On the relation between 2+1 Einstein gravity and Chern Simons theory

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Abstract
A simple example is given to show that the gauge equivalence classes of physical states in Chern Simons theory are not in one-to-one correspondence with those of Einstein gravity in three spacetime dimensions. The two theories are therefore not equivalent. It is shown that including singular metrics into general relativity has more, and in fact a quite counter-intuitive, impact on the theory than one naively expects.

1 Outline
It is often argued that Einstein gravity on a given three dimensional spacetime manifold $M$ is equivalent to a Chern Simons theory on the same manifold with the Poincaré group being the underlying gauge group [1, 2, 3]. In fact, writing the first order Einstein Hilbert action in the dreibein formulation of general relativity, one finds that this is, up to a total derivative, equal to the Chern Simons action. The dreibein and the spin connection are then interpreted as the translational and rotational components of a Poincaré algebra valued gauge field. The only difference between the two theories is that in Einstein gravity the dreibein is restricted to be invertible, whereas no such restriction exists in Chern Simons theory. In this sense, Chern Simons theory can be thought of as an extension of Einstein gravity including singular metrics, or vice versa Einstein gravity is considered as a restricted version of Chern Simons theory.
The purpose of this article is to show that this equivalence holds at the level of field configurations only. There is indeed a one-to-one relation between off shell as well as on shell field configurations of Einstein gravity, and those field configurations of Chern Simons theory which have a non-vanishing dreibein determinant everywhere on $\mathcal{M}$. However, there is no such relation at the level of *gauge equivalence classes* of physical states. After dividing out the gauge symmetries of the two theories, one ends up with two different reduced phase spaces. Both are finite dimensional manifolds, that is, both theories are *topological* in the sense that they have only finitely many physical degrees of freedom. But there is no one-to-one correspondence between the reduced states of Chern Simons theory and those of Einstein gravity.

The basic idea of the proof is the following. Assume that $\mathcal{Q}$ is the configuration space of Chern Simons theory, that is, the set of all Poincaré algebra valued one-forms on a given spacetime manifold $\mathcal{M}$. They are typically subject to some boundary conditions, but these are not of any importance here. Assume further, that there is an action principle defined on $\mathcal{Q}$, which provides a set of field equations that singles out a submanifold $\mathcal{P} \subset \mathcal{Q}$ of *physical states*, or on shell field configurations. This set of solutions $\mathcal{P}$ can be identified with the *physical phase space* of Chern Simons theory. The action defines a canonical symplectic two-form $\Omega_\mathcal{P}$ thereon. It is highly degenerate, because only finitely many dimensions of the infinite dimensional phase space $\mathcal{P}$ are actually physical degrees of freedom. All other dimensions, namely the null directions of $\Omega_\mathcal{P}$, are directions of gauge transformations.

In general, two states $\phi_1, \phi_2 \in \mathcal{P}$ are gauge equivalent, denoted by $\phi_1 \sim \phi_2$, if they lie on the same null orbit of $\Omega_\mathcal{P}$, that is, if they can be joint by a curve in $\mathcal{P}$ whose tangent vector lies in the kernel of the two-form $\Omega_\mathcal{P}$ everywhere along the curve. This is the definition of a *smoothly generated* gauge symmetry. In the case of Chern Simons theory, the gauge symmetries are the local Poincaré transformations. Depending on the topology of the spacetime manifold $\mathcal{M}$, some of them might not be smoothly generated. These are called *large* local Poincaré transformations. Large gauge transformations are symmetries of the physical phase space that map those states onto each other, which are physically indistinguishable but do not lie on the same null orbit of the symplectic two-form. It is thereby important to note that the existence of large gauge symmetries can not be inferred from the given action.

Instead, it depends on the physical interpretation of the model whether two states related by a large transformation should be considered as gauge equivalent or not. In particular, one has to decide which quantities can be measured and which cannot. Depending on this choice, an equivalence class of physical states, in the following also called a *gauge orbit*, consist of one or several null orbits of $\Omega_\mathcal{P}$. If there are no large gauge symmetries, then each gauge orbit is a connected submanifold of $\mathcal{P}$ and consists of exactly one null orbit of $\Omega_\mathcal{P}$. Otherwise, the gauge orbits are disconnected submanifolds of $\mathcal{P}$ consisting of more than one null orbit of $\Omega_\mathcal{P}$. In any case, the reduced phase space $\mathcal{R} = \mathcal{P}/\sim$ is defined to be the quotient space of the physical phase space modulo gauge symmetries. Provided that the gauge orbits on $\mathcal{P}$ behave regularly enough, it is a manifold whose dimension is equal to the rank of $\Omega_\mathcal{P}$. A unique, non-degenerate symplectic two-form $\Omega_\mathcal{R}$ can then be defined on $\mathcal{R}$, such that $\Omega_\mathcal{P}$ is its pullback on $\mathcal{P}$.
By definition, each gauge invariant function on $\mathcal{P}$, usually called an observable, can be written as a function on $\mathcal{R}$ and vice versa. Hence, the reduced phase space $\mathcal{R}$ contains the full physically relevant information about the system. In other words, if Einstein gravity is equivalent to Chern Simons theory, or a restricted version thereof, then the reduced phase space of Einstein gravity should be equal to, or a subset of the reduced phase space of Chern Simons theory. But this, as we are now going to see, is not the case. Let me briefly explain what goes wrong, before making things more explicit. In the configuration space $\mathcal{Q}$ of Chern Simons theory, there is a submanifold of singular metrics, consisting of all field configurations with vanishing dreibein determinant at some point in $\mathcal{M}$. The configuration space $\tilde{\mathcal{Q}}$ of Einstein gravity is obtained from $\mathcal{Q}$ but taking away this submanifold. The action and also the equations of motion remain the same, which implies that the same holds for the physical phase space $\tilde{\mathcal{P}}$ of Einstein gravity. It is the subset of $\mathcal{P}$ obtained by taking away the submanifold of singular metrics. But now something crucial is happening to the gauge orbits. First of all, we have $\Omega_{\tilde{\mathcal{P}}} = \Omega_{\mathcal{P}}$, and therefore the null orbits of $\Omega_{\tilde{\mathcal{P}}}$ in Einstein gravity are the same as the null orbits of $\Omega_{\mathcal{P}}$ in Chern Simons theory. The problem is that they intersect with the submanifold of singular metrics. This is illustrated in figure 1. As a consequence, when the submanifold is taken away, the null orbits fall apart into disconnected components. For example, the states $\phi_1$ and $\phi_2$, which are related by a smoothly generated gauge symmetry in Chern Simons theory, no longer lie on the same null orbit of $\Omega_{\tilde{\mathcal{P}}}$, and thus they are not related by a smoothly generated gauge symmetry in Einstein gravity. As a result, the reduced phase space $\tilde{\mathcal{R}} = \tilde{\mathcal{P}}/\sim$ of Einstein gravity is different from that of Chern Simons theory, $\mathcal{R} = \mathcal{P}/\sim$.

Now, one might argue that the equivalence of the two theories can be easily restored by considering the states $\phi_1$ and $\phi_2$ in Einstein gravity as being related by a large gauge symmetry. Each gauge orbit then consists of several disconnected parts, namely exactly those that were
formerly, in Chern Simons theory, a single null orbit. In Einstein gravity, we would then say that \( \phi_1 \) and \( \phi_2 \), or \( \phi_3 \) and \( \phi_4 \), are related by a large local Poincaré transformation. From the mathematical point of view, this would be perfectly consistent, and the resulting reduced phase spaces \( \mathcal{R} = \mathcal{P}/\sim \) and \( \tilde{\mathcal{R}} = \tilde{\mathcal{P}}/\sim \) would indeed be identical. The only difference would be that a reduced state of Einstein gravity has less representatives in \( \tilde{\mathcal{P}} \) than the corresponding reduced state in Chern Simons theory has in \( \mathcal{P} \). However, remember that the question whether large gauge symmetries exist was a physical question rather than a mathematical one. So, what we should ask is whether the states \( \phi_1 \) and \( \phi_2 \) are physically indistinguishable.

Unfortunately, this is not the case. A simple example will show that declaring two such states as gauge equivalent in Einstein gravity leads to a straight contradiction to our intuitive understanding of the very basic principles of general relativity. To express it more drastically, if nobody would know about the relation between Einstein gravity and Chern Simons theory, then nobody would have ever suggested to declare the states \( \phi_1 \) and \( \phi_2 \) in figure 1 to be gauge equivalent. Or, phrased differently, given only the phase space \( \mathcal{P} \) of Einstein gravity, why should we glue two \( \text{a priori} \) unrelated regions of this space together along the dashed line, in the way shown in the figure, especially if we do not know how to interpret the states on that line? In any case, we have to conclude that the similarity between Einstein gravity and Chern Simons theory can be quite useful, but when we speak about gauge symmetries and the reduced phase space, we have to decide which of the two theories we are dealing with.

Without making this decision, some authors run into yet another problem [7]. It arises if one tries to include too many features of both Einstein gravity and Chern Simons theory into a kind of mixture of the two theories, which then has too many large gauge symmetries. In Einstein gravity, an essential part of the gauge group are the diffeomorphisms of the underlying spacetime manifold \( \mathcal{M} \). In the spirit of \textit{general relativity}, one assumes that the individual points of \( \mathcal{M} \) do not have any particular physical meaning. This implies that every transformation that interchanges them without destroying the differentiable structure of \( \mathcal{M} \) should be considered as a gauge symmetry. For a generic manifold \( \mathcal{M} \), there are smoothly generated diffeomorphisms as well as large ones. Now, the orbits of the smoothly generated diffeomorphisms in phase space are included in the orbits of smoothly generated local Poincaré transformations. This is because the infinitesimal generator of any diffeomorphisms can be written as a local Poincaré transformation with a special parameter.

However, it turns out that the large diffeomorphisms of \( \mathcal{M} \) have nothing to do with the local Poincaré transformations. They act on the physical phase space in a completely different way. For example, the state \( \phi_2 \) in the figure could be mapped onto \( \phi_3 \) by some large diffeomorphism, lying on a different null orbit of \( \Omega_P \). In Einstein gravity, including the large diffeomorphism as gauge symmetries does not cause a problem. What happens is that several null orbits of \( \Omega_P \) are united into a single gauge orbit. However, the null orbits to the left and to the right of the dashed line in the figure are thereby identified in a different way, as compared to the way they are glued together by including the submanifold of singular metrics. A problem arises if we now consider the large diffeomorphisms as gauge symmetries in Chern Simons theory, or if both the large
Poincaré transformations and the large diffeomorphisms are considered as gauge symmetries in Einstein gravity.

In both cases, the states \( \phi_1 \) through \( \phi_4 \) in the figure all become gauge equivalent, and we can extend this sequence into both directions, as indicated by the arrows. Now, assume that the plane shown in the figure is actually a part of a cylinder, and after a finite number of steps we get back to this region, but we never again hit the null orbit we started from. The gauge orbit then fills the plane densely, and the resulting reduced phase space is no longer a manifold. It is in particular not Hausdorff, because two such gauge orbits which both fill the plane densely cannot be separated by open neighbourhoods. Using the example already mentioned above, we shall see that this is a quite typical situation. It arises because one tries to keep too many features of both theories. If one either excludes singular metrics, sticking to Einstein gravity, or does not regard large diffeomorphisms as gauge symmetries, for which there is actually no motivation in Chern Simons theory, then this problem is absent.

2 Chern Simons theory and Einstein gravity

Let us now make things more explicit. In the following, we assume that \( M \) is an orientable, three dimensional manifold with a trivial tangent bundle, equipped with a Levi Civita tensor \( \varepsilon^{\mu \nu \rho} \). On \( M \), we introduce two \( \mathfrak{sl}(2) \) valued one-forms \( \omega_\mu \) and \( e_\mu \). In Chern Simons theory, these are interpreted as the rotational and the translational components of a Poincaré algebra valued gauge field. Note that \( \mathfrak{sl}(2) \) is the spinor representation of the three dimensional Lorentz algebra, but it is also, as a vector space, isomorphic to Minkowski space. The Poincaré field strength also splits into an \( \mathfrak{sl}(2) \) valued rotational component

\[
F_{\mu \nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu],
\]

and an also \( \mathfrak{sl}(2) \) valued translational component

\[
T_{\mu \nu} = \partial_\mu e_\nu - \partial_\nu e_\mu + [\omega_\mu, e_\nu] - [\omega_\nu, e_\mu].
\]

In Einstein gravity, \( \omega_\mu \) is interpreted as the spin connection and \( e_\mu \) is the dreibein, defining the metric and the determinant,

\[
g_{\mu \nu} = \frac{1}{2} \text{Tr}(e_\mu e_\nu), \quad e = \frac{1}{12} \varepsilon^{\mu \nu \rho} \text{Tr}(e_\mu e_\nu e_\rho).
\]

The latter is required to be different from zero everywhere on \( M \). Hence, the configuration space \( Q \) of Chern Simons theory is the set of all field configurations \((\omega_\mu, e_\mu)\) on \( M \), and the configuration space \( \tilde{Q} \) of Einstein gravity is the set of all \((\omega_\mu, e_\mu)\) with \( e \neq 0 \). In both cases, they are subject to some additional restrictions such as smoothness and fall off conditions at infinity. A more detailed discussion, taking into account all such boundary conditions for a specific model, will be given in [8]. Here, we only need to know that all these extra conditions
can be chosen to be the same in both theories, such that $\tilde{Q}$ is the subset of $Q$ with the singular metrics excluded. The vacuum Einstein Hilbert action can then be written as

$$\frac{1}{16\pi G} \int d^3 x \, \varepsilon^\mu\nu\rho \, \text{Tr}(e_\mu F_\nu\rho).$$ (2.4)

Up to a total derivative, this coincides with the Chern Simons action [1, 2]. The equations of motions are

$$F_\mu\nu = 0, \quad T_\mu\nu = 0,$$ (2.5)

defining the physical phase space $P$, respectively $\tilde{P}$. In Chern Simons theory, they state that the Poincaré field strength vanishes. In Einstein gravity they are interpreted as the torsion the curvature equation, stating that the dreibein is covariantly constant and that the metric is flat. All we need to know about the symplectic form $\Omega^P = \Omega_{\tilde{P}}$ is that its null directions are the generators of the local symmetries of the action (2.4). These are the local Poincaré transformations,

$$\delta \omega_\mu = \partial_\mu \lambda + [\omega_\mu, \lambda], \quad \delta e_\mu = \partial_\mu \zeta + [\omega_\mu, \zeta] + [e_\mu, \lambda],$$ (2.6)

where $\lambda : M \to \mathfrak{sl}(2)$ is the generating parameter of a local Lorentz rotation and $\zeta : M \to \mathfrak{sl}(2)$ that of a local translation. It is easy to see that the generators of spacetime diffeomorphisms are included in these transformations. If we choose the parameters to be

$$\lambda = \xi^\mu \omega_\mu, \quad \zeta = \xi^\mu e_\mu,$$ (2.7)

for some vector field $\xi^\mu$ on $M$, then the transformation (2.6) becomes

$$\delta \omega_\mu = \mathcal{L}_\xi \omega_\mu + F_\mu\nu \xi^\nu, \quad \delta e_\mu = \mathcal{L}_\xi e_\mu + T_\mu\nu \xi^\nu,$$ (2.8)

where $\mathcal{L}_\xi$ denotes the Lie derivative with respect to the vector field $\xi^\mu$. As the field strength, respectively the curvature and the torsion vanishes on the physical phase space, this is equal to the infinitesimal generator of a diffeomorphism of $M$ with generating vector field $\xi^\mu$. What we also need in the following are the finite versions of the gauge transformations. A finite local Lorentz rotation is given by

$$\omega_\mu \mapsto k^{-1}(\partial_\mu + \omega_\mu)k, \quad e_\mu \mapsto k^{-1}e_\mu k,$$ (2.9)

where $k : M \to \text{SL}(2)$ is a parameter field taking values in the spinor representation of the Lorentz group SL(2). At this point, however, we already have to be a little bit more careful. Not every transformation of the form (2.9) can be written as a smooth deformation of the fields generated by (2.6). This is the case if and only if the map $k : M \to \text{SL}(2)$ is contractible, that is, if it can be smoothly deformed into the trivial map $k = 1$. Otherwise the given transformation is a large local Lorentz rotation. So, already here we have to make a decision, namely whether to consider large local Lorentz rotations as gauge symmetries or not.
Both in Einstein gravity as well as in Chern Simons theory, it is physically reasonable to include them. Doing this essentially means that the rotational component of the gauge field, respectively the spin connection, can only be measured by parallel transport of vectors or spinors along closed loops in $\mathcal{M}$. One can show that this leads to the conclusion that all transformations of the form (2.9) are gauge symmetries [5]. But this is not the crucial point regarding the difference between Einstein gravity and Chern Simons theory, and therefore it is actually not relevant how to deal with the large local Lorentz rotations. The more interesting gauge symmetries are the local translations. A finite local translation is given by

$$\omega_\mu \mapsto \omega_\mu, \quad e_\mu \mapsto e_\mu + \partial_\mu n + [\omega_\mu, n],$$

where $n : \mathcal{M} \to \mathfrak{sl}(2)$ is actually the same parameter field as $\zeta$ in (2.6), but now it is a finite parameter rather than an infinitesimal generator. One can easily check that together with the local Lorentz rotations (2.9), the local translation form a local Poincaré group. It is also useful to note that there are no large local translations in Chern Simons theory. Every transformation of the form (2.10) can be written as a smooth deformation by simply replacing $n$ by $\epsilon n$, with $0 \leq \epsilon \leq 1$. For Einstein gravity, things are a bit more evolved, as we shall see below. Before coming to this, let us also write down the action of a diffeomorphism $h : \mathcal{M} \to \mathcal{M}$ on the field configuration. The one-forms are thereby replaced by their pullbacks, so that the finite version of (2.8) reads

$$\omega_\mu \mapsto h_* \omega_\mu, \quad e_\mu \mapsto h_* e_\mu.$$  (2.11)

From the relations between the infinitesimal generators of local Poincaré transformations (2.6) and diffeomorphisms (2.8), we know that the action of every smoothly generated diffeomorphism (2.11) can be written as a combination of a local Lorentz rotation (2.9) and a local translation (2.10), both being smoothly generated. This is valid in Einstein gravity as well as in Chern Simons theory. In Einstein gravity, however, there is yet another relation between the infinitesimal generators which is not present in Chern Simons theory. If the dreibein is invertible, then (2.7) provides a one-to-one relation between the generating parameter $\zeta$ of local translations and the generating vector field $\xi^\mu$ of spacetime diffeomorphisms. Hence, in Einstein gravity we can consider the spacetime diffeomorphisms instead of the local translations as the basic gauge symmetries.

Regarding the smoothly generated ones, this is just a reparametrization. However, it leads us to a different set of large transformations. To see that the situation is indeed as shown in figure 1, consider the action of a local translation (2.10) in Einstein gravity. Given a field configuration and a parameter $n$, there is no guarantee that the transformed dreibein is still invertible. In fact, it is easy to show that for any given field configuration it is also always possible to find a parameter $n$ such that the transformed dreibein becomes singular. So, the gauge orbits indeed intersect with the submanifold of singular metrics, as indicated in the figure. Moreover, given two states $\phi_1$ and $\phi_2$ with non-singular metric, and related by a local translation (2.10) for some parameter $n$, the simple trick to show that they can be smoothly transformed into each other
does not work, because when we replace $n$ by $\epsilon n$, then some of the intermediate states might have a singular metric.

This does not yet prove that the gauge orbits fall apart into disconnected components, because the phase space is infinite dimensional, and therefore the two dimensional picture does not necessarily give the correct impression. There might still be a null curve of $\Omega_{P}$ that connects $\phi_{1}$ and $\phi_{2}$ without passing through the submanifold of singular metrics. A possible way to prove that such a path does not exist is to show that the states $\phi_{1}$ and $\phi_{2}$ are not related by a combination of a diffeomorphism and a local Lorentz rotation. They are then, in particular, not related by a smoothly generated diffeomorphism and a local Lorentz rotation. On the other hand, we know that in Einstein gravity every smoothly generated gauge symmetry can be written as a combination of a smoothly generated diffeomorphism and a local Lorentz rotations. If this is not the case for the two given states, it follows that they are not related by a smoothly generated gauge symmetry, and therefore they do not lie on a connected component of a gauge orbit.

From this we have to conclude that in Einstein gravity some of the local translations are large, and we can ask the question whether these should be considered as gauge symmetries or not. It is important to note that in Chern Simons theory this question never arises, because all local translations are smoothly generated. Hence, if we decide not to consider the large local translation as gauge symmetries of Einstein gravity, and we shall shortly see that our intuitive understanding of general relativity forces us to do so, then the two theories are not equivalent. We shall give an explicit example of two states $\phi_{1}$ and $\phi_{2}$, related by a local translation, but representing two spacetimes which are obviously not diffeomorphic and therefore physically clearly distinguishable. From this we infer that, first of all, the two states are not related by a smoothly generated gauge symmetry, and secondly, that we must not consider large translations as gauge symmetries of Einstein gravity if we want to stick to its original physical interpretation.

3 The counter example

To give the explicit example, let us first have a closer look at the solutions to the field equations (2.5). It is well known that locally, within a simply connected region $\mathcal{U} \subset \mathcal{M}$, they can be parametrized by two fields $g: \mathcal{U} \to SL(2)$ and $f: \mathcal{U} \to sl(2)$, such that $[1, 3, 4]$

$$\omega_{\mu} = g^{-1} \partial_{\mu} g, \quad e_{\mu} = g^{-1} \partial_{\mu} f g.$$ (3.1)

In Chern Simons theory, this means that locally the fields are pure gauge. In fact, setting $k = g^{-1}$ in (2.9) and then $n = -f$ in (2.10), we can always transform the fields into the trivial solution $\omega_{\mu} \mapsto 0$ and $e_{\mu} \mapsto 0$. It follows, in particular, that there are no local physical degrees of freedom. In Einstein gravity, there is a restriction on $f$, and the interpretation of the parameters is as follows. The spacetime metric and the dreibein determinant can be written as

$$g_{\mu\nu} = \frac{1}{2} \text{Tr}(\partial_{\mu} f \partial_{\nu} f), \quad e = \frac{1}{12} \varepsilon^{\mu\nu\rho} \text{Tr}(\partial_{\mu} f \partial_{\nu} f \partial_{\rho} f).$$ (3.2)

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The second equation states that the dreibein determinant is equal to the Jacobian of $f$. If this is non-zero, then $f$ is locally one-to-one. Hence, in contrast to Chern Simons theory, $f$ is not completely arbitrary. According to the first equation, it then provides an isometric embedding of the simply connected subset $\mathcal{U}$ of spacetime into Minkowski space. The components of the vector $f$ are called Minkowski coordinates on $\mathcal{U}$. So, even though we cannot transform to $\omega_\mu \mapsto 0$ and $e_\mu \mapsto 0$, because then the metric is singular, we can still see that there are no local gravitational degrees of freedom, because the spacetime is always flat. The action of the gauge symmetries on the fields $g$ and $f$ is as follows. Under a local Lorentz rotation with parameter $k : \mathcal{M} \to \text{SL}(2)$, they transform as

$$g \mapsto gk, \quad f \mapsto f.$$  

(3.3)

For a local translation with parameter $n : \mathcal{M} \to \text{sl}(2)$, we find that

$$g \mapsto g, \quad f \mapsto f + gn g^{-1}.$$  

(3.4)

And finally, a spacetime diffeomorphism $h : \mathcal{M} \to \mathcal{M}$ acts as

$$g \mapsto g \circ h, \quad f \mapsto f \circ h.$$  

(3.5)

To specify a physical state completely, using the field $g$ and $f$ as parameters, we first have to fix a spacetime manifold $\mathcal{M}$, cover it by a set of simply connected regions, and define the fields $g$ and $f$ independently within in these regions. Let us do this for a special example. As a spacetime manifold, we choose the direct product $\mathcal{M} = \mathbb{R} \times \mathcal{N}$ of a real line, which is going to represent the time coordinate $t$, with the space manifold $\mathcal{N}$ shown in figure 2. It is a plane with four punctures, that is, four points are taken away. It is not simply connected, but it can be covered by two simply connected, closed subsets $\mathcal{N}_\pm$, overlapping along a sequence of five lines $\lambda_k$, with $k = 0, \ldots, 4$, as indicated in the figure.

A physical state can then be parametrized by two pairs of fields $g_\pm : \mathcal{M}_\pm \to \text{SL}(2)$ and $f_\pm : \mathcal{M}_\pm \to \text{sl}(2)$, where $\mathcal{M}_\pm = \mathbb{R} \times \mathcal{N}_\pm$. These fields are subject to a boundary condition.

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Figure 2: The space manifold $\mathcal{N}$ is a plane with four punctures. It can be covered by two simply connected regions $\mathcal{N}_\pm$, overlapping along their common boundary, which consists of five lines $\lambda_0$ through $\lambda_4$. 
on the overlap surfaces \( \mathbb{R} \times \lambda_k \) of \( M_+ \) and \( M_- \). For the dreibein and the spin connection to be well defined, the right hand sides of (3.1) must be the same in \( M_+ \) and \( M_- \). This is the case if and only if the values of \( g_\pm \) and \( f_\pm \) in the overlap region are related by

\[
g_- = u_k^{-1} g_+, \quad f_- = u_k^{-1} (f_+ - v_k) u_k,
\]

(3.6)

for some constants \( u_k \in \text{SL}(2) \) and \( v_k \in \mathfrak{sl}(2) \). The second equation states that \( f_- \) is related to \( f_+ \) by a rigid Poincaré transformation, that is, an isometry of the embedding Minkowski space, consisting of a translation \( v_k \) and a Lorentz rotation \( u_k \). This implies that the spacetime metric induced by \( f_+ \) is the same as that induced by \( f_- \). The first equation then ensures that not only the metric but also the dreibein and the spin connection are continuous on \( M \). If the overlap region splits into several disconnected components, then there are independent constants for each part. In our case, we have five independent transition functions \( u_k \) and \( v_k \), with \( k = 0, \ldots, 4 \).

Using this, we can formulate the following statement, which holds in Chern Simons theory, but not in Einstein gravity. Given two physical states, parametrized by fields \( g \) and \( f \), respectively \( g' \) and \( f' \), such that the transition functions \( u_k \) and \( v_k \) coincide, then the two states are gauge equivalent. The proof is quite simple. Define a field \( k = g^{-1} g' \). By assumption, this is a continuous field on \( M \), which follows from the first equation in (3.6). Inserted into (3.3), it transforms \( g \) into \( g' \) by a local Lorentz rotation. Next, assume that \( g = g' \) and define \( n = g^{-1} (f' - f) g \). According to the second equation in (3.6), this is again a continuous field on \( M \), and when we take this as a parameter of a local translation (3.4), \( f \) is transformed into \( f' \). Hence, given two states with coinciding transition functions, we can always find a combination of a local Lorentz rotation and a local translation that maps the two states onto each other.

What remains to be done to prove the non-equivalence of Einstein gravity and Chern Simons theory is to find two states of Einstein gravity with coinciding transition functions, which are however physically distinguishable and therefore not gauge equivalent. They are then related by a local translation. But this cannot be a smoothly generated one because then the two states would also be related by a diffeomorphism. Hence, it must be a large local translation, and we have to conclude that the large local translations are not to be considered as gauge symmetries of Einstein gravity. To give two such states explicitly, we have to introduce some more notation. Let \( \gamma_a \), with \( a = 0, 1, 2 \), be some orthonormal basis of \( \mathfrak{sl}(2) \), such that \( \gamma_a \gamma_b = \eta_{ab} \mathbf{1} - \varepsilon_{abc} \gamma^c \), where \( \eta_{ab} \) is the Minkowski metric with signature \((-++,+)\) and \( \varepsilon_{abc} \) is the Levi Civita tensor with \( \varepsilon_{012} = -1 \). This implies, for example, that conjugation of a vector \( \nu \) with a group element \( e^{\nu} \in \text{SL}(2) \) provides a spatial rotation of \( \nu \) by \( 2\alpha \). This is all we need to know about the algebra of the gamma matrices. To define an explicit solution to the field equations, we first specify the following transition functions,

\[
\begin{align*}
    u_0 &= e^{-2m\gamma_0}, \quad &u_1 &= e^{-m\gamma_0}, \quad &u_2 &= 1, \quad &u_3 &= e^{m\gamma_0}, \quad &u_4 &= e^{2m\gamma_0},
\end{align*}
\]

(3.7)

where \( m = \pi/12 \). These group elements represent spatial rotations by \(-60^\circ, -30^\circ, 0^\circ, 30^\circ, \) and \( 60^\circ \). Then, we choose two arbitrary, smooth fields \( g_\pm : M_\pm \rightarrow \text{SL}(2) \), such that the boundary
Figure 3: The images of the maps $\vec{f}_\pm$ on the spatial plane in Minkowski space. The geometry of the space manifold can be read off by gluing the two flat surfaces together along their edges. The space obtained in (a) is a cone with four tips in a row, which is different from that in (b), where the tips form a parallelogram.

conditions (3.6) are satisfied. Their precise form does not matter. They are all related by local Lorentz rotations anyway. All we need to know is that such fields exist, which can be easily verified. Finally, we have to define the fields $f_\pm$. We decompose them into time and space components,

$$f_\pm(t, z) = t \gamma_0 + \vec{f}_\pm(z) \cdot \vec{\gamma}, \quad \text{(3.8)}$$

where $\vec{f}_\pm(z) = (f_1(z), f_2(z))$ maps the points $z \in \mathcal{N}_\pm$ onto points in the Euclidean plane, considered as the spatial plane in Minkowski space, which is spanned by $\vec{\gamma} = (\gamma_1, \gamma_2)$. If the functions $\vec{f}_\pm$ are independent of $t$, it follows from (3.8) and (3.2) that the resulting spacetime is static, and that the time coordinate $t$ is the physical time. It is then sufficient to specify the spatial metric to fix the geometry of the whole spacetime manifold.

We choose the maps $\vec{f}_\pm$ to be one-to-one, and such that their images $\vec{f}_\pm(\mathcal{N}_\pm)$ are the subsets of the spatial plane shown in figure 3(a). It is, once again, not important to know what these maps precisely looks like. In fact, the images determine them up to a diffeomorphism of $\mathcal{N}$, which can even be shown to be a smoothly generated one. Hence, we have the statement that given the transition functions $u_k$ and the images of $\vec{f}_\pm$, the state is fixed up to a local Lorentz rotation and a smoothly generated diffeomorphism. What is not immediately obvious is that, given the images, we can choose $\vec{f}_\pm$ such that the boundary conditions (3.6) are satisfied. The condition for this to be possible is that the edges of the upper and lower surface in the figure are mapped onto each other by isometries of the Euclidean plane, consisting of a translation and a rotation by $-60^\circ, -30^\circ, 0^\circ, 30^\circ, \text{and } 60^\circ$, respectively. In particular, we must have the following relations between the corner points $x_k$ and $y_k$ in the figure and the transition functions,

$$x_k = u_k^{-1}(y_k - v_k) \ u_k, \quad x_k = u_{k-1}^{-1}(y_k - v_{k-1}) \ u_{k-1}, \quad \text{(3.9)}$$

for $k = 1, \ldots, 4$. These relations are indeed satisfied, with the $u_k$ taken from (3.7) and for some
suitable chosen values of $v_k$. The first equation follows from evaluating (3.6) at the beginning of the line $\lambda_k$, and the second from the same condition at the end of $\lambda_{k-1}$.

To see what the geometry of the spacetime manifold looks like, we have to cut out the surface in figure 3(a), and glue them together along their edges. Note that they always fit together, because the corresponding edges of the two half spaces are mapped onto each other by isometries of the embedding Minkowski space. In particular, we always obtain a locally flat spacetime. The resulting space manifold can be described as a cone with four tips, arranged in a row. Its total deficit angle is $8m = 120^\circ$, and at each tip there is a deficit angle of $2m = 30^\circ$. The spacetime obtained by taking the direct product with a real line can be interpreted as a universe containing four point particles with mass $m$, being at rest with respect to each other. The mass is thereby measured in units of the Planck mass $1/(4\pi G)$ [5, 6, 8].

It is now possible to construct a different universe, containing the same particles, which is not diffeomorphic to the first, but with all transition function being the same, so that the two states of Einstein gravity are related by a large translation. It is thereby sufficient to note that the transition functions are already determined by the corner points $x_k$ and $y_k$ in the figure, and the directions of the edges extending to infinity. This follows from (3.9), or simply from the fact that a Poincaré transformation, which is in this case just an isometry of the Euclidean plane, is fixed if its action on two points is known. If we, for example, choose the maps $f_{\pm}$ such that their images are the surfaces in figure 3(b), then the transition functions remain the same, because the corner points and the directions of the edges extending to infinity are unchanged.

In fact, one can easily see that the rotations and translations needed to map the corresponding edges of the upper surface onto those of the lower surface as the same in figure 3(a) and (b). But if we now cut out the surfaces and glue them together along the edges, then we find that the particles are no longer arranged in a row. Instead, they form a parallelogram. It is immediately obvious that the two static spacetimes are not diffeomorphic, and therefore they are not gauge equivalent states of Einstein gravity. This completes the proof that there are states on the gauge orbits of Chern Simons theory $\phi_1$ and $\phi_2$, as shown in figure 1, which do not lie on the same gauge orbit of Einstein gravity, and it also explains why it is contradictory to the usual interpretation of general relativity to consider large translations as gauge symmetries of Einstein gravity.

An important point in this consideration is that it does not depend on the boundary conditions imposed on the fields. They usually play a very important role in the precise definition of the phase space, because they have to guarantee that the action is finite, and they also have some impact on what is a gauge symmetry and what is not [8]. For example, a transformation with a constant parameter $\lambda$ or $\zeta$ is typically not a gauge symmetry when certain fall off conditions are imposed on the fields. However, all this has nothing to do with our simple counter example. The equivalence of the two theories can not be restored by imposing boundary condition on the fields, neither at infinity nor in the neighbourhood of the punctures. The crucial transformation, namely the large translation that takes us from (a) to (b) in figure 3, takes place within a compact region of $\mathcal{N}$. The fields $f_{\pm}$ can be chosen to be the same in (a) and (b), say, outside the finite region.
shown in the figure, and by slightly deforming the images in the neighbourhoods of the corner points \( x_k \) and \( y_k \), we can even achieve that the fields are unchanged in a finite neighbourhood of the particles.

What remains to be seen is that regarding the large diffeomorphisms as gauge symmetries of Chern Simons theory, or to consider both large diffeomorphisms and large translations as gauge symmetries of Einstein gravity, makes the gauge orbits behave such that certain regions of the phase space are filled densely by a single orbit, so that the reduced phase space becomes non-Hausdorff. To see this, observe that there are infinitely many inequivalent ways to draw the edges of the surfaces in figure 3. The only conditions are that the corners are connected in the correct ordering, and that the corresponding edges of the lower surface are mapped onto those of the upper surfaces by the given isometries (3.6). But even if we, for example, fix the edges extending to infinity and the one between the second and the third corner, then there is still an infinite series of possibilities, two of which are shown. The edge starting at \( x_1 \), for example, can wind around \( x_3 \) any number of times before it gets to \( x_2 \). We can label the different states by this winding number. All these states are then related to each other by local translations.

How do the resulting spacetimes look like? They always consist of four identical point particles being at rest. Two of them, number 1 and 4, are at fixed positions, and the other two, number 2 and 3, are located in between them, with varying angular orientation. Each time we increase the winding number, the inner particles are rotated by \( 4m \) with respect to the outer ones. For the example given, we have \( 4m = 60^\circ \), which means that increasing the winding number by six, or three if we consider the particles as being identical, leads us back to the same geometry. A closer analysis shows that the two states are then related by a large diffeomorphism. But what happens if the mass \( m \) of the particles is an irrational multiple of \( \pi \)? We can then make the same construction, but the possible angular orientations take countably many different values for the different winding numbers. The possible locations of the inner particles with respect to the outer ones fill a circle densely.

In other words, a single gauge orbit fills a certain region of the phase space densely, are there are different gauge orbits that do this for the same region. These cannot be separated by open neighbourhoods. As a result, the quotient space is not Hausdorff. Once again, it should be emphasized that this is a consequence of the assumption that both the large local translations and the large diffeomorphisms are considered as gauge symmetries. If either of them is dropped, that is, if one sticks to the original definition of Chern Simons theory, which does not know anything about large diffeomorphism, or to Einstein gravity, without singular metrics and without large local translations as gauge symmetries, then this problem is absent. In the former case, the states with the particles filling the circle densely are in fact not close to each other in phase space, and in the second case they are not gauge equivalent, so in neither case a single gauge orbit fills a region of phase space densely.
References


