\[ \mathcal{V}_i a \in (N, M) \text{ representation appears, and it forms the potential} \]
\[
\mathcal{V} = \frac{1}{4} \left[ \xi_N \text{tr}_N \left( F_{k^{-1}}(x, N) F^{\dagger k^{-1}}(x, N) \right) + \xi_M \text{tr}_M \left( F_{k^{-1}}(x, M) F^{\dagger k^{-1}}(x, M) \right) \right],
\]
where
\[
F_{k^{-1}}(x, N) \equiv \delta_i^j - H_i^a H^r_{kj}^a, \quad F_{k^{-1}}(x, M) \equiv \delta^b_a - H_i^b H^r_{rj}^i .
\]
The index \( k \) \( (k^{-1}) \) represents the translation from \( N \) to \( M \) \( (M \) to \( N \). When the constants \( \xi_N \) and \( \xi_M \) are positive, the vacuum minimizing this potential preserves the gauge symmetry \( SU(N - M) \times U(1)^{M-1} \).

In some cases, it is effective to identify some of the vector spaces \( \mathcal{V} \) (acted on by the gauge rotation) associated with the elements of the discrete space, in order to obtain phenomenological models. For an example, let us identify the two in the above set up (therefore take \( N \) equal to \( M \)).
The Higgs field \( \Sigma \) is then in the adjoint representation, and the Higgs potential is
\[
\mathcal{V} = \frac{\xi}{2} \left( N - 2 \text{tr} \Sigma^2 + \text{tr} \Sigma^4 \right) (2-4)
\]
Referring to the result of Ref. 6), the vacuum of the potential (2-4) breaks the gauge symmetry \( SU(N) \) down to \( SU(N - n) \times SU(n) \times U(1), \) where \( n \) is the greatest integer not larger than \( N/2 \).
§3. *SU*(5) model

A fascinating feature of the *SU*(N) model (2-4) is that in the case *N* = 5 we obtain the standard model gauge symmetry, *SU*(3) × *SU*(2) × *U*(1). The gauge group *SU*(5) is one of the candidates for grand unification, and the adjoint Higgs Σ is just the one needed for the symmetry breaking in the *SU*(5) GUT.

In the usual minimal *SU*(5) GUT, one introduces a five-dimensional Higgs field *H* in order to realize the electroweak symmetry breaking, as well as Σ. In the context of Ref. 3, this matter content is supplied\(^1\) by adopting a space-time *M*₄ × *Z*₃. This is shown schematically in Fig. 2. Vector spaces are assigned as follows: *V*(*C*₅) on *M*₄⁽¹⁾, *V*(*C*⁵) on *M*₄⁽²⁾ and *V*(*R*) on *M*₄⁽³⁾. We have assumed that the vector spaces on *M*₄⁽¹⁾ and *M*₄⁽²⁾ are the same as before. Hence the five-dimensional Higgs fields *H* which connect two *M*₄’s as *M*₄⁽¹⁾ → *M*₄⁽³⁾ and *M*₄⁽²⁾ → *M*₄⁽³⁾ are the same. **

![Fig. 2. A schematic representation for obtaining also the five-dimensional Higgs fields.](image)

Let us calculate the potential for the Higgs fields. The Higgs field strengths read from the combinations of the translation in Fig. 2 are as follows:

\[
\begin{align*}
F_{kl} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{lh} &= H_1^{tα} \Sigma^{α}_r, \\
F_{kh} &= \Sigma^{α}_r, \\
F_{h-1} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{k-1} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{k-1}^{tα} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{k-1}^{tβ} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{h-1}^{tα} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r, \\
F_{h-1}^{tβ} &= H_1^{tα} - H_1^{tβ} \Sigma^{α}_r.
\end{align*}
\]

(3-1)

Here, the indices *h*, *k* and *l* denote the directions of the translation (see Fig. 3). Note that we have two scale parameters, *d*₁ and *d*₉, the distance between two Minkowski spaces (the other parameter *d*₉ is equal to *d*₉ because of the identification of *M*₄⁽¹⁾)

---

\(^1\) Though fermions and their interaction terms can be introduced as in Ref. 7, we concentrate on the Higgs potential in this paper.

** The gauge fields on *M*₄⁽¹⁾ and *M*₄⁽²⁾ are also the same. The gauge field on *M*₄⁽³⁾ does not appear, since we put a real vector space on it. Thus the gauge group is only *SU*(5).
and $M_4^{(2)}$). Since the Higgs fields in the definition (2.2) are dimensionless, we associate them with the length of the translation: $\tilde{H}(x, p, p + k) = H(x, p, p + k)/d_k$. Moreover, the Higgs potential should have dimension four. Hence we make the naive assumption for the normalization parameter that $\xi_{p, k} = a_{p, k}/(d_k^2 d_k^2)$, where $a_{p, k}$ is a dimensionless parameter expected to be of order 1. So as to obtain the hierarchical structure of the VEVs of the Higgs fields, we assume $\epsilon \equiv d_k^2/d_k^2 \ll 1$.

From the relations (3.1) we obtain the Higgs potential as$^4$:

$$V = \frac{1}{4 d_k^4} \left[ V_1 + \epsilon V_2 + \epsilon^2 V_3 \right], \quad (3.2)$$

where

$$V_1 = 2 b_1 \left( 5 - 2 \text{tr} (\Sigma^2) + \text{tr} (\Sigma^4) \right), \quad V_2 = 2 (b_2 + b_3) (H^\dagger H - 2 H^\dagger \Sigma H + H^\dagger \Sigma^2 H),$$

$$V_3 = b_4 \left( \text{tr} (\Sigma^2) - 2 H^\dagger \Sigma H + (H^\dagger H)^2 \right) + (b_5 + b_6)(d_k^2 - H^\dagger H)^2 + 4 b_5. \quad (3.3)$$

$^4$ The potential (3.2) is different from that obtained in Ref. 4). The latter was given through another configuration of vector spaces on $M_4 \times Z_5$:

$$V = \frac{1}{4} \left[ a \left\{ \left( \text{tr} (\Sigma^2) \right)^2 - \text{tr} (\Sigma^4) \right\} - (8a - 3c) \text{tr} (\Sigma^2) + \left( b + 4c \right) (H^\dagger H)^2 - 4 (b - 4d) H^\dagger H \right. \right.$$

$$\left. + d \left\{ H^\dagger \Sigma^2 H + \text{tr} (\Sigma^2) \right\} (H^\dagger H) \right] + (-6c - 12d + e) H^\dagger \Sigma H.$$

Though the authors of Ref. 4) introduced five arbitrary parameters, $a, \ldots, e$, let us associate these with the distances between the Minkowski spaces, in order to give an appropriate dimension to the potential:

$$a = \frac{1}{4d_k^4}, \quad b = \frac{2}{4d_k^4}, \quad c = \frac{1}{4d_k^4}, \quad d = \frac{1}{4d_k^4}, \quad e = 0.$$

Here $a, \ldots, d$ are of order 1. The parameter $\epsilon$ does not stem from the kinetic terms of the Yang-Mills type, and therefore we set it to zero. The symmetry breaking of the GUT scale can be seen in the limit $d_k \to \infty$. (This corresponds to the limit $\epsilon \to 0$.) Assuming that the resultant breaking possesses the appropriate hierarchical structure $\langle H \rangle \sim \sqrt{\epsilon} \langle \Sigma \rangle$ ($\langle H \rangle \sim \langle \Sigma \rangle$), in this limit we have only the $\alpha$-term and thus there remains only the adjoint Higgs $\Sigma$.

For positive $\alpha$, the potential of the $\alpha$-part gives rise to a symmetry-breaking pattern $SU(5) \to SU(4) \times U(1)$ (see Ref. 6), and for negative $\alpha$, the potential turns out not to be bounded below. Therefore, this model has no realistic vacuum if the coefficients of the potential are set in the manner above.
Here we have used relations for the coefficients coming from the identification

\[
\begin{align*}
    a_{1, l^{-1} l} &= a_{2, l^{-1} l} \equiv b_1, \\
    a_{1, k^{-1} k} &= a_{2, k^{-1} k} \equiv b_2, \\
    a_{3, kl} &= a_{3, k^{-1} l^{-1}} = b_3, \\
    a_{1, h^+} &= a_{2, k^{-1} h} \equiv b_4, \\
    a_{1, h^{-1} h} &= a_{2, k^{-1} h^{-1}} = b_5, \\
    a_{3, h^{-1} h} &= a_{3, k^{-1} h^{-1}} = b_6.
\end{align*}
\]

(3.4)

As for the adjoint Higgs, the \((tr \Sigma^2)^2\) term, which would be phenomenologically acceptable, does not appear in the potential. This is characteristic of our model, due to the structure of the field strength \(H\) (3.1).

To investigate this potential, it is useful to write the VEV of \(H\) explicitly as \((h_1, h_2, h_3, h_4, h_5)\) and put \(\langle \Sigma \rangle = \text{diag}(p_1, p_2, p_3, p_4, p_5)\). This is possible with the use of the \(SU(5)\) rotation. When solving the stationary condition for \(V\), the constraint \(tr \Sigma = 0\) should be taken into account. Thus we add to the above \(V\) a term \(\lambda \sum p_i\), where \(\lambda\) is a Lagrange multiplier. Then, the necessary conditions for minimizing the potential \(V\) are

\[
\begin{align*}
    \frac{\partial V}{\partial h_i} &= 0 \quad \Leftrightarrow \quad (b_2 + b_3)(1 - p_i^2) - \epsilon \left[ b_4 p_i + (b_4 + b_5 + b_6) \sum_j |h_j|^2 + (b_5 + b_6) \right] = 0 \\
    &\text{or} \quad h_i = 0, \\
    \frac{\partial V}{\partial p_i} &= 0 \quad \Leftrightarrow \quad -4b_1(p_i - p_i^3) - \epsilon \cdot 2(b_2 + b_3)|h_i|^2(1 - p_i^2) + \epsilon^2 \cdot b_4(|p_i| - |h_i|)^2 = 0, \\
    \frac{\partial V}{\partial \lambda} &= 0 \quad \Leftrightarrow \quad \sum p_i = 0.
\end{align*}
\]

(3.5)

(3.6)

(3.7)

The pattern of the symmetry breaking of interest is required to have a hierarchical structure. Thus we add the constraint \(\Sigma \gg \bar{H}\), or in other words,

\[
O(p_i) \gg O(h_i) \sqrt{\epsilon}.
\]

(3.8)

The vacuum should satisfy Eqs. (3.5)-(3.8).

It is natural for the coefficients to be given by \(b_1 = b_2 = b_3 = b_4 = b_5 = b_6 = 1\). But adopting these values, it turns out that the potential \(V\) is minimized at \(\langle H \rangle = 0\). Hence unfortunately, the electroweak breaking does not occur. This is mainly due to the hierarchical structure of the potential (3.2). The global structure is given by the part \(V_1\) (leading us to the GUT breaking), and the next order term \(V_2\) determines the breaking pattern for \(H\). As seen from the form of the potential (3.3), \(V_2\) is bilinear in \(H\), and therefore gives \(\langle H \rangle = 0\) (or, if the coefficient in front of \(H^+ H\) is negative, it gives a large \(\langle H \rangle\) together with \(V_3\), and the hypothesis of the hierarchy for VEVs is violated).

Therefore, instead, let us assume the relation \(b_2 + b_3 = 0\). Then setting \(b_1 = b_4 = b_5 = b_6 = 1\) for simplicity, a straightforward calculation shows that the whole potential \(V\) is minimized at

\[
\begin{align*}
    \langle \Sigma \rangle &= \frac{1}{\sqrt{4}} \text{diag}(-2, -2, -2, 3, 3), \\
    \langle H \rangle &= (0, 0, 0, 0, h_5), \quad \text{where} \quad |h_5| = \sqrt{\frac{2}{3} + \frac{1}{\sqrt{\epsilon}}}. 
\end{align*}
\]

(3.9)

(3.10)
In this way, we obtain a hierarchical structure of the gauge symmetry breaking of \( SU(5) \) GUT: \( SU(5) \rightarrow SU(3)_C \times SU(2)_W \times U(1)_Y \rightarrow SU(3)_C \times U(1)_{em}. \)

§4. Discussion

In this paper, we have studied the patterns of the gauge symmetry breaking in models constructed along the proposal of Ref. 3. Applying the proposal to a simple \( SU(5) \) model\(^1\) with an adjoint Higgs field, the obtained breaking pattern, \( SU(5) \rightarrow SU(3)_C \times SU(2)_W \times U(1)_Y \), is found to be very preferable phenomenologically. Then, this model was extended so as to include a five-dimensional Higgs field for electroweak symmetry breaking. This is made possible by adding another Minkowski space, and under the assumption for the coefficients of the potential, the vacuum breaks the gauge symmetry down to \( SU(3)_C \times U(1)_{em} \) with a hierarchical structure.\(^2\)

Though the connection Higgs formalism seems to be a nice idea to realize and explain the Higgs mechanism, it suffers from a problem of non-renormalizability. Actually, this formalism brings about a coupling reduction, and in general it is impossible to renormalize the obtained Higgs potential. One way to get rid of this difficulty is to regard the couplings in the Higgs potential as the one at a certain energy scale, i.e. the renormalization point.\(^3\) But then, what does this energy scale represent? A possible scenario\(^4\) is to think of the Minkowski spaces separated from each other as D-branes, and consider the energy scale as the string scale. In string phenomenology, new methods for model construction have been exploited with use of D-branes, where our four-dimensional space-time is realized as a worldvolume of the D-brane.\(^5\)

If we regard three parallel Minkowski spaces as D-branes, the identification used in this paper and many other models in NCG becomes very natural. The situation of Fig. 3 can be translated into a configuration of string theory\(^6\) as follows: consider 5 coincident D3-branes (the blobs in Fig. 3) and a parallel orbifold singular surface (gray line) located apart from those D3-branes. The gauge group on the D-branes is \( SU(5) \). A gauge group \( SO(10) \) is obtained when the location of the orbifold singular surface coincides with that of the D3-branes. The Higgs field \( \Sigma \) can be regarded as a sort of Wilson line,\(^7\) since it is a translational operator. The path generating a non-trivial Wilson line is the same as the configuration of a Dirichlet open string in a twisted sector. The scale of this system is the distance \( d_{1/2} \) between the D-branes and the orbifold singularity. Another Minkowski space \( M^3 \) (the circle) is a single D-brane on the orbifold singular surface. The scale for symmetry breaking is determined by the distance between D-branes.\(^8\)

Although the above points, D-branes provide a natural interpretation of the configuration of this paper, D-branes are themselves supersymmetric objects. In the context of Ref. 3 and NCG, it is difficult to incorporate \( \mathcal{N} = 1 \) supersymmetry.\(^9\) Therefore, relations between string theory and NCG should be further clarified.

\(^1\) Minimal \( SU(5) \) GUTs in the context of Refs. 1-3, 7-8 are completely different from ours.

\(^2\) We have not considered radiative corrections in this paper.

\(^3\) A similar situation can be found in Ref. 12 in string theory.
Gauge Symmetry Breaking in Models Inspired by NCG

(see related discussion in Ref. 15) to understand gauge symmetry breaking, using recently found non-BPS brane configurations,\(^{16}\) for example.

Acknowledgments

The author would like to thank S. Heusler, T. Hirayama, S. Yatsui, and K. Yoshioka for valuable discussions, and T. Kugo for insightful comments. This work is supported in part by a Grant-in-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture (\#3160).

References