J \epsilon_{KIJ} = 0, \quad (50)$$

where $\| \cdot \|$ is the $O(3) \times O(2)$ -invariant norm. As before, the condition obtained is actually invariant under $SL(3) \times SL(2)$, so we obtain the moduli independent condition of Ref. [19]:

$$\epsilon^{\alpha\beta} q_{\alpha}^I q_{\beta}^J \epsilon_{KIJ} = 0. \quad (51)$$

Cases $d=3,4$. In these cases the matrix Z is four dimensional, but because of its symplectic property (34), there are only two independent eigenvalues (two pairs of equal eigenvalues), $\lambda_{1,2}$. Indeed, we find

$$\begin{aligned} \text{Tr } ZZ^{\dagger} &= 2(\lambda_1^2 + \lambda_2^2), \\ \text{Tr } (ZZ^{\dagger})^2 &= 2(\lambda_1^4 + \lambda_2^4). \end{aligned} \quad (52)$$

The characteristic equation (or, better, its square root) is

$$\lambda^2 - \frac{1}{2} \text{Tr } ZZ^{\dagger} \lambda + (\det ZZ^{\dagger})^{1/2} = 0, \quad (53)$$

with

$$(\det ZZ^{\dagger})^{1/2} = \frac{1}{8} (\text{Tr } ZZ^{\dagger})^2 - \frac{1}{4} \text{Tr } (ZZ^{\dagger})^2. \quad (54)$$

The roots are

$$\lambda_{1,2} = \frac{1}{2} \left(\frac{1}{2} \text{Tr } ZZ^{\dagger} \pm \sqrt{\text{Tr } (ZZ^{\dagger})^2 - \frac{1}{4} (\text{Tr } ZZ^{\dagger})^2} \right). \quad (55)$$

We consider now the two cases separately.

In the $d=3$ case we can switch from $Sp(4)$ to $O(5)$ indices by setting

$$Z_{ab} = Z_{IJ} (\gamma^{IJ})_{ab} \quad (I, J = 1, \dots, 5), \quad (56)$$

where Z_{IJ} is real and antisymmetric and

$$\gamma^{IJ} = \frac{1}{2} [\gamma^I, \gamma^J]. \quad (57)$$

It is clear that Z_{IJ} can be skew diagonalized with an $O(5)$ transformation, so Z_{ab} can be diagonalized with a $USp(4)$ transformation. From the relation

$$\frac{1}{2} \{ \gamma^{IJ}, \gamma^{KL} \} = (g^{IJ} g^{KL} - g^{JK} g^{IL} + \epsilon^{IJKL} \gamma_P), \quad (58)$$

it follows that

$$ZZ^{\dagger} = Z^2 \mathbf{I} + Z^P \gamma_P, \quad (59)$$

where

$$Z^2 = 2Z^P Q Z_{PQ}, \quad Z^P = \epsilon^{PIJKL} Z_{IJ} Z_{KL}. \quad (60)$$

From this, we have

$$\begin{aligned} \text{Tr } ZZ^{\dagger} &= 4Z^2, \\ \text{Tr } (ZZ^{\dagger})^2 &= 4Z^4 + 4Z^P Z_P. \end{aligned} \quad (61)$$

The 1/2 BPS condition becomes

$$\text{Tr } (ZZ^{\dagger})^2 - \frac{1}{4} (\text{Tr } ZZ^{\dagger})^2 = 4Z^P Z_P = 0, \quad (62)$$

which is $O(5)$ invariant and implies

$$Z^P = \epsilon^{PIJKL} Z_{IJ} Z_{KL} = 0. \quad (63)$$

Equation (63) is $SL(5)$ invariant when Z^{IJ} is in the ten-dimensional representation of $SL(5)$, and therefore it is moduli independent, giving the result of Ref. [19]:

$$\epsilon^{PIJKL} q_{IJ} q_{KL} = 0. \quad (64)$$

The $d=4$ case was already discussed in Refs. [5,7], but we outline it here for completeness. In this case, the matrix Z_{ab} is a general $O(5)$ bispinor. However, since its square is Hermitian, it decomposes as

$$\begin{aligned} ZZ^{\dagger} &= Z^2 \mathbf{I} + Z_{(l)}^P \gamma_P, \quad p = 1, \dots, 5, \\ Z^{\dagger} Z &= Z^2 \mathbf{I} + Z_{(r)}^P \gamma_P \quad \text{with } Z_{(l)}^P Z_{(l)P} = Z_{(r)}^P Z_{(r)P}, \end{aligned} \quad (65)$$

where the subindices l, r refer to the two $O(5)$ factors of the R symmetry group. It follows that

$$\text{Tr } ZZ^{\dagger} = 4Z^2, \quad \text{Tr } (ZZ^{\dagger})^2 = 4Z^4 + 4Z_{(l)}^P Z_{(l)P}. \quad (66)$$

The 1/2 BPS condition is then

$$\text{Tr } (ZZ^{\dagger})^2 - 1/4 (\text{Tr } ZZ^{\dagger})^2 = 4Z_{(l)}^P Z_{(l)P} = 0. \quad (67)$$

The equations $Z_{(l)}^P Z_{(l)P} = Z_{(r)}^P Z_{(r)P} = 0$ imply that the $O(5)$ vectors $Z_{(r)}^P, Z_{(l)}^P$ vanish. This is an $O(5,5)$ -invariant statement. $(Z, Z^{\dagger})$ form the 16-dimensional (chiral spinor) representation of $O(5,5)$ and the $O(5,5)$ ten dimensional representation (lightlike vector), $(\text{Tr } \gamma Z Z^{\dagger}, \text{Tr } \gamma Z^{\dagger} Z)$, then vanishes when $|Z_{(l)}| = |Z_{(r)}| = 0$. We then retrieve the condition of Ref. [19] on the quantized charges in the spinor representation of $O(5,5)$.

IV. BPS SPECTRUM FOR THE $d=5,6$ DIMENSIONAL CASES

In this section we will examine the more interesting cases of $d=5,6$, corresponding to supergravity compactified down to $D=4,5$ dimensions respectively.

The different BPS states, preserving some fraction of supersymmetry, are classified by the orbits of $E_{6(6)}$ and $E_{7(7)}$, respectively, as given in Table III.

To put this analogy in perspective it is useful to parametrize the set of BPS charges allowed by the duality constraints by their eigenvalues and some angular variables (which can be removed by an R -symmetry transformation [20]). The duality constraints which follow from the BPS conditions are precisely those constraints which do not depend on these extra angular variables and which can be removed by an H transformation in G . These constraints will give different orbits corresponding to different BPS conditions on the 0-brane charges.

The different orbits of different BPS levels will correspond to different solutions of the characteristic equation of the central charge matrix (or its square). These different solutions will be characterized by invariant constraints which are moduli independent in spite of the fact that the eigenvalues of the matrix are moduli dependent. Because of this, the orbits are simply given by invariant constraints on the ‘‘quantized’’ charges, as found in Ref. [19].

Case $d=6$. We consider the E_7 quartic invariant [22,30,31]

TABLE III. Orbits of the BPS energy levels for $d=5,6$.

	Orbit	Dimension	Eigenvalue	Angles
$d=5$				
1/2 BPS	$E_{6(6)}/O(5,5) \propto \mathbf{R}^{16}$	17	1	$16 = \dim[\text{USp}(8)/O(5) \times O(5)]$
1/4 BPS	$E_{6(6)}/O(4,5) \propto \mathbf{R}^{16}$	26	2	$24 = \dim[\text{USp}(8)/O(4) \times O(4)]$
1/8 BPS	$E_{6(6)}/F_{4(4)}$	$26+1$	3	$24 = \dim[\text{USp}(8)/\text{USp}(2)^4]$
$d=6$				
1/2 BPS	$E_{7(7)}/E_{6(6)} \propto \mathbf{R}^{27}$	28	1	$27 = \dim[\text{SU}(8)/\text{USp}(8)]$
1/4 BPS	$E_{7(7)}/[O(5,6) \propto \mathbf{R}^{32}] \times \mathbf{R}$	45	2	$43 = \dim[\text{SU}(8)/\text{USp}(4)^2]$
1/8 BPS	$E_{7(7)}/F_{4(4)} \propto \mathbf{R}^{26}$	55	4	$51 = \dim[\text{SU}(8)/\text{USp}(2)^4]$
	$E_{7(7)}/E_{6(2)}$	$55+1$	5	$51 = \dim[\text{SU}(8)/\text{USp}(2)^4]$

$$I = 4\text{Tr}(Z\bar{Z})^2 - (\text{Tr} Z\bar{Z})^2 + 2^4(PfZ + Pf\bar{Z}), \quad (68)$$

where

$$PfZ = \frac{1}{2^4 4!} \epsilon^{ABCD R P G H} Z_{AB} Z_{CD} Z_{RP} Z_{GH}. \quad (69)$$

We want to consider second derivatives of the above quartic invariant that could give us covariant equations. The anti-symmetric matrix Z_{AB} is in the 28-dimensional representation of $\text{SU}(8)$, while we can express symbolically $Z_{56} = (Z_{AB}, \bar{Z}^{AB})$, in the 56-dimensional representation of E_7 ($\mathbf{56} = \mathbf{28} + \overline{\mathbf{28}}$). Taking the second derivative

$$\left. \frac{\partial^2 I}{\partial Z_{56} \partial \bar{Z}_{56}} \right|_{\text{Adj}_{E_7}} \quad (70)$$

will give us a quadratic polynomial which is a symmetric tensor, in the $(\mathbf{56} \times \mathbf{56})_S = \mathbf{1596}$ representation of E_7 which is not irreducible and decomposes as $\mathbf{1463} + \mathbf{133}$. $\mathbf{133}$ is the Adj_{E_7} , so we can project on that space as indicated above Eq. (70). Since $\mathbf{133}$ decomposes as $\mathbf{63} + \mathbf{70}$ under $\text{SU}(8)$, the expression (70) splits into the two following $\text{SU}(8)$ covariant polynomials:

$$\left. \frac{\partial^2 I}{\partial Z_{AB} \partial \bar{Z}^{CB}} \right|_{\text{Adj}_{\text{SU}(8)}} \approx \left(Z_{AB} \bar{Z}^{CB} - \frac{1}{8} \delta_A^C Z_{PQ} \bar{Z}^{PQ} \right) = V_A^C, \quad (71)$$

$$\frac{\partial^2 I}{\partial Z_{[AB} \partial \bar{Z}_{CD]} } - \frac{1}{4!} \epsilon^{ABCD P Q R S} \frac{\partial^2 I}{\partial \bar{Z}^{[AB} \partial \bar{Z}^{CD]} } = V_{[ABCD]}^+. \quad (72)$$

The 1/2 BPS condition is the E_7 -invariant statement $V_A^C = 0$ and $V_{[ABCD]}^+ = 0$. This is the constraint imposed in Ref. [19] on the quantized 56 electric and magnetic charges defining a 1/2 BPS configuration. The equation $V_A^C = 0$ implies that the matrix $ZZ^\dagger$ has four coinciding eigenvalues (that is, it is a multiple of the identity), while the equation $V_{[ABCD]}^+ = 0$ implies that the eigenvalues of Z are real.

The vanishing of Eq. (72) follows from the vanishing of Eq. (71) and the differential relations (29) satisfied by Z_{AB} [28], which in this case take the form

$$\nabla_{\text{SU}(8)} Z_{AB} = \frac{1}{2} \epsilon_{ABCD} \bar{Z}^{CD}. \quad (73)$$

We now want to consider more general cases. The characteristic equation (or, better, its square root) is given by

$$\sqrt{\det(ZZ^\dagger - \lambda \mathbf{I})} = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d = 0, \quad (74)$$

where

$$\begin{aligned} a &= -(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) = -\frac{1}{2} \text{Tr} ZZ^\dagger, \\ b &= \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 \\ &= \frac{1}{4} \left[\frac{1}{2} (\text{Tr} ZZ^\dagger)^2 - \text{Tr}(ZZ^\dagger)^2 \right], \\ c &= -(\lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_2 \lambda_4 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4) \\ &= -\frac{1}{6} \left(\frac{1}{8} (\text{Tr} ZZ^\dagger)^3 + \text{Tr}(ZZ^\dagger)^3 \right. \\ &\quad \left. - \frac{3}{4} \text{Tr} ZZ^\dagger \text{Tr}(ZZ^\dagger)^2 \right), \\ d &= \lambda_1 \lambda_2 \lambda_3 \lambda_4 = \frac{1}{4} \left(\frac{1}{96} (\text{Tr} ZZ^\dagger)^4 + \frac{1}{8} [\text{Tr}(ZZ^\dagger)^2]^2 \right. \\ &\quad \left. + \frac{1}{3} \text{Tr}(ZZ^\dagger)^3 \text{Tr} ZZ^\dagger \right. \\ &\quad \left. - \frac{1}{2} \text{Tr}(ZZ^\dagger)^4 - \frac{1}{8} (\text{Tr} ZZ^\dagger)^2 \text{Tr}(ZZ^\dagger)^2 \right). \end{aligned} \quad (75)$$

In the case of two pairs of equal roots we have

$$\prod_{i=1}^4 (\lambda - \lambda_i) = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2. \quad (76)$$

This implies among the coefficients the relations

$$c = \frac{1}{2} a \left(b - \frac{1}{4} a^2 \right),$$

$$d = \frac{1}{4} \left(b - \frac{1}{4} a^2 \right)^2, \quad (77)$$

which imply the following relations among the invariants:

$$\begin{aligned} \frac{32}{3} \text{Tr}(ZZ^\dagger)^3 &= 4 \text{Tr} ZZ^\dagger \text{Tr}(ZZ^\dagger)^2 - \frac{1}{3} (\text{Tr} ZZ^\dagger)^3, \\ (\det ZZ^\dagger)^{1/2} &= \frac{1}{64} \left[\text{Tr}(ZZ^\dagger)^2 - \frac{1}{4} (\text{Tr} ZZ^\dagger)^2 \right]^2. \end{aligned} \quad (78)$$

The eigenvalues are given by the expression

$$\lambda_{1,2} = \frac{1}{8} \text{Tr} ZZ^\dagger \pm \frac{1}{2} \sqrt{\frac{1}{2} \text{Tr}(ZZ^\dagger)^2 - \frac{1}{16} (\text{Tr} ZZ^\dagger)^2}, \quad (79)$$

being the BPS mass, m_{BPS}^2 the highest eigenvalue (+ sign).

We want now to show how the 1/4 BPS condition follows from the E_7 invariance. Let us consider the E_7 -covariant constraint

$$\frac{\partial I}{\partial Z_{AB}} = 0 \quad \left(\Rightarrow \frac{\partial I}{\partial \bar{Z}^{AB}} = 0 \right), \quad (80)$$

where I is the invariant from Eq. (68). From this, the following quartic $SU(8)$ -invariant equations follow:

$$\frac{\partial I}{\partial Z_{AB}} Z_{AB} + \frac{\partial I}{\partial \bar{Z}^{AB}} \bar{Z}^{AB} = 4I = 0, \quad (81)$$

$$\frac{\partial I}{\partial Z_{AB}} Z_{AB} - \frac{\partial I}{\partial \bar{Z}^{AB}} \bar{Z}^{AB} = 0. \quad (82)$$

The second equation implies that the Pfaffian of Z is real, so

$$PfZ = PfZ^\dagger, \quad (83)$$

and therefore

$$(PfZ)^2 = (\det ZZ^\dagger)^{1/2}. \quad (84)$$

Plugging Eq. (84) into Eq. (81) and squaring, it gives $(\det ZZ^\dagger)^{1/2}$ as in Eq. (78).

In the same way one can show that the equation giving $\text{Tr}(ZZ^\dagger)^3$ as in Eq. (78) is the $SU(8)$ -invariant equation

$$\frac{\partial I}{\partial Z_{AB}} \frac{\partial I}{\partial \bar{Z}^{AB}} = 0. \quad (85)$$

In the generic case the 1/8 BPS states will correspond to four different eigenvalues. They are explicitly given as follows. Define the quantities

$$\begin{aligned} u &= b^2 + 12d - 3ca, \\ v &= 2b^3 + 27c^2 - 72bd - 9abc + 27da^2, \\ w &= \left(v + \frac{\sqrt{v^2 - 4u^3}}{2} \right)^{1/3}, \\ s &= \sqrt{\frac{a^2}{4} - \frac{2b}{3} + \frac{u}{3w} + \frac{w}{3}}. \end{aligned} \quad (86)$$

Then,

$$\begin{aligned} \lambda_{1,2} &= -\frac{a}{4} + \frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} - \frac{a^3 - 4ab + 8c}{4s} - \frac{u}{3w} - \frac{w}{3}}, \\ \lambda_{3,4} &= -\frac{a}{4} - \frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{a^2}{2} - \frac{4b}{3} + \frac{a^3 - 4ab + 8c}{4s} - \frac{u}{3w} - \frac{w}{3}}. \end{aligned} \quad (87)$$

The BPS mass m_{BPS}^2 is λ_1 . This is the actual determination of the energy spectrum for 1/8 BPS states in terms of the duality invariant quantities (75).

It is amusing that analytic expressions for the roots of a polynomial exist only up to quartic equations, as found by Galois [29], and this is precisely what is required by maximal supersymmetry ($N=8$ at $D=5,6$).

Case $d=5$. The central charge $\hat{Z}_{AB}$ is a symplectic, Ω -traceless antisymmetric matrix:

$$\bar{\hat{Z}} = -\Omega \hat{Z} \Omega, \quad \hat{Z}^T = -\hat{Z}, \quad \text{Tr} \hat{Z} \Omega = 0. \quad (88)$$

This implies that the matrix

$$Z = \hat{Z} \Omega \quad (89)$$

is Hermitian traceless. The characteristic equation for Z becomes

$$\sqrt{\det Z - \lambda \mathbf{I}} = \prod_{i=1}^4 (\lambda - \lambda_i) = \lambda^4 + b\lambda^2 + c\lambda + d = 0, \quad (90)$$

where

$$\begin{aligned} b &= -\frac{1}{4} \text{Tr} Z^2, \\ c &= -\frac{1}{6} \text{Tr} Z^3, \\ d &= \frac{1}{8} \left(\frac{1}{4} (\text{Tr} Z^2)^2 - \text{Tr} Z^4 \right). \end{aligned} \quad (91)$$

A 1/4 BPS state is a state for which $c=0$. This is an E_6 -invariant statement since $c=I_3$ is the E_6 cubic invariant. In this case we get

$$2\lambda_{1,2}^2 = \frac{1}{4} \text{Tr} Z^2 \pm \sqrt{\frac{1}{2} \text{Tr} Z^4 - \frac{1}{16} (\text{Tr} Z^2)^2}. \quad (92)$$

The discriminant is related to the modulus of the $USp(8)$ (and $E_{6(6)}$) vector

$$V_B^A = \frac{\partial I}{\partial Z_A^B} \approx Z_A^C Z_C^B - \frac{1}{8} Z_D^C Z_C^D \delta_A^B. \quad (93)$$

Indeed,

$$\text{Tr} V^2 = \text{Tr} Z^4 - \frac{1}{8} (\text{Tr} Z^2)^2. \quad (94)$$

The condition for 1/2 BPS is that the discriminant vanish. Therefore, this implies, by positivity, $V=0$, which is an E_6 -invariant statement:

$$\frac{\partial I}{\partial Z_A^B} = 0. \quad (95)$$

We therefore have retrieved the results of Ferrara and Maldacena [19].

For the 1/8 BPS state the four roots are given by

$$\begin{aligned}\lambda_{1,2} &= \frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{-4b}{3} - \frac{2c}{s} - \frac{u}{3w} - \frac{w}{3}}, \\ \lambda_{3,4} &= -\frac{s}{2} \pm \frac{1}{2} \sqrt{\frac{-4b}{3} + \frac{2c}{s} - \frac{u}{3w} - \frac{w}{3}},\end{aligned}\quad (96)$$

where

$$\begin{aligned}u &= b^2 + 12d, \quad z = 2b^3 + 27c^2 - 72bd, \\ w &= \left(z + \frac{\sqrt{z^2 - 4u^3}}{2} \right)^{1/3}, \quad s = \sqrt{\frac{w}{3} + \frac{u}{3w} - \frac{2b}{3}}.\end{aligned}\quad (97)$$

The BPS mass is therefore given by the highest root λ_1 .

V. BPS CONDITIONS FOR THEORIES WITH 16 SUPERSYMMETRIES

In this last section we will extend our analysis to theories with 16 supersymmetries. These theories are obtained in three different ways: by compactifying heterotic string theory on T^d ($1 \leq d \leq 6$), from M-theory compactified on K_3 ($D=7$), and from type IIA theory compactified on K_3 ($D=6$).

In the theories where matter vector fields exist, the duality group G depends on the matter content and on the space-time dimension D . Its maximal compact subgroup is $H_R \times H_M$ where H_R is the R symmetry and H_M is the group acting on the matter multiplets. In our case, $H_M = O(n)$, where n is the number of matter multiplets. G is of the form $O(10-D, n) \times O(1,1)$ for $5 \leq D \leq 9$ while for $D=4$ it is $SL(2) \times O(6, n)$. The R -symmetry groups are $O(10-D)$ for $5 \leq D \leq 9$ and $O(6) \times O(2) \approx SU(4) \times U(1)$ for $D=4$. The last result can easily be understood from the geometric symmetry of heterotic string on T^6 , where G is enlarged by the electric-magnetic duality for 0-branes.

The G and H_R representations of the 0-branes are given in Eqs. (98) and (99):

Central charge representation of H_R

$$\begin{aligned}d=1 & \quad \mathbf{1} \quad O(1) = \mathbf{I}, \\ d=2 & \quad \mathbf{1}^c \quad \text{complex} \quad U(1) \approx O(2), \\ d=3 & \quad \mathbf{3} \quad \text{real} \quad SU(2) \approx USp(2), \\ d=4 & \quad \mathbf{4} \quad \text{real} \quad O(4) \approx USp(2) \times USp(2), \\ d=5 & \quad \mathbf{1+5} \quad \text{real} \quad O(5) \approx USp(4), \\ d=6 & \quad \mathbf{6}^c \quad \text{complex} \quad O(6) \times O(2) \\ & \approx SU(4) \times U(1).\end{aligned}\quad (98)$$

From the above, and according our previous analysis, it follows that the central charge matrix Z_a has only one independent eigenvalue for $d=1, \dots, 4$ and two independent eigenvalues for $d=5, 6$. Therefore, for $d=1, \dots, 4$ only 1/2 BPS states can occur while for $d=5, 6$ both, 1/2 and 1/4 BPS states can occur.

0-brane representation of G

$$\begin{aligned}d=1, \dots, 4 & \quad \mathbf{d+n} \quad \text{real vector} \quad O(d, n) \times O(1, 1), \\ d=5 & \quad (\mathbf{1, 2}) + (\mathbf{5+n, -1}) \quad (\text{singlet} + \text{vector}) \quad O(5, n) \\ & \quad \times O(1, 1), \\ d=6 & \quad (\mathbf{2, 6+n}) \quad SL(2) \times SO(6, n).\end{aligned}\quad (99)$$

We consider now separately the two cases $d=5, 6$.

Case $d=5$. This is the case which corresponds to heterotic string on T^5 or M-theory (type IIA, type IIB) on $K_3 \times T^2$ ($K_3 \times S^1$). In such compactifications, $n=21$, so $G = O(5, 21) \times O(1, 1)$ but our analysis is independent of this specific number n .

The central charge $\hat{Z}$ is an antisymmetric symplectic matrix. The Hermitian matrix $Z = \hat{Z}\Omega$ decomposes as

$$Z = Z^a \gamma_a + Z^0 \mathbf{I}, \quad (100)$$

where γ_a are the $O(5)$ γ -matrices and Z^a, Z^0 are real. It follows that

$$\begin{aligned}\text{Tr } Z &= 4Z^0, \\ (\det Z)^{1/2} &= (Z^0)^2 - \bar{Z}^2 = \frac{1}{8}(\text{Tr } Z)^2 - \frac{1}{4}\text{Tr } Z^2.\end{aligned}\quad (101)$$

The characteristic equation (or, better, its square root) is

$$\lambda^2 - \frac{1}{2}\text{Tr } Z \lambda + (\det Z)^{1/2} = 0, \quad (102)$$

implying that Z has two coinciding eigenvalues (in absolute value) either if

$$\text{Tr } Z = 0 \quad \text{or} \quad \frac{1}{4}(\text{Tr } Z)^2 = 4(\det Z)^{1/2}. \quad (103)$$

Using Eq. (101), the above equation directly implies

$$Z_0 Z_a = 0. \quad (104)$$

The eigenvalues are given by

$$\lambda_{1,2} = \frac{1}{2} \left(\frac{1}{2}\text{Tr } Z \pm \sqrt{\text{Tr } Z^2 - \frac{1}{4}(\text{Tr } Z)^2} \right), \quad (105)$$

being the plus sign the mass squared of the BPS state.

We discuss now the covariance of Eq. (104). Since $Z_0 = e^{2\sigma} m$ where $e^{2\sigma}$ parametrizes $O(1, 1)$ and m is the charge associated with Z_0 , $Z_0 = 0$ implies $m = 0$ which is an $O(5, n)$ singlet, so it is G invariant.

According to Eq. (99) we write the projection of the coset representative over the $(\mathbf{5+n, -1})$ representation as $e^{-\sigma} L_a^\Lambda$ where σ parametrizes $O(1, 1)$ and L_a^Λ is the coset representative of $O(5+n)/O(5) \times O(n)$. If Z_I , $I=1, \dots, n$ are the matter charges associated with the n matter multiplets, we have that, because of Eq. (29),

$$\nabla_{O(5)} Z_a = \frac{1}{4} \text{Tr}(\gamma_a P_I) Z^I - Z_a d\sigma. \quad (106)$$

Therefore $Z_a=0$ implies $Z_I=0$. This is also an $O(5,n)$ -invariant statement since it comes by differentiating the quadratic invariant polynomial:

$$I = \sum_{a=1}^5 Z_a Z^a - \sum_{I=1}^M Z_I Z^I. \quad (107)$$

Therefore, $Z_a=Z_I=0$ implies $q^\Lambda=0$ where q^Λ , $\Lambda=1, \dots, 5+n$, is a fixed charge vector of $O(5,M)$, as found in [19].

Case $d=6$ ($D=4$). We now consider theories with 16 supersymmetries in $D=4$, as heterotic string compactified on T^6 , type II on $K_3 \times T^2$, or M-theory on $K_3 \times T^3$. The new phenomenon which occurs here is the electric-magnetic duality of 0-branes which are assigned to the $(2,6+n)$ representation of $SU(1,1) \times O(6,n)$.

The central charge is a four-dimensional complex matrix Z_{AB} , antisymmetric in the $SU(4) \approx O(6)$ indices. Therefore, the matrix $ZZ^\dagger$ has two independent eigenvalues, given by the characteristic equation

$$\begin{aligned} [\det(ZZ^\dagger - \lambda \mathbf{I})]^{1/2} &= 0, \\ \lambda^2 - \frac{1}{2} \text{Tr} ZZ^\dagger \lambda + (\det ZZ^\dagger)^{1/2} &= 0, \end{aligned} \quad (108)$$

with the solution

$$\lambda_{1,2} = \frac{1}{2} \left(\frac{1}{2} \text{Tr} ZZ^\dagger \pm \sqrt{[\text{Tr}(ZZ^\dagger)]^2 - \frac{1}{4} [\text{Tr}(ZZ^\dagger)]^2} \right). \quad (109)$$

A generic 1/4 BPS state has m_{BPS}^2 equal to the eigenvalue with + sign above. The 1/2 BPS configuration corresponds to a vanishing discriminant, i.e., $\lambda_1 = \lambda_2$. We would like to show how this condition is $SU(1,1) \times O(6,n)$ invariant in the sense that it is moduli independent in spite of the fact that the discriminant is moduli dependent.

For this purpose we proceed as for the maximally supersymmetric case in $D=4$. If the two eigenvalues of $ZZ^\dagger$ coincide, then the Hermitian traceless matrix

$$V_A^C = Z_{AC} \bar{Z}^{BC} - \frac{1}{4} \delta_A^B Z_{PQ} \bar{Z}^{PQ} \quad (110)$$

vanishes. [The discriminant is just $\text{Tr} V^2$, the invariant norm of the $SU(4)$ vector V_A^C .]

Consider now the $SU(1,1) \times O(6,n)$ quartic invariant

$$I = I_1^2 - I_2 \bar{I}_2, \quad (111)$$

where

$$\begin{aligned} I_1 &= Z_{AB} \bar{Z}^{AB} - Z_I \bar{Z}^I, \\ I_2 &= \frac{1}{4} \epsilon^{ABCD} Z_{AB} Z_{CD} - \bar{Z}_I \bar{Z}^I. \end{aligned} \quad (112)$$

The fact that I is an invariant was derived in Ref. [30] and can be easily understood from the fact that $(I_1, I_2, \bar{I}_2)$ is a triplet of $SU(1,1) \approx O(1,2)$, each of the entries being $O(6,n)$ invariant.

Equation (110) can be seen as the second derivative of I projected onto the adjoint representation of $SU(4)$:

$$V_A^C \approx \left. \frac{\partial^2 I}{\partial Z_{AB} \partial \bar{Z}^{CD}} \right|_{\text{Adj}_{SU(4)}}. \quad (113)$$

Indeed, let us call U the $(\mathbf{2}, \mathbf{6} + \mathbf{n})$ of $Sl(2)$ vector constructed with (Z_{AB}, Z_I) and its complex conjugate. The quantity

$$\frac{\partial^2 I}{\partial U \partial U} \quad (114)$$

is in the symmetric product $[(\mathbf{2}, \mathbf{6} + \mathbf{n}) \times (\mathbf{2}, \mathbf{6} + \mathbf{n})]_S$, which decomposes under $O(6, M)$ as $(\mathbf{3}, \mathbf{Sym}) + (\mathbf{1}, \mathbf{Adj}_{O(6,n)}) = (\mathbf{3}, \mathbf{1}) + (\mathbf{3}, \mathbf{TrSym}) + (\mathbf{1}, \mathbf{Adj}_{O(6,n)})$, where $\mathbf{Sym}$ is the two-fold symmetric representation, and $\mathbf{TrSym}$ is the traceless symmetric representation. To show that the $V_A^C=0$ is a G -invariant statement we use the fact that $\mathbf{Adj}_{O(6,n)}$ decomposes under $O(6) \times O(n)$ as $\mathbf{Adj}_{O(6,n)} \rightarrow (\mathbf{Adj}_{O(6)}, \mathbf{1}) + (\mathbf{1}, \mathbf{Adj}_{O(n)}) + (\mathbf{6}, \mathbf{n})$. We will show that the vanishing of the projection onto $\mathbf{Adj}_{O(6)} \approx \mathbf{Adj}_{SU(4)}$ of Eq. (114) implies the vanishing of the projection onto $(\mathbf{1}, \mathbf{Adj}_{O(n)})$ and $(\mathbf{6}, \mathbf{n})$. In fact, differentiating $V_A^C=0$ and using the differential identities (29) one also finds

$$Z_I \bar{Z}_J - \bar{Z}_I Z_J = 0, \quad (115)$$

$$Z_{AB} Z_J - \frac{1}{4} \epsilon_{ABCD} \bar{Z}^{CD} \bar{Z}_J = 0. \quad (116)$$

The vanishing of the three equations $V_A^C=0$, Eq. (115), and Eq. (116) implies that the projection of Eq. (114) on $\mathbf{Ad}_{O(6,n)}$ vanishes. This is a $SU(1,1) \times SO(6,n)$ invariant and therefore the moduli dependence drops out. These three equations can be rewritten in terms of the fixed charges (q_Λ, p_Λ) , in the $(\mathbf{6} + \mathbf{n})$ of $O(6,n)$ times the fundamental representation of $Sl(2) \approx SU(1,1)$ as

$$T_{\Lambda\Sigma}^{(A)} = q_\Lambda p_\Sigma - p_\Lambda q_\Sigma = 0. \quad (117)$$

Note that in this basis the projection of Eq. (114) onto the representation $(\mathbf{3}, \mathbf{Sym})$ is

$$T_{\Lambda\Sigma}^{(S)} = \left(q_\Lambda q_\Sigma, p_\Lambda p_\Sigma, \frac{1}{2} (q_\Lambda p_\Sigma + p_\Lambda q_\Sigma) \right), \quad (118)$$

whose trace part is the $Sl(2)$ triplet $(q^2 \cdot p^2, q \cdot p)$. It can be written as a matrix

$$T^{(0)} = \begin{pmatrix} q^2 & q \cdot p \\ q \cdot p & p^2 \end{pmatrix}. \quad (119)$$

The invariant I can be written either as $T_{\Lambda\Sigma}^{(A)} T^{(A)\Lambda\Sigma}$ or as $\det T^{(0)}$, and its square root is the entropy formula for 1/4 BPS 0-branes in theories with 16 supersymmetries [17,32].

As a final remark, let us comment on the orbits of the $O(5,n)$ and $O(6,n)$ vectors for the BPS configurations discussed above.

For 1/2 BPS states at $d=5$ we have $m q_\Lambda = 0$, so either m or q_Λ vanishes. In the former case, the BPS condition requires q_Λ to be time like or light like ($q_\Lambda q^\Lambda \geq 0$) [19], so the

orbit is either $O(5,n)/O(4) \times O(n)$ or $O(5,n)/IO(4,n-1)$. If $m \neq 0$ then $q_\Lambda = 0$, so the orbit is a point since the little group is $O(5,n)$ itself.

Let us consider now the $d=6$ case. The BPS condition corresponds to the statement that the matrix $T^{(0)}$ is positive semidefinite. This implies

$$\det T^{(0)} = q^2 p^2 - (q \cdot p)^2 \geq 0, \quad \text{Tr } T^{(0)} = q^2 + p^2 \geq 0. \quad (120)$$

From this it follows that $q^2 \geq 0$ and $p^2 \geq 0$.

$\det T^{(0)} = 0$ corresponds to $1/2$ BPS states; this happens when $q = \lambda p$, ($\lambda \geq 0$).

For $\det T^{(0)} > 0$, $q^2 > 0$ and $p^2 > 0$ and the generic $1/4$ BPS configuration will depend on five parameters, since p, q , by an $O(2)$ transformation in $SL(2)$ can be made orthogonal ($q_\Lambda p^\Lambda = 0$). Indeed, the first vector can be put in the form

$(p_1, 0, \dots, 0, p_{n+1}, 0, \dots, 0)$ and the second in the form $(q_1, q_2, 0, \dots, 0, q_{n+1}, q_{n+2}, 0, \dots, 0)$ by an $O(6) \times O(n)$ transformation. The orthogonality condition is used to eliminate one of the six parameters. The remaining $7 + 2n$ parameters are the ‘‘angles’’ in $O(2) \times O(6) \times O(n)/O(4) \times O(n-2)$. The little group in G of the two time like vectors is $O(4) \times O(n)$.

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