We introduce a general class of generating functionals for the calculation of quantum-mechanical expectation values of arbitrary functionals of fluctuating paths with fixed end points in configuration or momentum space. The generating functionals are calculated explicitly for harmonic oscillators with time-dependent frequency, and used to derive a smearing formulas for correlation functions of polynomial and nonpolynomials functions of time-dependent positions and momenta. These formulas summarize the effect of thermal and quantum fluctuations, and serve to derive generalized Wick rules and Feynman diagrams for perturbation expansions of nonpolynomial interactions.

1. INTRODUCTION

A useful technique for describing compactly the properties of a quantum mechanical system is to define a suitable generating functional of some external source or current \( j(t) \). The desired properties are obtained from functional derivatives with respect to \( j(t) \). For example, the correlation functions and the quantum mechanical density matrix in one space
dimension $x$ is determined by a generating functional which is a path integral in configuration space over all paths $x(t)$ with fixed end points $x(t_a) = x_a, x(t_b) = x_b$ [1, Chap. 2]:

$$
(x_b t_b | x_a t_a) [j(t)] = \int_{x_a, t_a}^{x_b, t_b} D x(t) \exp \left\{ \frac{i}{\hbar} \mathcal{A}[x(t); j(t)] \right\},
$$

(1.1)

where the exponent contains the classical action $\mathcal{A}[x(t)]$ plus a source term linear in $x(t)$:

$$
\mathcal{A}[x(t); j(t)] = \mathcal{A}[x(t)] + \int_{t_a}^{t_b} dt x(t) j(t).
$$

(1.2)

In this note we set up a useful alternative expression for the generating functional (1.1) and a related one in momentum space. This alternative expression is obtained by extending the current $j(t)$ by singular sources proportional to $\delta(t_b - t)$ and $\delta(t - t_a)$, and by reducing the path integral (1.1) with fixed end points in configuration space to one with vanishing end points. This will permit us to simplify considerably the calculation of quantum mechanical correlation functions. To see this simplification explicitly, consider a harmonic oscillator whose action reads

$$
\mathcal{A}[x(t)] = \int_{t_a}^{t_b} dt \left[ \frac{M}{2} \dot{x}^2(t) - \frac{M}{2} \omega^2 x^2(t) \right],
$$

(1.3)

for which the generating functional can be calculated [1, Eq. (3.89)] as follows:

$$
(x_b t_b | x_a t_a) [j(t)] = \sqrt{\frac{M \omega}{2 \pi i \hbar \sin \omega(t_b - t_a)}} \exp \left\{ \frac{i M \omega [(x_b^2 + x_a^2) \cos \omega(t_b - t_a) - 2 x_a x_b]}{2 \hbar \sin \omega(t_b - t_a)} \right\}
\times \exp \left\{ \frac{i}{\hbar} \int_{t_a}^{t_b} dt \frac{x_a \sin \omega(t_b - t) + x_b \sin \omega(t - t_a)}{\sin \omega(t_b - t_a)} j(t) \right. \\
- \left. \frac{i}{\hbar M \omega} \int_{t_a}^{t_b} dt \int_{t_a}^{t} dt' \frac{\sin \omega(t_b - t) \sin \omega(t' - t_a)}{\sin \omega(t_b - t)} j(t) j(t') \right\}.
$$

(1.4)

The nonzero end points $x_a$ and $x_b$ make this expression quite involved. For vanishing end points, however, it simplifies to
\[
(0 \ t_b \ | \ 0 \ t_a) [j(t)] = \sqrt{\frac{M \omega}{2\pi i \hbar \sin \omega(t_b - t_a)}} 
\times \exp \left\{ \frac{i}{\hbar M \omega} \int_{t_a}^{t_b} (t') \sin \omega(t_b - t') \sin \omega(t - t') \sin \omega(t_b - t) \right\} \ . \tag{1.5}
\]

The observation which motivates the present paper relies on replacing in the simple expression (1.5) the current \( j(t) \) by

\[
j'(t) = j(t) + M x_a \dot{\delta}(t - t_a) - M x_b \dot{\delta}(t_b - t), \tag{1.6}
\]

where the delta functions are understood as \( \dot{\delta}(t - t_a + \epsilon) \) and \( \dot{\delta}(t_b - \epsilon - t) \) in the limit \( \epsilon \to 0 \). By performing some partial integrations, this replacement reproduces all terms in the complicated generating functional (1.4), except for a rather trivial additional singular phase factor. The important relation is

\[
(x_b \ t_b \ | \ x_a \ t_a) [j(t)] = (x_b = 0 \ t_b \ | \ x_a = 0 \ t_a) [j(t) + M x_a \dot{\delta}(t - t_a) - M x_b \dot{\delta}(t_b - t)] 
\times \exp \left\{ \frac{i M}{2 \hbar} (x_b^2 + x_a^2) \delta(0) \right\} \ . \tag{1.7}
\]

In Section II we prove that the relation (1.7) holds for an arbitrary quantum-mechanical system whose Hamiltonian has the standard form

\[
H_0(p, x, t) = \frac{p^2}{2M} + V(x, t) . \tag{1.8}
\]

In Section III we calculate explicit amplitudes for a harmonic oscillator with arbitrary time-dependent frequency, and as an important application we derive in Section IV from the new form of the generating functional a smearing formula for calculating expectation values of polynomial and nonpolynomial potentials functions. In particular, this result allows to calculate expectation values appearing in perturbation expansions for nonlinear interactions, as for example for the nonlinear \( \sigma \)-model. In Section V we show that our smearing formula
generalizes Wick rules and Feynman diagrams for harmonic expectation values from products of variables to mixtures of nonpolynomial functions and polynomials. In Section VI, we finally specialize our generating functional simplifies to periodic paths.

II. GENERATING FUNCTIONALS

We begin by setting up phase space path integrals for generating functionals with fixed end points in either configuration or momentum space. The action contains additional currents $k(t)$ and $j(t)$ coupled linearly to momentum $p(t)$ and position $x(t)$. By extending the currents with singular $\delta$-functions as in Eq. (1.6), we reduce the path integrals with fixed end points to those with vanishing end points. Our procedure applies to arbitrary Hamiltonians $H_0(p, x, t)$, with certain simplification resulting from a standard Hamiltonian (1.8).

A. General Phase Space Formulation

Consider a quantum-mechanical particle coupled to a momentum and position source $k(t)$ and $j(t)$ with the classical Hamiltonian

$$H(p, x, t) = H_0(p, x, t) - pk(t) - xj(t),$$

(2.1)

where the corresponding action reads

$$A[p(t), x(t); k(t), j(t)] = \int_{t_a}^{t_b} dt \{ p(t)\dot{x}(t) - H(p(t), x(t), t) \}.$$  

(2.2)

The total time evolution amplitude between fixed space points $x_a$ and $x_b$ is given by the path integral
\[(x_b t_b | x_a t_a)[k(t), j(t)] = \int_{x_a, t_a}^{x_b, t_b} \frac{Dp(t) Dx(t)}{2\pi \hbar} \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\}. \tag{2.3}\]

A Fourier transformation with respect to \(x_a\) and \(x_b\) produces the time evolution amplitude in momentum space
\[(p_b t_b | p_a t_a)[k(t), j(t)] = \int_{-\infty}^{+\infty} dx_a \int_{-\infty}^{+\infty} dx_b e^{-i(p_b x_b - p_a x_a)/\hbar} (x_b t_b | x_a t_b)[k(t), j(t)]. \tag{2.4}\]

Here the initial and final momenta \(p_a\) and \(p_b\) are held fixed, so that the right-hand side may be written as the path integral
\[(p_b t_b | p_a t_a)[k(t), j(t)] = \int_{p_a, t_a}^{p_b, t_b} \frac{Dp(t) Dx(t)}{2\pi \hbar} \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\}. \tag{2.5}\]

We remark that both path integrals (2.3) and (2.5) are properly defined as continuum limits of ordinary integrals after a time-slicing procedure. Since end points of paths are fixed in coordinate and momentum space, respectively, the discretized expressions for the path integrals turn out to be slightly asymmetric in \(p(t)\) and \(x(t)\) [1, Chap. 2].

The time evolution amplitudes (2.3) and (2.5) with fixed end points can now be reduced to corresponding ones with vanishing end points. For this, we shift the current \(k(t)\) in (2.1) by a source term \(x_b \delta(t_b - t) - x_a \delta(t - t_a)\) and observe that this produces by (2.2) and (2.5) an overall phase factor:
\[
(p_b t_b | p_a t_a)[k(t) + x_b \delta(t_b - t) - x_a \delta(t - t_a), j(t)] = \exp \left[ \frac{i}{\hbar} (p_b x_b - p_a x_a) \right] (p_b t_b | p_a t_a)[k(t), j(t)]. \tag{2.6}\]

By inverting the Fourier transformation (2.4), the configuration space amplitude (2.3) is seen to satisfy
\[
(x_b t_b | x_a t_a)[k(t) + x'_b \delta(t_b - t) - x'_a \delta(t - t_a), j(t)]
= (x_b + x'_b t_b | x_a + x'_a t_a)[k(t), j(t)], \tag{2.7}\]
where again the delta functions are understood as $\delta(t_b - \epsilon - t)$ and $\delta(t - t_a + \epsilon)$ in the limit $\epsilon \to 0$. Because of this relation, the amplitude (2.3) can be reduced to a path integral with vanishing end points but additional $\delta$-terms in the current $k(t)$:

$$\langle x_b t_b | x_a t_a \rangle[k(t), j(t)] = (x_b = 0 t_b | x_a = 0 t_a)[k(t) + x_b \delta(t_b - t) - x_a \delta(t - t_a), j(t)].$$  \hspace{1cm} (2.8)

A similar expression exists, if momentum end points are fixed in momentum space by adding $p_a \delta(t - t_a) - p_b \delta(t_b - t)$ to the current $j(t)$:

$$\langle p_b t_b | p_a t_a \rangle[k(t), j(t)] = (p_b = 0 t_b | p_a = 0 t_a)[k(t), j(t) + p_a \delta(t - t_a) - p_b \delta(t_b - t)].$$  \hspace{1cm} (2.9)

We now explore the consequences of these two relations for the calculation of correlation functions.

### B. Correlation Functions

The functional dependence of the time evolution amplitudes (2.3) and (2.5) on the currents $k(t)$ and $j(t)$ allows us to calculate expectation values of arbitrary functionals $F[p(t), x(t)]$ from the path integral

$$\langle F[p(t), x(t)][k(t), j(t)]_{v_b, t_b} \rangle = \frac{1}{(v_b t_b | v_a t_a)[k(t), j(t)]} \times \int_{v_a, t_a} \frac{Dp(t) Dx(t)}{2\pi \hbar} F[p(t), x(t)] \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\},$$  \hspace{1cm} (2.10)

where the variable $v$ may be $p$ or $x$. The usual correlation functions

$$\langle p(t_1) \cdots p(t_n) x(t_1) \cdots x(t_m) \rangle[k(t), j(t)]_{v_b, t_b} = \frac{1}{(v_b t_b | v_a t_a)[k(t), j(t)]} \times \int_{v_a, t_a} \frac{Dp(t) Dx(t)}{2\pi \hbar} p(t_1) \cdots p(t_n) x(t_1) \cdots x(t_m) \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\},$$  \hspace{1cm} (2.11)
are special cases of (2.10), so we shall call the general expectation values (2.10) \textit{correlation functionals}. The sources \(k(t)\) and \(j(t)\) permit us to express (2.10) in terms of functional derivatives:

\[
\langle F[p(t), x(t)] \rangle [k(t), j(t)]_{v_a, t_a} = \frac{F \left[ \frac{\hbar}{i} \delta \frac{\partial}{\partial k(t)}, \frac{\hbar}{i} \delta \frac{\partial}{\partial j(t)} \right] (v_b t_b | v_a t_a) [k(t), j(t)]}{(v_b t_b | v_a t_a) [k(t), j(t)]}. \tag{2.12}
\]

Recalling (2.8) and (2.9), we shall rewrite the functionals \((v_b t_b | v_a t_a) [k(t), j(t)]\) in a unified common way as follows

\[
(v_b t_b | v_a t_a) [k(t), j(t)] = \int_{v_a = 0, t_a}^{v_b = 0, t_b} \frac{Dp(t) Dx(t)}{2\pi \hbar} \delta(v(t_a) - v_a) \delta(v(t_b) - v_b) \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\}, \tag{2.13}
\]

where the paths \(v(t)\) stand either for \(p(t)\) or for \(x(t)\). In each of these cases, the paths \(w(t)\) denote the conjugate variables \(x(t)\) or \(p(t)\), respectively. In this form, the path integral possesses the advantage that usual correlation functions (2.11) can be determined by path averages, in which intermediate and end points are treated on equal footing. Indeed, inserting delta functions according to

\[
\langle p(t_1) \cdots p(t_n) x(t_1) \cdots x(t_m) \rangle [k(t), j(t)]_{v_a, t_a}^{v_b, t_b} = \frac{1}{(v_b t_b | v_a t_a) [k(t), j(t)]} \int_{-\infty}^{+\infty} dp_1 \cdots \int_{-\infty}^{+\infty} dp_n \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_m p_1 \cdots p_n x_1 \cdots x_m
\]

\[
\times \int_{v_a, t_a}^{v_b, t_b} \frac{Dp(t) Dx(t)}{2\pi \hbar} \delta(p(t_1) - p_1) \cdots \delta(p(t_n) - p_n)
\]

\[
\times \delta(x(t_1) - x_1) \cdots \delta(x(t_m) - x_m) \exp \left\{ \frac{i}{\hbar} A[p(t), x(t); k(t), j(t)] \right\}, \tag{2.14}
\]

we obtain with a similar reasoning

\[
\langle p(t_1) \cdots p(t_n) x(t_1) \cdots x(t_m) \rangle [k(t), j(t)]_{v_a, t_a}^{v_b, t_b}
\]
\[
\begin{align*}
&= \frac{1}{(v_b t_b | v_a t_a)[k(t), j(t)]} \int_{-\infty}^{+\infty} dp_1 \cdots \int_{-\infty}^{+\infty} dp_n \int_{-\infty}^{+\infty} dx_1 \cdots \int_{-\infty}^{+\infty} dx_m p_1 \cdots p_n x_1 \cdots x_m \\
&\times \int_{w_b=0,t_b}^{w_a=0,t_a} \frac{Dp(t) Dx(t)}{2\pi\hbar} \delta(v(t_a) - v_a) \delta(p(t_1) - p_1) \cdots \delta(p(t_n) - p_n) \\
&\times \delta(x(t_1) - x_1) \cdots \delta(x(t_m) - x_m) \delta(v(t_b) - v_b) \exp \left\{ \frac{i}{\hbar} A[x(t); k(t), j(t)] \right\} .
\end{align*}
\]

\[ \tag{2.15} \]

C. Standard Hamiltonian

The above formalism can be made more specific for the standard Hamiltonian (1.8). Then the path integrals over the momentum paths \( p(t) \) in (2.3) and (2.5) becomes harmonic and can be explicitly evaluated. The phase space integral (2.3), for instance, reduces to the configuration space path integral

\[
(x_b t_b | x_a t_a)[k(t), j(t)] = \int_{x_a t_a}^{x_b t_b} dx(t) \exp \left\{ \frac{i}{\hbar} A[x(t); k(t), j(t)] \right\} ,
\]

where the current \( k(t) \) couples linearly to the path momentum \( M \dot{x}(t) \) in the action

\[
A[x(t); k(t), j(t)] = \int_{t_a}^{t_b} dt \left\{ \frac{M}{2} \dot{x}^2(t) - V(x(t), t) + x(t)j(t) + M \dot{x}(t)k(t) + \frac{M}{2} k^2(t) \right\} .
\]

\[ \tag{2.17} \]

A subsequent partial integration transforms the current \( k(t) \) to an effective coordinate current with an extra phase factor:

\[
(x_b t_b | x_a t_a)[k(t), j(t)]
= (x_b t_b | x_a t_a)[0, j(t) - M \dot{k}(t)] \exp \left\{ \frac{iM}{\hbar} \left[ x_b k_b - x_a k_a + \frac{1}{2} \int_{t_a}^{t_b} dt \dot{k}^2(t) \right] \right\} .
\]

\[ \tag{2.18} \]

In the next section we determine the generating functional \( (x_b t_b | x_a t_a)[0, j(t)] \) for a harmonic oscillator with arbitrary time-dependent frequency \( \Omega(t) \) and use (2.18) to construct the full generating functional \( (x_b t_b | x_a t_a)[k(t), j(t)] \).
Consider a standard Hamiltonian (1.8) with a harmonic potential containing an arbitrary time-dependent frequency:

\[ V(x, t) = \frac{M}{2} \Omega^2(t)x^2. \]  

(3.1)

The generating functionals (2.3) and (2.5) are then expressible in terms of two fundamental solutions \( D_a(t), D_b(t) \) of the corresponding classical equation of motion with particular boundary conditions [2]

\[ \hat{K}(t) D_a(t) = 0; \quad D_a(t_a) = 0, \quad \dot{D}_a(t_a) = 1, \]  

(3.2)

\[ \hat{K}(t) D_b(t) = 0; \quad D_b(t_b) = 0, \quad \dot{D}_b(t_b) = -1, \]  

(3.3)

where \( \hat{K}(t) \) denotes the operator

\[ \hat{K}(t) = -\partial_t^2 - \Omega^2(t). \]  

(3.4)

Since the time derivative of the Wronski determinant

\[ W(t) = D_a(t)\dot{D}_b(t) - \dot{D}_a(t)D_b(t) \]  

(3.5)

vanishes, we observe the identity

\[ D_a(t_b) = D_b(t_a). \]  

(3.6)

Note that a similar identity does not hold for the time derivatives of the two fundamental solutions \( D_a(t) \) and \( D_b(t) \). Indeed, partially integrating the differential equation for \( \dot{D}_a(t) \) and taking into account (3.2)-(3.3), we deduce
\[
\dot{D}_b(t_a) + \dot{D}_a(t_b) = -2 \int_{t_a}^{t_b} dt \, \Omega(t) \dot{\Omega}(t) \, D_a(t) D_b(t).
\] 

Let us now determine the time evolution amplitude (2.16) in configuration space for a vanishing current \( k(t) \) as defined in (2.16). We decompose the paths \( x(t) \) into the classical path \( x_{\text{cl}}^j(t) \) and the quantum fluctuations \( \delta x(t) \) around it:

\[
x(t) = x_{\text{cl}}^j(t) + \delta x(t).
\] 

The classical path \( x_{\text{cl}}^j(t) \) solves the boundary value problem

\[
\tilde{K}(t) \, x_{\text{cl}}^j(t) = -\frac{j(t)}{M}; \quad x_{\text{cl}}^j(t_a) = x_a, \ x_{\text{cl}}^j(t_b) = x_b,
\] 

and the fluctuations \( \delta x(t) \) vanish at the endpoints:

\[
\delta x(t_a) = \delta x(t_b) = 0.
\]

Inserting the decomposition (3.8) into the action (2.17), we observe that due to (3.8) and (3.9) the total action decomposes into a classical part

\[
\mathcal{A}[x_{\text{cl}}^j(t); 0, j(t)] = \frac{M}{2} \left[ x_b \dot{x}_{\text{cl}}^j(t_b) - x_a \dot{x}_{\text{cl}}^j(t_a) \right] + \frac{1}{2} \int_{t_a}^{t_b} dt \, x_{\text{cl}}^j(t) j(t),
\]

and a fluctuation part, which is simply the classical action evaluated for the fluctuations \( \delta x(t) \) at \( j = 0 \):

\[
\mathcal{A}[x(t); 0, j(t)] = \mathcal{A}[x_{\text{cl}}^j(t); 0, j(t)] + \mathcal{A}[\delta x(t); 0, 0].
\]

Inserting this into the original path integral (2.16), it factorizes into the product of a classical amplitude with the classical action (3.11), and an additional fluctuation factor which is equal to the amplitude at vanishing end points:

\[
(x_b \, t_b \, | \, x_a \, t_a \, )[0; j(t)] = \exp \left\{ \frac{i}{\hbar} \, \mathcal{A}[x_{\text{cl}}^j(t); 0, j(t)] \right\} \, (x_b = 0 \, t_b \, | \, x_a = 0 \, t_a \, )[0, 0].
\]
A. Classical Action

The classical action in the presence of currents can be expressed in terms of the solutions $D_a(t), D_b(t)$ of the time-dependent harmonic boundary value problems (3.2) and (3.3). First we decompose the solution of the boundary value problem (3.9) in the presence of into a homogeneous and an inhomogeneous contribution:

$$x_{cl}^j(t) = x_{cl}(t) + \Delta x_{cl}^j(t).$$

(3.14)

The homogeneous solution reads

$$x_{cl}(t) = \frac{D_b(t)x_a + D_a(t)x_b}{D_a(t_b)},$$

(3.15)

while the inhomogeneous one is given by

$$\Delta x_{cl}^j(t) = -\frac{1}{M} \int_{t_a}^{t_b} dt' G_{jj}^x(t, t') j(t'),$$

(3.16)

where $G_{jj}^x(t, t')$ denotes the Green function of the classical equation of motion

$$\dot{K}(t) G_{jj}^x(t, t') = \delta(t - t'),$$

(3.17)

with Dirichlet boundary conditions

$$G_{jj}^x(t_a, t') = G_{jj}^x(t_b, t') = 0.$$  

(3.18)

From (3.17) we deduce that the Green function $G_{jj}^x(t, t')$ solves the homogeneous differential equation for $t \neq t'$:

$$\dot{K}(t)G_{jj}^x(t, t') = 0,$$

(3.19)

and that its first derivative $\partial_t G_{jj}^x(t, t')$ is discontinuous at $t = t'$:
\[
\lim_{\epsilon \to 0} \left[ \partial_t G^x_{jj}(t, t') \bigg|_{t = t' + \epsilon} - \partial_t G^x_{jj}(t, t') \bigg|_{t = t' - \epsilon} \right] = -1. \tag{3.20}
\]

The Green function itself is continuous around \( t = t' \):

\[
\lim_{\epsilon \to 0} \left[ G^x_{jj}(t, t') \bigg|_{t = t' + \epsilon} - G^x_{jj}(t, t') \bigg|_{t = t' - \epsilon} \right] = 0. \tag{3.21}
\]

The solution of (3.18)-(3.21) is given by Wronski’s famous expression

\[
G^x_{jj}(t, t') = \Theta(t - t') D_a(t) D_b(t) + \Theta(t' - t) D_a(t') D_b(t'), \tag{3.22}
\]

where \( \Theta(t - t’) \) denotes the Heaviside function which vanishes for \( t < t' \) and is equal to unity for \( t > t' \). Inserting (3.14) and (3.16) we obtain for the classical action (3.11)

\[
A[x^j_{cl}(t); 0, j(t)] = \frac{M}{2D_a(t_b)} \left[ \dot{D}_a(t_b)x_b^2 - \dot{D}_b(t_a)x_a^2 - 2x_ax_b \right] + \int_{t_a}^{t_b} dt \, x_{cl}(t) j(t) - \frac{1}{2M} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \, G^x_{jj}(t, t') j(t) j(t'), \tag{3.23}
\]

where \( x_{cl}(t) \) and \( G^x_{jj}(t, t') \) are given by (3.15) and (3.22), respectively.

**B. Fluctuation Factor**

Now we calculate the fluctuation factor in (3.13). Recalling the path representation (2.16) with the action (2.17), we have to evaluate

\[
(x_b = 0_{t_b} \mid x_a = 0_{t_a})[0, 0] = \int \mathcal{D} \delta x(t) \exp \left[ \frac{iM}{2\hbar} \int_{t_a}^{t_b} dt \delta x(t) \hat{K}(t) \delta x(t) \right]. \tag{3.24}
\]

To this end we decompose the fluctuations \( \delta x(t) \) in (3.24) into eigenfunctions \( x_n(t) \) of the operator \( \hat{K}(t) \) of (3.4) with Dirichlet boundary conditions

\[
\hat{K}(t) \, x_n(t) = \lambda_n \, x_n(t); \quad x_n(t_a) = x_n(t_b) = 0 \tag{3.25}
\]
which satisfy the orthonormality and completeness relations

\[ \int_{t_a}^{t_b} dt \, x_n(t) x_{n'}(t) = \delta_{n,n'} , \]  
(3.26)

\[ \sum_n x_n(t) x_n(t') = \delta(t - t') , \]  
(3.27)

as follows:

\[ \delta x(t) = \sum_n c_n x_n(t) . \]  
(3.28)

The path integral over all possible fluctuations \( \delta x(t) \) in (3.24) amounts to a product of integrals over all expansion coefficients \( c_n \):

\[ \delta x(t_a) = 0 \int \delta x(t_b) = 0 D \delta x(t) = J \left\{ \prod_n \int_{-\infty}^{+\infty} dc_n \right\} . \]  
(3.29)

The Jacobi determinant \( J \) of the transformation (3.28) is an irrelevant constant. Applying (3.25)-(3.29), the path integral (3.24) is finally determined by

\[ (x_b = 0 t_b \mid x_a = 0 t_a)[0,0] = \frac{J}{\sqrt{\text{Det} \hat{K}(t)}} , \]  
(3.30)

where the determinant of the operator \( \hat{K}(t) \) is equal to the product of its eigenvalues

\[ \text{Det} \hat{K}(t) = \prod_n \lambda_n . \]  
(3.31)

C. Operator Determinant

In order to calculate the operator determinant (3.31) it is advantageous to introduce a one-parameter family of operators [3,4]

\[ \hat{K}^g(t) = -\partial_t^2 - g \Omega^2(t) , \]  
(3.32)
depending linearly on a coupling strength parameter \( g \in [0, 1] \), and coinciding with the original operator \( \hat{K}(t) \) in (3.4) for \( g = 1 \). It is possible to relate the operator determinant \( \text{Det} \hat{K}^g(t) \) to the fundamental solutions \( D^g_a(t), D^g_b(t) \), and to the Green function \( G^x_{jj}(t, t') \) emerging from (3.2), (3.3) and (3.17), (3.18). For this we substitute the operator \( \hat{K}(t) \) by \( \hat{K}^g(t) \), and differentiate the \( g \)-dependent version of the eigenvalue problem (3.25) with respect to \( g \):

\[
\hat{K}^g(t) \frac{\partial x^g_n(t)}{\partial g} - \Omega^2(t) x^g_n(t) = \frac{\partial \lambda^g_n}{\partial g} x^g_n(t) + \lambda^g_n \frac{\partial x^g_n(t)}{\partial g}.
\]  

(3.33)

Multiplying (3.33) with \( x^g_n(t)/\lambda^g_n \) and performing a summation over \( n \) plus an integration with respect to \( t \), we obtain with (3.25), (3.26) and (3.31):

\[
\frac{\partial}{\partial g} \ln \text{Det} \hat{K}^g(t) = - \int_{t_a}^{t_b} dt \Omega^2(t) G^x_{jj}(t, t).
\]  

(3.34)

In the last step we have used the spectral decomposition of the Green function

\[
G^x_{jj}(t, t') = \sum_n x^g_n(t) x^g_n(t') \lambda^g_n.
\]  

(3.35)

To solve the differential equation (3.34), we differentiate the boundary value equation (3.2) for \( D^g_a(t) \) with respect to \( g \), and obtain the inhomogeneous initial value problem

\[
\hat{K}^g(t) \frac{\partial D^g_a(t)}{\partial g} = \Omega^2(t) D^g_a(t); \quad \frac{\partial D^g_a(t)}{\partial g} \bigg|_{t=t_a} = \frac{\partial}{\partial t} \frac{\partial D^g_a(t)}{\partial g} \bigg|_{t=t_a} = 0.
\]  

(3.36)

Generalizing (3.22) from \( g = 1 \) to arbitrary values \( g \in [0, 1] \), the solution of (3.36) is given by

\[
\frac{\partial}{\partial g} \ln D^g_a(t_b) = - \int_{t_a}^{t_b} dt \Omega^2(t) G^x_{jj}(t, t).
\]  

(3.37)

This shows that (3.34) is solved by
\[ \text{Det } \hat{K}^g(t) = C \, D_a^g(t_b), \]  

(3.38)

where \( C \) denotes some constant. Due to this result, the ratio of two fluctuation factors with two different parameters \( g_1 \) and \( g_2 \) can be rewritten as

\[
\frac{(x_b = 0 \, t_b \mid x_a = 0 \, t_a)[0,0]^{g_1}}{(x_b = 0 \, t_b \mid x_a = 0 \, t_a)[0,0]^{g_2}} = \frac{\sqrt{D_a^{g_2}(t_b)}}{\sqrt{D_a^{g_1}(t_b)}}.
\]

(3.39)

This serves to determine the fluctuation factor of the initial time-dependent harmonic oscillator at \( g_1 = 1 \) in terms of the fluctuation factor of the free particle \( g_2 = 0 \). The latter is well-known and may be calculated explicitly, for instance, via time-slicing [1, Chap. 2] as

\[
(x_b = 0 \, t_b \mid x_a = 0 \, t_a)[0,0]^{g_2=0} = \sqrt{\frac{M}{2\pi i \hbar (t_b - t_a)}}.
\]

(3.40)

Since the obvious solution of (3.2) at \( g_2 = 0 \) reads \( D_a^{g_2=0}(t_b) = t_b - t_a \), we obtain the famous Gelfand-Yaglom formula for Dirichlet boundary conditions [5]:

\[
(x_b = 0 \, t_b \mid x_a = 0 \, t_a)[0,0] = \sqrt{\frac{M}{2\pi i \hbar D_a(t_b)}}.
\]

(3.41)

Note that similar results can also be derived for periodic and antiperiodic boundary conditions [3,4].

### D. Full Generating Functional

Having obtained the generating functional \( (x_b \, t_b \mid x_a \, t_a)[0, j(t)] \) of the harmonic oscillator with arbitrary frequency with vanishing current \( k(t) \), we now make use of the relation (2.18) to derive the full generating functional \( (x_b \, t_b \mid x_a \, t_a)[k(t), j(t)] \). The terms containing the current velocity \( \dot{k}(t) \) can be turned into functionals of \( k(t) \) itself with the help of several partial integrations. These turn out to remove the extra phase factor in (2.18). As a result,
the time evolution amplitude in the configuration representation is determined by a Van Vleck-Pauli-Morette type of formula \[1, \text{Chap. 4}\]

\[
(x_b, t_b | x_a, t_a)[k(t), j(t)] = \sqrt{\frac{i}{2\pi\hbar}} \frac{\partial^2 \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)]}{\partial x_b \partial x_a} \exp \left\{ \frac{i}{\hbar} \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] \right\} \tag{3.42}
\]

with the action

\[
\mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] = \frac{M}{2D_a(t_b)} \left[ \dot{D}_a(t_b) x_b^2 - \dot{D}_b(t_a) x_a^2 - 2x_a x_b \right]
+ \int_{t_a}^{t_b} \! \! dt \left[ x_{cl}(t) j(t) + p_{cl}(t) k(t) \right] - \frac{1}{2} \int_{t_a}^{t_b} \! \! dt \int_{t_a}^{t_b} \! \! dt' \left[ \frac{1}{M} G_{jj}^{\pi}(t, t') j(t) j(t') \right.
+ \left. G_{jk}^{\pi}(t, t') j(t) k(t) + G_{kk}^{\pi}(t, t') k(t) k(t') \right]. \tag{3.43}
\]

The homogeneous classical solution \(x_{cl}(t)\) is given in (3.15), and \(p_{cl}(t)\) denotes the classical momentum \(p_{cl}(t) \equiv M\dot{x}_{cl}(t)\). The Green function \(G_{jj}^{\pi}(t, t')\) is given by (3.22), while the others are

\[
G_{jk}^{\pi}(t, t') = \frac{\Theta(t - t') \dot{D}_b(t) \dot{D}_a(t') + \Theta(t' - t) \dot{D}_a(t) \dot{D}_b(t')}{D_a(t_b)} = G_{kj}^{\pi}(t', t), \tag{3.44}
\]

\[
G_{kk}^{\pi}(t, t') = \frac{\Theta(t - t') \dot{D}_b(t) \dot{D}_a(t') + \Theta(t' - t) \dot{D}_a(t) \dot{D}_b(t')}{D_a(t_b)} = G_{kk}^{\pi}(t', t). \tag{3.45}
\]

By differentiating (3.43) functionally with respect to \(j\) and \(k\), we see that the Green functions correspond to the correlation functions

\[
\langle x_b | \tilde{x}(t) \tilde{x}(t') | x_a \rangle = \frac{i\hbar}{M} G_{jj}^{\pi}(t, t'), \tag{3.46}
\]

\[
\langle x_b | \tilde{x}(t) \tilde{p}(t') | x_a \rangle = i\hbar G_{jk}^{\pi}(t, t') = i\hbar G_{kj}^{\pi}(t', t), \tag{3.47}
\]

\[
\langle x_b | \tilde{p}(t) \tilde{p}(t') | x_a \rangle = i\hbar M G_{kk}^{\pi}(t, t') \tag{3.48}
\]

with \(\tilde{x}(t) = x(t) - x_{cl}(t)\) and \(\tilde{p}(t) = p(t) - p_{cl}(t)\). These results can be summarized by the mnemonic rule that the Green functions involving a momentum current \(k(t)\) once or twice
follow from \( G^x_{jj}(t, t') \) by one or two time derivatives if the time derivatives of the Heaviside functions are neglected:

\[
G^x_{jk}(t, t') = \frac{\partial G^x_{jj}(t, t')}{\partial t}, \quad G^x_{kj}(t, t') = \frac{\partial G^x_{jj}(t, t')}{\partial t'}, \quad G^x_{kk}(t, t') = \frac{\partial^2 G^x_{jj}(t, t')}{\partial t \partial t'}, \quad (3.49)
\]

A complete analogous expression to (3.43) is found for the time evolution amplitude in the momentum representation. The Fourier transformation (2.4) of (3.42) yields a Van Vleck-Pauli-Morette type of formula

\[
(p_b t_b | p_a t_a)[k(t), j(t)] = \sqrt{\frac{2\pi i \hbar}{\partial_{p_b} \partial_{p_a}} \exp \left\{ \frac{i}{\hbar} \mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)] \right\}}
\]

where the action is the Legendre transform of (3.43)

\[
\mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)] = \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] - p_b x_b + p_a x_a,
\]

calculated for the conjugate variables

\[
p_b = \frac{\partial \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)]}{\partial x_b}, \quad p_a = -\frac{\partial \mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)]}{\partial x_a}.
\]

This brings (3.51) to the form

\[
\mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)] = \frac{D_a(t_b) \left[ \frac{\partial}{\partial p_a} \mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)] \right] - \frac{\partial}{\partial p_a} \mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)]}{2M[1 + \dot{D}_a(t_b)D_b(t_a)]}
\]

\[
+ \int_{t_a}^{t_b} dt \left[ \frac{\partial}{\partial j(t)} \mathcal{A}(p_b, t_b; p_a, t_a)[k(t), j(t)] \right] - \frac{1}{2} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \left[ \frac{1}{M} G^p_{jj}(t, t')j(t)j(t') - G^p_{jk}(t, t')j(t)k(t') + MG^p_{kk}(t, t')k(t)k(t') \right],
\]

where the classical solution now reads

\[
\tilde{x}_{cl}(t) = \frac{p_a[D_a(t) + D_b(t)\dot{D}_a(t_b)] + p_b[D_a(t)\dot{D}_b(t_a) - D_b(t)]}{M[1 + \dot{D}_a(t_b)D_b(t_a)]},
\]

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and $\bar{p}_c(t)$ denotes the associated classical momentum $\bar{p}_c(t) \equiv M \dot{x}_c(t)$. The Green functions in (3.53) turn out to be

$$
G_{jj}^p(t, t') = \Theta(t - t') \frac{[D_b(t) \dot{D}_a(t) + D_a(t)][D_a'(t') \dot{D}_b(t_a) - D_b'(t')]}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} + \Theta(t' - t) \frac{D_a(t) [D_a'(t') \dot{D}_b(t_a) - D_b'(t')]}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} = G_{jj}^p(t', t), \quad (3.55)
$$

$$
G_{jk}^p(t, t') = \Theta(t - t') \frac{D_b(t) \dot{D}_a(t_a) + D_a(t)}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} + \Theta(t' - t) \frac{D_a(t) \dot{D}_b(t_a) - D_b(t)}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} = G_{jk}^p(t', t), \quad (3.56)
$$

$$
G_{kk}^p(t, t') = \Theta(t - t') \frac{\dot{D}_b(t) \dot{D}_a(t_a) + D_a(t)}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} + \Theta(t' - t) \frac{\dot{D}_a(t) \dot{D}_b(t_a) - D_b(t)}{D_a(t_b)[1 + D_a(t_b) D_b(t_a)]} = G_{kk}^p(t', t). \quad (3.57)
$$

The relation to the correlation functions is similar to (3.46)–(3.48):

$$
\langle p_b \mid \bar{x}(t) \bar{x}(t') \mid p_a \rangle = \frac{i\hbar}{M} G_{jj}^p(t, t'), \quad (3.58)
$$

$$
\langle p_b \mid \bar{x}(t) \bar{p}(t') \mid p_a \rangle = i\hbar G_{jk}^p(t, t') = i\hbar G_{kj}^p(t', t), \quad (3.59)
$$

$$
\langle p_b \mid \bar{p}(t) \bar{p}(t') \mid p_a \rangle = i\hbar MG_{kk}^p(t, t'). \quad (3.60)
$$

The relation between the similar-looking actions (3.43) and (3.53) becomes more transparent by reexpressing both in terms of partial derivatives of the classical solutions $x_c(t), \bar{x}_c(t), p_c(t), \bar{p}_c(t)$ with respect to the end points $x_b, x_a$ and $p_b, p_a$, respectively. In the configuration representation we obtain

$$
\mathcal{A}(x_b, t_b; x_a, t_a)[k(t), j(t)] = \frac{1}{2}(x_b, x_a) \left( \begin{array}{c} \frac{\partial p_b}{\partial x_b} \\ \frac{\partial p_b}{\partial x_a} \end{array} \right) \begin{pmatrix} x_b \\ x_a \end{pmatrix} + \int_{t_a}^{t_b} dt \left( x_b, x_a \right) \left( \begin{array}{c} \frac{\partial p_c(t)}{\partial x_b} \\ \frac{\partial p_c(t)}{\partial x_a} \end{array} \right) \begin{pmatrix} x_b \\ x_a \end{pmatrix} \left( \begin{array}{c} k(t) \\ j(t) \end{array} \right)
$$

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\[-\frac{1}{2} \frac{\partial x_b}{\partial p_a} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' (k(t), j(t)) \left[ \Theta(t - t') \begin{pmatrix} \frac{\partial p_{cl}(t)}{\partial x_a} & \frac{\partial p_{cl}(t)}{\partial x_b} \\ \frac{\partial x_{cl}(t)}{\partial x_a} & \frac{\partial x_{cl}(t)}{\partial x_b} \end{pmatrix} \begin{pmatrix} \frac{\partial p_{cl}(t)}{\partial x_a} & \frac{\partial p_{cl}(t)}{\partial x_b} \\ \frac{\partial x_{cl}(t)}{\partial x_a} & \frac{\partial x_{cl}(t)}{\partial x_b} \end{pmatrix} \right] \begin{pmatrix} \frac{\partial x_{cl}(t)}{\partial x_a} & \frac{\partial x_{cl}(t)}{\partial x_b} \\ \frac{\partial x_{cl}(t)}{\partial x_a} & \frac{\partial x_{cl}(t)}{\partial x_b} \end{pmatrix} \right] \begin{pmatrix} k(t') \\ j(t') \end{pmatrix} \right]. \tag{3.61}

The momentum representation, on the other hand, has the analogous form with \( x \) and \( p \) interchanged:

\[
A(p_b, t_b; p_a, t_a)[k(t), j(t)] = \frac{1}{2} (p_b, p_a) \begin{pmatrix} \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \\ \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \end{pmatrix} \begin{pmatrix} k(t) \\ j(t) \end{pmatrix}
+ \int_{t_a}^{t_b} dt (p_b, p_a) \begin{pmatrix} \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \\ \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \end{pmatrix} \begin{pmatrix} k(t) \\ j(t) \end{pmatrix}
+ \frac{1}{2} \frac{\partial x_b}{\partial p_a} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' (k(t), j(t)) \left[ \Theta(t - t') \begin{pmatrix} \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \\ \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \end{pmatrix} \begin{pmatrix} \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \\ \frac{\partial x_{cl}(t)}{\partial p_a} & \frac{\partial x_{cl}(t)}{\partial p_b} \end{pmatrix} \right] \begin{pmatrix} k(t') \\ j(t') \end{pmatrix} \right]. \tag{3.62}

These expressions for the generating functionals (3.42) and (3.50) exhibit clearly the symmetry properties (2.8) and (2.9).

\section*{IV. SMEARING FORMULA FOR HARMONIC FLUCTUATIONS}

As a first application of the generating functional (3.42) we derive a general rule for calculating correlation functions of polynomial or nonpolynomial functions of \( x(t) \) and \( p(t) \).
The result will be expressed in the form of a smearing formula. This formula will represent an essential tool for calculating perturbation expansions with nonpolynomial interactions. Such expansions serve in variational perturbation theory to obtain convergent approximations for quantum-statistical partition functions [6] or density matrices [7].

Consider the correlation functions of a product of local functions for vanishing currents

$$\langle F_1(x(t_1)) F_2(x(t_2)) \ldots F_N(x(t_N)) F_{N+1}(p(t_{N+1})) F_{N+2}(p(t_{N+2})) \ldots F_{N+M}(p(t_{N+M})) \rangle^{x_b \ldots x_a}_\Omega$$

$$= \frac{1}{(x_b t_b \mid x_a t_a)} \int \frac{dx dp}{2\pi \hbar} \prod_{n=1}^N [F_n(x(t_n)) \prod_{m=1}^M [F_{N+m}(p(t_{N+m}))] \exp \left\{ \frac{i}{\hbar} A[p, x; 0, 0] \right\}, \quad (4.1)$$

where the harmonic time evolution amplitude with zero external currents $(x_b t_b \mid x_a t_a)[0, 0]$ is written as $(x_b t_b \mid x_a t_a)$. By Fourier transforming the functions $F_n(x(t_n))$ and $F_{N+m}(p(t_{N+m}))$ according to

$$F_n(x(t_n)) = \int_{-\infty}^{+\infty} dx_n F_n(x_n) \delta(x_n - x(t_n)) = \int_{-\infty}^{+\infty} dx_n F(x_n) \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} \exp \{i\xi_n(x_n - x(\tau_n))\}$$

(4.2)

and

$$F_{N+m}(p(t_{N+m})) = \int_{-\infty}^{+\infty} \frac{dp_m}{2\pi \hbar} F_{N+m}(p_m) \delta(p_m - p(t_{N+m})) = \int_{-\infty}^{+\infty} \frac{dp_m}{2\pi \hbar} F_{N+m}(p_m) \int_{-\infty}^{+\infty} dk_m e^{-i\kappa_m(p_m - p(t_{N+m}))/\hbar}, \quad (4.3)$$

the correlation functions (4.1) may be reexpressed as

$$\langle F_1(x(t_1)) \ldots F_{N+M}(p(t_{N+M})) \rangle^{x_b \ldots x_a}_\Omega = \frac{1}{(x_b t_b \mid x_a t_a)} \prod_{n=1}^N \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} e^{i\xi_n x_n} \right]$$

$$\times \prod_{m=1}^M \left[ \int_{-\infty}^{+\infty} \frac{dp_m}{2\pi \hbar} F_{N+m}(p_m) \int_{-\infty}^{+\infty} dk_m e^{-i\kappa_m p_m/\hbar} \right] (x_b t_b \mid x_a t_a) [k, j], \quad (4.4)$$

where the generating functional is given by (3.42). The currents $j(t)$ and $k(t)$ are specialized to
\[ j(t) = -\hbar \sum_{n=1}^{N} \xi_n \delta(t - t_n), \quad k(t) = \sum_{m=1}^{M} \kappa_m \delta(t - t_{N+m}). \] (4.5)

Inserting these equations into the action (3.43) and the Green functions (3.22), (3.44) and (3.45), we find the Fourier decomposition of the the generating functional (3.42), so that the correlation functions (4.4) become

\[
\langle F_1(x(t_1)) \ldots F_{N+M}(p(t_{N+M})) \rangle_{\Omega}^{x_0,x_0} = \prod_{n=1}^{N} \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \right] \int_{-\infty}^{+\infty} \frac{d\xi_n}{2\pi} e^{i\xi_n(x_n - x_{cl}(t_n))}
\]
\[
\times \prod_{m=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dp_m}{2\pi \hbar} F_{N+m}(p_m) \right] e^{-i\kappa_m(p_m - p_{cl}(t_{N+m}))}/\hbar
\]
\[
\times \exp \left\{ -\frac{i\hbar}{2M} \sum_{n,n'=1}^{N} \xi_n G_{jj}^{nn'} \xi_{n'} + \frac{i}{2} \sum_{m=1}^{M} \sum_{n=1}^{N} \xi_n G_{jk}^{nn} \kappa_m - \frac{iM}{2\hbar} \sum_{m,m'=1}^{M} \kappa_m G_{kk}^{mm'} \kappa_{m'} \right\}, \quad (4.6)
\]

where we used the abbreviations

\[
G_{jj}^{nn'} = G_{jj}^{x}(t_n, t_{n'}), \quad G_{jk}^{nn} = G_{jk}^{x}(t_n, t_{N+m}), \quad G_{kk}^{mm'} = G_{kk}^{x}(t_{N+m}, t_{N+m'}). \quad (4.7)
\]

To proceed, it is more convenient to write expression (4.6) as a convolution integral

\[
\langle F_1(x(t_1)) \ldots F_{N+M}(p(t_{N+M})) \rangle_{\Omega}^{x_0, x_0} = \prod_{n=1}^{N} \left[ \int_{-\infty}^{+\infty} dx_n F_n(x_n) \right] \prod_{m=1}^{M} \left[ \int_{-\infty}^{+\infty} \frac{dp_m}{2\pi \hbar} F_{N+m}(p_m) \right]
\]
\[
\times \left( \frac{M\Omega}{\hbar} \right)^{(N-M)/2} P(x_1, \ldots, x_N, p_1, \ldots, p_M) \quad (4.8)
\]

involving the Gaussian distribution

\[
P(x_1, \ldots, p_M) \equiv \frac{1}{(2\pi)^N} \int d^{N+M}v \exp \left\{ i\mathbf{w}^T \mathbf{v} - \frac{i}{2} \mathbf{v}^T \mathbf{G} \mathbf{v} \right\}. \quad (4.9)
\]

The dimensionless vectors \( \mathbf{v} \) and \( \mathbf{w} \) have \( N + M \) components and are defined as

\[
\mathbf{v}^T = \left( \sqrt{\frac{\hbar}{M\Omega}} \xi_1, \ldots, \sqrt{\frac{\hbar}{M\Omega}} \xi_N, \sqrt{\frac{M\Omega}{\hbar}} \kappa_1, \ldots, \sqrt{\frac{M\Omega}{\hbar}} \kappa_M \right) \quad (4.10)
\]

and
\[ w^T = \left( \sqrt{\frac{M\Omega}{\hbar}} [x_1 - x_{cl}(t_1)], \ldots, \sqrt{\frac{M\Omega}{\hbar}} [x_N - x_{cl}(t_N)], \right. \]
\[ \left. - \frac{1}{\sqrt{\hbar M\Omega}} [p_1 - p_{cl}(t_{N+1})], \ldots, - \frac{1}{\sqrt{\hbar M\Omega}} (p_M - p_{cl}(t_{N+M})) \right). \]  
\( \text{(4.11)} \)

The \((N + M) \times (N + M)\)-matrix of Green functions
\[ G = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix} \]  
\( \text{(4.12)} \)
can be decomposed into block matrices \(A\), \(B\), and \(C\). The \(N \times N\)-matrix \(A\) and the \(M \times M\)-matrix \(C\) are defined by
\[ A = \Omega \begin{pmatrix} G_{1j}^{11} & G_{1j}^{12} & \cdots & G_{1j}^{jN} \\ G_{1j}^{2j} & G_{1j}^{1j} & \cdots & G_{1j}^{j2} \\ \vdots & \vdots & \ddots & \vdots \\ G_{1j}^{1N} & G_{1j}^{2N} & \cdots & G_{1j}^{j1} \end{pmatrix}, \quad C = \frac{1}{\Omega} \begin{pmatrix} G_{1k}^{11} & G_{1k}^{12} & \cdots & G_{1k}^{1M} \\ G_{1k}^{2k} & G_{1k}^{1k} & \cdots & G_{1k}^{2k} \\ \vdots & \vdots & \ddots & \vdots \\ G_{1k}^{NM} & G_{1k}^{2M} & \cdots & G_{1k}^{1M} \end{pmatrix} \]  
\( \text{(4.13)} \)
and yield quadratic forms of the position and momentum variables, respectively. The \(N \times M\)-matrix
\[ B = \begin{pmatrix} -G_{1j}^{11} & -G_{1j}^{12} & \cdots & -G_{1j}^{1M} \\ -G_{1j}^{2j} & -G_{1j}^{11} & \cdots & -G_{1j}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ -G_{1j}^{jN} & -G_{1j}^{N2} & \cdots & -G_{1j}^{NM} \end{pmatrix} \]  
\( \text{(4.14)} \)
gives rise to quadratic terms which are linear in both position and momentum variables.

The multidimensional integral in (4.9) is of Fresnel type and can easily be done, yielding and explicit expression for the Gaussian distribution (4.8)
\[ P(x_1, \ldots, x_N, p_1, \ldots, p_M) = \frac{1}{\sqrt{i^{N+M}(2\pi)^{N-M} \det G}} \exp \left\{ \frac{i}{2} w^T G^{-1} w \right\}, \]  
\( \text{(4.15)} \)
where $G^{-1}$ represents the matrix inverse of (4.12) whose block form is

$$
G^{-1} = \begin{pmatrix}
X^{-1} & -X^{-1}BC^{-1} \\
-C^{-1}B^TX^{-1} & C^{-1} + C^{-1}B^TX^{-1}BC^{-1}
\end{pmatrix}
$$

(4.16)

with the abbreviation

$$
X = A - BC^{-1}B^T.
$$

(4.17)

Since the matrix $G$ may be decomposed as

$$
G = \begin{pmatrix}
1 & B \\
0 & C
\end{pmatrix}
\begin{pmatrix}
X & 0 \\
C^{-1}B^T & 1
\end{pmatrix}
$$

(4.18)

when the matrix $C$ is regular, the determinant of $G$ factorizes as follows

$$
\det G = \det C \det X.
$$

(4.19)

For singular matrix $C$ but $A$ regular, one may make use of another decomposition,

$$
G = \begin{pmatrix}
1 & 0 \\
B^TA^{-1} & X'
\end{pmatrix}
\begin{pmatrix}
A & B \\
0 & 1
\end{pmatrix},
$$

(4.20)

with $X' = C - B^TA^{-1}B$. Then the determinant of $G$ is given by

$$
\det G = \det X' \det A.
$$

(4.21)

With the Gaussian distribution (4.15), our result (4.8) constitutes a *smearing formula* which describes the effect of harmonic fluctuations upon arbitrary products of functions of space and momentum variables at different times.
In applications, there often occur correlation functions for mixtures of nonpolynomial functions $F(\tilde{x}(t))$ or $F(\tilde{p}(t))$ and powers according to

$$
\langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle^\Omega_{xb,xa}, \langle F(\tilde{p}(t_1)) \tilde{p}^n(t_2) \rangle^\Omega_{xb,xa},
$$

$$
\langle F(\tilde{p}(t_1)) \tilde{x}^n(t_2) \rangle^\Omega_{xb,xa}, \langle F(\tilde{p}(t_1)) \tilde{p}^n(t_2) \rangle^\Omega_{xb,xa}.
$$

(5.1)

In order to evaluate such correlation functions, we derive in this section generalized Wick rules and Feynman diagrams on the basis of the smearing formula (4.8).

A. Ordinary Wick Rules

It is well known that if one has to calculate expectation values of polynomials with even power, Wick’s rule can be written as the sum over all possible permutations of products of two-point functions. We shortly recall to this expansion by considering the case of a position-dependent $n$-point correlation function, $n$ even, defined as

$$
G^{(n)}(t_1, \ldots, t_n) = \langle \tilde{x}(t_1) \cdots \tilde{x}(t_n) \rangle^\Omega_{xb,xa}.
$$

(5.2)

Note that it will be sufficient to study only the correlation functions involving the deviations from the classical path, respectively. This expectation value can be decomposed with the help of Wick’s expansion

$$
G^{(n)}(t_1, \ldots, t_n) = \sum_{\text{pairs}} G^{(2)}(t_{p(1)}, t_{p(2)}) \cdots G^{(2)}(t_{p(n-1)}, t_{p(n)}),
$$

(5.3)

where $p$ denotes the operation of pairwise index permutation. Thereby, the Green function $G^{(2)}(t_1, t_2)$ is already given by (3.46). Note that Eq. (5.3) may be considered as a consequence of a simple derivative rule.
\[ \langle F(\vec{x}(t_1)) \vec{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = \langle \vec{x}(t_1) \vec{x}(t_2) \rangle_{\Omega}^{x_b,x_a} \langle F'(\vec{x}(t_1)) \rangle_{\Omega}^{x_b,x_a} \quad (5.4) \]

with \( F'(\vec{x}) = \partial F(\vec{x})/\partial x \). By applying this recursively, one eventually obtains (5.3). And conversely, the derivative rule (5.4) can be proved for polynomial functions \( F(\vec{x}(t)) \), following directly from Wick’s theorem (5.3).

The two-point Green functions \( G^{(2)}(t_1, t_2) \), occurring in (5.3), can be considered as a Wick contraction which we introduce as follows:

\[
\begin{align*}
\langle \vec{x}(t_1) \vec{x}(t_2) \rangle & = \langle \vec{x}(t_1) \vec{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = \frac{i\hbar}{M} G_{jj}(t_1, t_2), \\
\langle \vec{x}(t_1) \vec{p}(t_2) \rangle & = \langle \vec{x}(t_1) \vec{p}(t_2) \rangle_{\Omega}^{x_b,x_a} = i\hbar G_{jk}(t_1, t_2), \\
\langle \vec{p}(t_1) \vec{x}(t_2) \rangle & = \langle \vec{p}(t_1) \vec{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = i\hbar G_{kj}(t_1, t_2) = i\hbar G_{jk}(t_2, t_1), \\
\langle \vec{p}(t_1) \vec{p}(t_2) \rangle & = \langle \vec{p}(t_1) \vec{p}(t_2) \rangle_{\Omega}^{x_b,x_a} = i\hbar MG_{kk}(t_1, t_2).
\end{align*}
\]

Decomposing polynomial correlations of \( \vec{x}(t) \) and \( \vec{p}(t) \) with the help of these contractions corresponding to Eq. (5.3) or successively applying the derivative rule (5.4) leads to following results

\[
\begin{align*}
\langle \vec{x}^n(t_1) \vec{x}^m(t_2) \rangle_{\Omega}^{x_b,x_a} &= \\
&= \sum_{\text{min}(n,m)} \left[ \frac{i\hbar}{M} G_{jj}(t_1, t_1) \right]^{(n-1)/2} \left[ \frac{i\hbar}{M} G_{jj}(t_1, t_2) \right]^l \left[ \frac{i\hbar}{M} G_{jj}(t_2, t_2) \right]^{(m-1)/2}, \\
\langle \vec{x}^n(t_1) \vec{p}^m(t_2) \rangle_{\Omega}^{x_b,x_a} &= \\
&= \sum_{\text{min}(n,m)} \left[ \frac{i\hbar}{M} G_{jj}(t_1, t_1) \right]^{(n-1)/2} [i\hbar G_{jk}(t_1, t_2)]^l [i\hbar MG_{kk}(t_2, t_2)]^{(m-1)/2}, \\
\langle \vec{p}^n(t_1) \vec{x}^m(t_2) \rangle_{\Omega}^{x_b,x_a} &= \\
&= \sum_{\text{min}(n,m)} [i\hbar MG_{kk}(t_1, t_1)]^{(n-1)/2} [i\hbar G_{jk}(t_2, t_1)]^l \left[ \frac{i\hbar}{M} G_{jj}(t_2, t_2) \right]^{(m-1)/2}, \\
\langle \vec{p}^n(t_1) \vec{p}^m(t_2) \rangle_{\Omega}^{x_b,x_a} &= \\
&= \sum_{\text{min}(n,m)} [i\hbar MG_{kk}(t_1, t_1)]^{(n-1)/2} [i\hbar MG_{kk}(t_1, t_2)]^l [i\hbar MG_{kk}(t_2, t_2)]^{(m-1)/2}.
\end{align*}
\]
with the multiplicity factor

\[ c_l = \frac{(n - l - 1)!!(m - l - 1)!!n!m!}{l!(n - l)!(m - l)!}. \]  

(5.13)

Note, that \((-k)!! \equiv 1\) for any positive integer \(k\). For nonvanishing correlation, the sum \(n + m\) must be even so that the regulation parameter \(\alpha\) is defined as follows:

\[ \alpha = \begin{cases} 0, & n, m \text{ even}, \\ 1, & n, m \text{ odd}. \end{cases} \]  

(5.14)

The contractions defined in (5.5)-(5.8) can be used to treat The desired derivative rules read

\[ \langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle^{x_a,x_b}_\Omega = \sum_{l=\alpha,\alpha+2,\alpha+4,...}^{n} \frac{n!}{(n-l)!!l!!} \left[ \frac{i\hbar}{M} G_{jj}(t_2, t_2) \right]^{(n-l)/2} \left[ \frac{i\hbar}{M} G_{jj}(t_1, t_2) \right]^l \langle F^{(l)}(\tilde{x}(t_1)) \rangle^{x_a,x_b}_\Omega, \]  

(5.15)

\[ \langle F(\tilde{p}(t_1)) \tilde{p}^n(t_2) \rangle^{x_a,x_b}_\Omega = \sum_{l=\alpha,\alpha+2,\alpha+4,...}^{n} \frac{n!}{(n-l)!!l!!} \left[ i\hbar M G_{kk}(t_2, t_2) \right]^{(n-l)/2} \left[ i\hbar M G_{kk}(t_1, t_2) \right]^l \langle F^{(l)}(\tilde{p}(t_1)) \rangle^{x_a,x_b}_\Omega, \]  

(5.16)

\[ \langle F(\tilde{p}(t_1)) \tilde{x}^n(t_2) \rangle^{x_a,x_b}_\Omega = \sum_{l=\alpha,\alpha+2,\alpha+4,...}^{n} \frac{n!}{(n-l)!!l!!} \left[ i\hbar M G_{jj}(t_2, t_2) \right]^{(n-l)/2} \left[ i\hbar M G_{jj}(t_1, t_2) \right]^l \langle F^{(l)}(\tilde{p}(t_1)) \rangle^{x_a,x_b}_\Omega. \]  

(5.17)

\[ \langle F(\tilde{p}(t_1)) \tilde{p}^n(t_2) \rangle^{x_a,x_b}_\Omega = \sum_{l=\alpha,\alpha+2,\alpha+4,...}^{n} \frac{n!}{(n-l)!!l!!} \left[ i\hbar M G_{jj}(t_2, t_2) \right]^{(n-l)/2} \left[ i\hbar M G_{jj}(t_1, t_2) \right]^l \langle F^{(l)}(\tilde{p}(t_1)) \rangle^{x_a,x_b}_\Omega. \]  

(5.18)

The parameter \(\alpha\) distinguishes between even and odd power \(n\):

\[ \alpha = \begin{cases} 0, & n \text{ even}, \\ 1, & n \text{ odd}. \end{cases} \]  

(5.19)

since even (odd) powers of \(n\) lead to even (odd) derivatives of the function \(F(\tilde{x}(t_1))\). The \(l\)th derivative \(F^{(l)}(\tilde{x}(t_1))\) is formed with respect to \(x(t_1)\), and \(F^{(l)}(\tilde{p}(t_1))\) is the \(l\)th derivative with respect to \(p(t_1)\). Note, that in the last line the Green function \(G_{jk}\) appears with
exchanged time arguments, which in this case happens to be inessential due to the symmetry
\[ G_{jk}(t_2, t_1) = G_{kj}(t_1, t_2). \]

**B. Generalized Wick Rule**

According to their derivation, the contractions (5.15)-(5.18) are only applicable to functions \( F(\tilde{x}(t)) \) and \( F(\tilde{p}(t)) \) which can be Taylor-expanded. In the following, we will show with the help of the smearing formula (4.8) that these derivative rules remain valid for functions \( F(\tilde{x}(t)) \) and \( F(\tilde{p}(t)) \) with Laurent expansions. Expectations of this type appear in variational perturbation theory (see for position-position coupling Ref. [7]). Since the proceeding is similar in all the cases (5.15)-(5.18), we shall only discuss the expectation value
\[
\langle F(\tilde{x}(t_1)) \tilde{p}^n(t_2) \rangle_{\Omega}^{x_b,x_a}
\]
in detail. For this we consider the generating functional of all such expectation values following from (4.8)
\[
\langle F(\tilde{x}(t_1)) e^{i\tilde{p}^n(t_2)} \rangle_{\Omega}^{x_b,x_a} = \frac{1}{\sqrt{-\det G}} \int_{-\infty}^{+\infty} dx F(x) \int_{-\infty}^{+\infty} \frac{dp}{2\pi\bar{\hbar}} e^{ip} \times \exp \left\{ \frac{i}{2 \det G} \left[ \frac{M}{\hbar} \frac{G_{kk}(t_2, t_2)}{x^2 - 2\frac{1}{\hbar} G_{jk}(t_1, t_2) xp + \frac{1}{\hbar M} G_{jj}(t_1, t_1) p^2} \right] \right\}. \tag{5.21}
\]

The \( p \)-integration can easily be done, leading to
\[
\langle F(\tilde{x}(t_1)) e^{i\tilde{p}^n(t_2)} \rangle_{\Omega}^{x_b,x_a} = e^{i\hbar M G_{kk}(t_2, t_2) j^2/2}
\times \int_{-\infty}^{+\infty} \frac{dx}{\sqrt{2\pi i\hbar G_{jj}(t_1, t_1)}} F(x + i\hbar G_{jk}(t_1, t_2) j) e^{iMx^2/2G_{jj}(t_1, t_1)}
\]
\[
= e^{i\hbar M G_{kk}(t_2, t_2) j^2/2} \sum_{l=0}^{+\infty} \frac{1}{l!} \left[ i\hbar G_{jk}(t_1, t_2) j \right]^l \langle F^{(l)}(\tilde{x}(t_1)) \rangle_{\Omega}^{x_b,x_a}. \tag{5.22}
\]

The correlation of two functions at different times has been reduced to a single-time expectation value of the \( l \)th derivative of the function \( F(\tilde{x}(t_1)) \) with respect to \( x(t_1) \), denoted by
\( F(t) (\tilde{x}(t_1)) \), with Green functions describing the dependence on the second time. Expanding both sides in powers of \( j \), we reobtain (5.16).

Now we demonstrate that the derivative rules (5.15)-(5.18) for Laurent-expandable functions \( F(\tilde{x}(t)) \) and \( F(\tilde{p}(t)) \) also follow from generalized Wick rules. Without restriction of universality, we only consider the expectation value
\[
\langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle^\Omega_{xb,xa}.
\]

The proceeding to reduce the power of the polynomial at the expense of the function \( F(\tilde{x}(t_1)) \) is as follows:

1a. If possible \( (n \geq 2) \), contract \( \tilde{x}(t_2) \tilde{x}(t_2) \) with multiplicity \( (n - 1) \), giving
\[
(n - 1) \tilde{x}(t_2) \tilde{x}(t_2) \langle F(\tilde{x}(t_1)) \tilde{x}^{n-2}(t_2) \rangle^\Omega_{xb,xa},
\]
else jump to 1b. directly.

1b. Contract \( F(\tilde{x}(t_1)) \tilde{x}(t_2) \) and let the remaining polynomial invariant. We define this contraction by the symbol
\[
F(\tilde{x}(t_1)) \tilde{x}(t_2) \tilde{x}^{n-1}(t_2) = \tilde{x}(t_1) \tilde{x}(t_2) \langle F'(\tilde{x}(t_1)) \tilde{x}^{n-1}(t_2) \rangle^\Omega_{xb,xa}.
\]

1c. Add the terms 1a. and 1b.

2. Repeat steps 1a.-1c. until only expectation values of \( F(\tilde{x}) \) or expectations of its derivatives remain.

Summarizing, we can express the first power reduction by the generalized Wick rule \( (n \geq 2) \)
\[
\langle F(\tilde{x}(t_1)) \tilde{x}^n(t_2) \rangle^\Omega_{xb,xa} = (n - 1) \tilde{x}(t_2) \tilde{x}(t_2) \langle F(\tilde{x}(t_1)) \tilde{x}^{n-2}(t_2) \rangle^\Omega_{xb,xa}
+ F(\tilde{x}(t_1)) \tilde{x}(t_2) \tilde{x}^{n-1}(t_2)
\]

(5.26)
with the contraction rules defined in (5.5) and (5.25). For \( n = 1 \), we obtain
\[
\langle F(x(t_1)) \bar{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = \bar{x}(t_1) \bar{x}(t_2) \langle F'(x(t_1)) \rangle_{\Omega}^{x_b,x_a},
\]
which is valid for any function \( F(\bar{x}(t)) \) generalizing the rule (5.4) that was proved for polynomial functions only. Recursively applying this power reduction, we finally end up with the derivative rule (5.15). Note that the generalization of Wick’s rule for mixed position-momentum or pure momentum couplings is done along similar lines, leading to the derivative rules (5.16)-(5.18).

C. New Feynman-like Rules for Nonpolynomial Interactions

Higher-order perturbation expressions become usually complicated. For simple polynomial interactions, Feynman diagrams are a useful tool to classify perturbative contributions with the help of graphical rules. Here, we are going to set up analogous diagrammatic rules for perturbation expansions for nonpolynomial interactions \( V(x(t), p(t)) \), whose contributions may be expressed as expectations values
\[
\int_{t_a}^{t_b} dt_n \cdots \int_{t_a}^{t_b} dt_1 \langle V(x(t_n), p(t_n)) \cdots V(x(t_1), p(t_1)) \rangle_{\Omega}^{x_b,x_a}
\]
From (5.5)-(5.8) follows that we have four basic propagators whose graphical representation may be defined as (setting \( \hbar = M = 1 \) from now on)

\[
\begin{align*}
t_1 \quad \cdots \quad t_2 & \equiv \langle \bar{x}(t_1) \bar{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = iG_{jj}(t_1, t_2), \\
t_1 \quad \cdots \quad t_2 & \equiv \langle \bar{p}(t_1) \bar{p}(t_2) \rangle_{\Omega}^{x_b,x_a} = iG_{kk}(t_1, t_2), \\
t_1 \quad \cdots \quad t_2 & \equiv \langle \bar{x}(t_1) \bar{p}(t_2) \rangle_{\Omega}^{x_b,x_a} = iG_{jk}(t_1, t_2), \\
t_1 \quad \cdots \quad t_2 & \equiv \langle \bar{p}(t_1) \bar{x}(t_2) \rangle_{\Omega}^{x_b,x_a} = iG_{kj}(t_1, t_2) = iG_{jk}(t_2, t_1).
\end{align*}
\]
A vertex is represented as usual by a small dot. The time variable is integrated over at a vertex in a perturbation expansion,

$$ \star \equiv \int_{t_a}^{t_b} dt. $$

We now introduce the diagrammatic representations of the expectation value of arbitrary functions $F(\bar{x}(t))$ or $F(\bar{p}(t))$ and their derivatives as

$$ \star \equiv \int_{t_a}^{t_b} dt \langle F(\bar{x}(t)) \rangle_{\Omega}^{x_{b-a}}, \quad \star \equiv \int_{t_a}^{t_b} dt \langle F(\bar{p}(t)) \rangle_{\Omega}^{x_{b-a}}, $$

$$ \star \equiv \int_{t_a}^{t_b} dt \langle F'(\bar{x}(t)) \rangle_{\Omega}^{x_{b-a}}, \quad \star \equiv \int_{t_a}^{t_b} dt \langle F'(\bar{p}(t)) \rangle_{\Omega}^{x_{b-a}}, $$

$$ \star \equiv \int_{t_a}^{t_b} dt \langle F''(\bar{x}(t)) \rangle_{\Omega}^{x_{b-a}}, \quad \star \equiv \int_{t_a}^{t_b} dt \langle F''(\bar{p}(t)) \rangle_{\Omega}^{x_{b-a}}, $$

$$ \vdots $$

With these elements, we can compose Feynman graphs for two-point correlation functions of the type (5.1) for arbitrary $n$ by successively applying the generalized Wick rule (5.26) or directly using the derivative relations (5.15)-(5.18). The general results become obvious by giving explicitly a graphical representation of the following four correlation functions

$$ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1)) \bar{x}(t_2) \rangle_{\Omega}^{x_{b-a}} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 iG_{jj}(t_1, t_2) \langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{x_{b-a}} $$

$$ = \star, $$

$$ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1)) \bar{x}^2(t_2) \rangle_{\Omega}^{x_{b-a}} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{jj}(t_2, t_1) \langle F(\bar{x}(t_1)) \rangle_{\Omega}^{x_{b-a}} + [iG_{jj}(t_1, t_2)]^2 \langle F''(\bar{x}(t_1)) \rangle_{\Omega}^{x_{b-a}} \right\} $$

$$ = \star \circ + \circ \equiv, $$

$$ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1)) \bar{x}^3(t_2) \rangle_{\Omega}^{x_{b-a}} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ 3iG_{jj}(t_1, t_2) iG_{jj}(t_2, t_1) \langle F'(\bar{x}(t_1)) \rangle_{\Omega}^{x_{b-a}} \right\} $$

$$ = \star \circ \circ \equiv. $$
\[ + \left[ iG_{jj}(t_1, t_2) \right]^3 \langle F'''(\bar{x}(t_1)) \rangle_{\Omega}^{x_b, x_a} \]  
\[ \equiv 3 \quad \begin{array}{c} \star \end{array} \quad \begin{array}{c} \circ \end{array} \quad + \quad \begin{array}{c} \star \end{array} \quad \begin{array}{c} \bigcirc \end{array} \quad , \]

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\bar{x}(t_1)) \bar{x}^4(t_2) \rangle_{\Omega}^{x_b, x_a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ \left[ iG_{jj}(t_2, t_2) \right]^2 \langle F(\bar{x}(t_1)) \rangle_{\Omega}^{x_b, x_a} \right. \]
\[ + 6 \left[ iG_{jj}(t_1, t_2) \right]^2 iG_{jj}(t_2, t_2) \langle F''(\bar{x}(t_1)) \rangle_{\Omega}^{x_b, x_a} \]
\[ + \left[ iG_{jj}(t_1, t_2) \right]^4 \langle F^{(4)}(\bar{x}(t_1)) \rangle_{\Omega}^{x_b, x_a} \left\} \right. \]
\[ \equiv \quad \begin{array}{c} \star \end{array} \quad \begin{array}{c} \bigcirc \end{array} \quad \begin{array}{c} \bigcirc \end{array} \quad + \quad 6 \quad \begin{array}{c} \star \end{array} \quad \begin{array}{c} \bigcirc \end{array} \quad + \quad \begin{array}{c} \star \end{array} \quad \begin{array}{c} \bigcirc \end{array} \quad . \]

Mixed position-momentum and momentum-momentum-correlations and their graphical representations are given in Appendix A.

The consideration of higher-order correlations with more than one function \( F(\bar{x}(t)) \) or \( F(\bar{p}(t)) \) can be reduced to the results (5.9)-(5.12) or (5.15)-(5.18) by expanding them with respect to the classical path or momentum, respectively. By expanding both functions in the expectation value, one obtains for example

\[ \langle F_1(\bar{x}(t_1)) F_2(\bar{x}(t_2)) \rangle_{\Omega}^{x_b, x_a} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{m! n!} f_{1,m} f_{2,n} \langle \bar{x}^m(t_1) \bar{x}^n(t_2) \rangle_{\Omega}^{x_b, x_a} \]  
(5.33)

with

\[ f_{i,m} = F^{(m)}(0), \quad i = 1, 2. \]  
(5.34)

But, constructing graphical rules for such general correlations is more involved due to the various summations over products of powers of propagators \( G_{jj}(t_i, t_j) \) with \( i, j = 1, 2 \).

Finally, we apply the diagrammatic rules to the anharmonic oscillator with \( \bar{x}^4 \)-interaction which is a powerful system being discussed in detail by the help of perturbation expansion

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[1, Chap. 3]. With the Green functions given by (3.22), (3.44), and (3.45), the two-point-correlation for anharmonic system with arbitrary time-dependent frequency can then be expressed graphically, yielding the known decomposition for the second-order perturbative contribution

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle \bar{x}^4(t_1) \bar{x}^4(t_2) \rangle \equiv 72 \quad + 24 \quad (5.35) \]

with subscript \( c \) indicating that we restrict to connected graphs only. Beyond this, our theory allows to describe nonstandard systems with polynomial interactions (5.28) depending on both, position and momentum, to higher-order. Finally, we want to give the graphs for a four-interation \( \bar{x}^2 \bar{p}^2 \) to second-order to see the variations of possible graphs in comparison with (5.35):

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle \bar{x}^2(t_1) \bar{p}^2(t_1) \bar{x}^2(t_2) \bar{p}^2(t_2) \rangle \equiv 2 \quad + 16 \quad + 2 \quad + 4 \quad + 16 \quad + 4 \quad + 16 \quad + 4 \quad . \quad (5.36) \]

We see, that we have the same class of graphs already occurring in (5.35), however, with different propagators connecting the vertices. Thus, both classes decay into subclasses with different multiplicities, but the total numbers remain 72 and 24 for each type of class, respectively. Furthermore, all graphs are vacuum-like graphs. Eventually, it is easy to construct the Feynman graphs for polynomial correlations higher than second order by applying Wick’s rule or the Feynman rules given in this section.
VI. SIMPLIFICATIONS FOR PERIODIC PATHS

Up to now, we discussed the harmonic time evolution amplitude with arbitrary frequency and external sources $j(t), k(t)$ and corresponding Green functions fulfilling Dirichlet boundary conditions. In the sense of the quantum mechanical partition function

$$ Z = \int_{-\infty}^{+\infty} dx \left( x_{t_b} | x_{t_a} \right), \quad (6.1) $$

which is an integral over the time evolution amplitude for closed paths, it is of interest to investigate the generating functional for closed paths. In analogy to (6.1), we define

$$ Z[j, k] = \int_{-\infty}^{+\infty} dx \left( x_{t_b} | x_{t_a} \right) [j, k], \quad (6.2) $$

with (3.42) for $x_a = x_b = x$. One immediately observes that $Z = Z[0, 0]$. The integral is easily done, giving

$$ Z[j, k] = \frac{1}{\sqrt{D_a(t_b) - D_b(t_a)}} \exp \left\{ -\frac{i}{2\hbar} \int_{t_a}^{t_b} dt \int_{t_a}^{t_b} dt' \left[ \frac{1}{M} j(t) \tilde{G}^x_{jj}(t, t') j(t') + \frac{1}{a(t_a, t_b)} g(t) g(t') + k(t) \tilde{G}^x_{jk}(t, t') j(t') + M k(t) \tilde{G}^x_{kk}(t, t') k(t') \right] \right\}. \quad (6.3) $$

The Green functions, expressed with fundamental solutions (3.2), (3.3), are found to be

$$ \tilde{G}^x_{jj}(t, t') = \frac{1}{D_a(t_b)} \left[ G^x_{jj}(t, t') + \frac{1}{a(t_a, t_b)} g(t) g(t') \right], \quad (6.4) $$

$$ \tilde{G}^x_{jk}(t, t') = \frac{1}{D_a(t_b)} \left[ G^x_{jk}(t, t') + \frac{1}{a(t_a, t_b)} g(t) \dot{g}(t') \right] = \tilde{G}^x_{kj}(t', t), \quad (6.5) $$

$$ \tilde{G}^x_{kk}(t, t') = \frac{1}{D_a(t_b)} \left[ G^x_{kk}(t, t') + \frac{1}{a(t_a, t_b)} \dot{g}(t) \dot{g}(t') \right], \quad (6.6) $$

with

$$ a(t, t') = \dot{D}_a(t') - \dot{D}_b(t) - 2. \quad (6.7) $$
Since the function
\[ g(t) = D_a(t) + D_b(t) \] (6.8)
is periodic, \( g(t_a) = g(t_b) \), due to conditions (3.2), (3.3), and (3.6), also the Green function \( \tilde{G}_j^x(t, t') \) becomes periodic,
\[ \tilde{G}_j^x(t_a, t') = \tilde{G}_j^x(t_b, t'). \] (6.9)

In analogy to the harmonic propagator without external sources (4.1) we can define expectation values consisting of \( N \) position-dependent functions and \( M \) momentum-dependent functions by
\[
\langle F_1(x(t_1)) F_2(x(t_2)) \cdots F_{N+M}(p(t_{N+M})) \rangle_\Omega = \\
\frac{1}{Z} \oint Dx \int Dp \ F_1(x(t_1)) F_2(x(t_2)) \cdots F_{N+M}(p(t_{N+M})) \exp \left\{ \frac{i}{\hbar} A[p, x; 0, 0] \right\}. \] (6.10)

We remark that the generalization of Wick’s rule and the graphical representation with the help of Feynman diagrams of such correlation functions is exactly the same as given in the last section after substituting the Green functions \( G(t, t') \) by \( \tilde{G}(t, t') \) and expectation values (4.1) by (6.10).

**VII. SUMMARY AND OUTLOOK**

We have reduced generating functionals with fixed end points to those with vanishing end points by adding special singular sources to the currents. The new generating functionals were calculated explicitly for the harmonic oscillator with time-dependent frequency. From this expression, a smearing formula was derived which serves to calculate correlation functions for arbitrary polynomial or nonpolynomial position- and momentum-dependent
couplings. We have further found a generalization of Wick’s theorem of decomposing correlation functions involving functions of the canonic variables of the system. This gives rise to certain generalized Feynman rules for position- and momentum-dependent expectation values.

Due to its universality, the theory should serve as a basis for investigating physical systems with nonstandard Hamiltonian via perturbation theory and its variational extension. Note, that a perturbation theory for momentum-dependent interactions arises in important field theories such as the nonlinear $\sigma$-model. Our work is supposed to prepare the grounds for a more efficient perturbation treatment of such theory.

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APPENDIX A: GENERALIZED CORRELATION FUNCTIONS

In this appendix we give the expectations for the correlation between a general position or momentum dependent function and a polynomial up to order $n = 4$:

**Position-Momentum-Coupling:**

$$\int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{x}(t_1)) \tilde{p}(t_2) \rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} \int_{t_a}^{t_b} iG_{jk}(t_1, t_2) \langle F'(\tilde{x}(t_1)) \rangle_{\Omega}^{x_b,x_a}$$

$$= \equiv \ldots,$$

$$\int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{x}(t_1)) \tilde{p}^2(t_2) \rangle_{\Omega}^{x_b,x_a} = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ iG_{kk}(t_2, t_2) \langle F(\tilde{x}(t_1)) \rangle_{\Omega}^{x_b,x_a} ight.$$  

$$+ [iG_{jk}(t_1, t_2)]^2 \langle F''(\tilde{x}(t_1)) \rangle_{\Omega}^{x_b,x_a} \right\}$$  

(A1)  

(A2)
Momentum-Position-Coupling:

\[ \int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{x}(t_1)) \tilde{p}(t_2) \rangle_{x_{b,x_a}} \]

\[ = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ 3 i G_{jk}(t_1,t_2) i G_{kk}(t_2,t_2) \langle F'(\tilde{x}(t_1)) \rangle_{x_{b,x_a}} \right. \]

\[ + [i G_{jk}(t_1,t_2)]^3 \langle F''(\tilde{x}(t_1)) \rangle_{x_{b,x_a}} \right\} \]

\[ \equiv 3 \quad \star \quad \circ \quad + \quad \bigcirc \quad , \]

\[ \int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{x}(t_1)) \tilde{p}(t_2) \rangle_{x_{b,x_a}} \]

\[ = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ [i G_{kk}(t_2,t_2)]^2 \langle F(\tilde{x}(t_1)) \rangle_{x_{b,x_a}} \right. \]

\[ + 6 [i G_{jk}(t_1,t_2)]^2 i G_{kk}(t_2,t_2) \langle F''(\tilde{x}(t_1)) \rangle_{x_{b,x_a}} \]

\[ + [i G_{jk}(t_1,t_2)]^4 \langle F^{(4)}(\tilde{x}(t_1)) \rangle_{x_{b,x_a}} \right\} \]

\[ \equiv \star \quad \bigcirc \quad + \quad \bigcirc \quad , \]

\[ \int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{p}(t_1)) \tilde{x}(t_2) \rangle_{x_{b,x_a}} \]

\[ = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ i G_{kj}(t_1,t_2) \langle F'(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right. \]

\[ + [i G_{kj}(t_1,t_2)]^2 \langle F''(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right\} \]

\[ \equiv \star \quad \bigcirc \quad + \quad \bigcirc \quad , \]

\[ \int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{p}(t_1)) \tilde{x}(t_2) \rangle_{x_{b,x_a}} \]

\[ = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ 3 i G_{kj}(t_1,t_2) i G_{jj}(t_2,t_2) \langle F'(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right. \]

\[ + [i G_{kj}(t_1,t_2)]^3 \langle F'''(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right\} \]

\[ \equiv \star \quad \bigcirc \quad + \quad \bigcirc \quad , \]

\[ \int_{t_a}^{t_b} \int_{t_a}^{t_b} \langle F(\tilde{p}(t_1)) \tilde{x}(t_2) \rangle_{x_{b,x_a}} \]

\[ = \int_{t_a}^{t_b} \int_{t_a}^{t_b} \left\{ 3 i G_{kj}(t_1,t_2) i G_{jj}(t_2,t_2) \langle F'(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right. \]

\[ + [i G_{kj}(t_1,t_2)]^3 \langle F'''(\tilde{p}(t_1)) \rangle_{x_{b,x_a}} \right\} \]
\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1)) \tilde{p}(t_2) \rangle_{\tilde{X}^{x}_{b,x}a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{jj}(t_2, t_2) \right\}^2 (F(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ 6 \left[ iG_{kj}(t_1, t_2) \right] ^2 iG_{jj}(t_2, t_2) (F''(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ [iG_{kj}(t_1, t_2)]^4 (F(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \right\} \]

\[ \equiv \star \bigcirc \bigcirc \bigcirc + 6 \star \bigcirc \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc ; \]

Momentum-Momentum-Coupling:

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1)) \tilde{p}(t_2) \rangle_{\tilde{X}^{x}_{b,x}a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 iG_{kk}(t_1, t_2) (F'(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \]

\[ \equiv \star \bigcirc \bigcirc \bigcirc , \]

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1)) \tilde{p}^2(t_2) \rangle_{\tilde{X}^{x}_{b,x}a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{kk}(t_2, t_2) (F(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ [iG_{kk}(t_1, t_2)]^2 (F''(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \right\} \]

\[ \equiv \star \bigcirc + \bigcirc \bigcirc , \]

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1)) \tilde{p}^3(t_2) \rangle_{\tilde{X}^{x}_{b,x}a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ 3 iG_{kk}(t_1, t_2) iG_{kk}(t_2, t_2) (F'(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ [iG_{kk}(t_1, t_2)]^3 (F'''(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \right\} \]

\[ \equiv 3 \star \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc , \]

\[ \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \langle F(\tilde{p}(t_1)) \tilde{p}^4(t_2) \rangle_{\tilde{X}^{x}_{b,x}a} = \int_{t_a}^{t_b} dt_1 \int_{t_a}^{t_b} dt_2 \left\{ iG_{kk}(t_2, t_2) \right\}^2 (F(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ 6 \left[ iG_{kk}(t_1, t_2) \right] ^2 iG_{kk}(t_2, t_2) (F''(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \\
+ [iG_{kk}(t_1, t_2)]^4 (F(\tilde{p}(t_1)))_{\tilde{X}^{x}_{b,x}a} \right\} \]

\[ \equiv 3 \star \bigcirc \bigcirc + \bigcirc \bigcirc \bigcirc \bigcirc ; \]
The case of position-position-coupling has already been calculated in Sect. V C.


