THE MOTIVATION FOR MAGNET MEASUREMENTS

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Abstract
Construction of magnets to precise tolerances and the subsequent checking of their manufacture by even more precise measurements is the key to the construction of reliable synchrotrons which outperform their specification. It is important to have a working knowledge of the transverse beam dynamics determining these tolerances in order to intelligently analyse measurements and design more efficient procedures for assembly and measurement.

1. INTRODUCTION
The first step in designing an accelerator or storage ring is to choose an optimum pattern of focusing and bending magnets, the lattice. At this stage, non-linearities in the guide field are ignored. It is assumed that the bending magnets are identical and have a pure dipole field. Gradient magnets or quadrupoles have radial field shapes which have a constant slope, unperturbed by higher-order multipole terms.

Quadrupole magnets provide the restoring fields and are usually embedded in the lattice of bending magnets in an alternating pattern. Half focus the beam, while the other half defocus the beam. The lattice has a strong influence on the aperture of these magnets which are usually the most expensive single system in the accelerator. We see in Fig. 1 the pattern of one cell of the CERN SPS which is repeated 108 times around the circumference. Focusing and defocusing quadrupoles are labelled F and D and for obvious reasons this focusing structure is called FODO.

![Diagram](image)

**Fig. 1** One cell of the CERN SPS representing 1/108 of the circumference. The pattern of dipole (B) magnets and quadrupole (F and D) lenses is shown.

![Diagram](image)

**Fig. 2** The paths of particles within a FODO lattice are within the envelope of betatron motion are always closer in the D quadrupoles so they receive a net focusing effect. The phase space ellipse is tall and narrow at the D lens where the beam has a large divergence spread.
Particles make betatron oscillations within the envelopes indicated in Fig. 2. The envelope of these oscillations follows the function $\beta(s)$ which has waists near each defocusing magnet and has a maximum at the centres of F quadrupoles. Since F quadrupoles in the horizontal plane are D quadrupoles vertically, and vice versa, the two functions $\beta_H(s)$ and $\beta_V(s)$ are one half-cell out of register in the two transverse planes. For formal reasons $\beta$ has the dimensions of length but the units bear no relation at this stage to physical beam size.

Imperfections in the field of bending magnets can cause vertical and horizontal distortions to the central closed orbit of the machine -- the horizontal axis of Fig. 2 -- and errors in the gradient of the quadrupoles can cause distortion of the shape of the envelope of the betatron oscillations shown in Fig. 1. Both these effects can dominate the aperture required in the magnets unless magnet field errors are kept strictly under control in the manufacturing process. Hence, before going too far in fixing parameters, the practical difficulties in designing the magnets must be considered and the tolerances which can be reasonably written into the engineering specification determined. Estimates must be made of the non-linear departures from pure dipole or gradient field shape, and of the statistical fluctuation of these errors around the ring at each field level.

We must take into consideration that the remanent field of a magnet may have quite a different shape from that defined by the pole geometry; that steel properties may vary during the production of laminations; that eddy currents in vacuum chamber and coils may perturb the linear field shape. Mechanical tolerances must ensure that asymmetries do not creep in.

At high field the linearity may deteriorate owing to saturation and variations in packing factor can become important. Superconducting magnets will have strong error fields due to persistent currents in their coils.

When these effects have been reviewed, tolerances and assembly errors may have to be revised and measures taken to mix or match batches of laminations with different steel properties or coils made from different batches of superconductor. It may be necessary to place magnets in a particular order in the ring in the light of production measurements of field uniformity or to shim some magnets at the edge of the statistical distribution. Even when all these precautions have been taken, non-linear errors may remain whose effect it is simpler to compensate with auxiliary multipole magnets.

All these effects tend to become more serious in rings of larger radius and in machines with superconducting magnets the conductors whose position defines the field shape must be located with very tight tolerances.

2. **MAGNETIC RIGIDITY**

When calculating the effect of errors it is conventional to consider the small angular deflection produced by a magnet of length, $\ell$, and strength, $\Delta B$:

$$\theta = \frac{\ell \Delta B}{(B\rho)}$$

where the denominator is the “magnetic rigidity” of the particle and related to its relativistic momentum, $p$, and the particles charge, $e$, by

$$(B\rho) = \frac{p}{e}.$$  

The common convention in charged particle dynamics is to quote $p$ in units of GeV and $B\rho$ in Tesla.meters. Whereupon:

$$(B\rho)[\text{T.m}] = 3.3356 p[\text{GeV}]$$
Figure 3 shows the trajectory of a particle in a bending magnet or dipole of length $\ell$. Usually the magnet is placed symmetrically about the arc which is the particle's path. One may see immediately that:

$$\sin(\theta/2) = \frac{\ell}{2\rho} = \frac{\ell B}{2(B\rho)},$$

and if $\theta \ll \pi/2$

$$\theta = \frac{\ell B}{(B\rho)}.$$

The ends of bending magnets are often parallel but in some machines are designed to be normal to the beam. There is a focusing effect at the end which depends on the angle of these faces.

3. **FOCUSBING**

The principal focusing elements in a modern synchrotron are quadrupole magnets. The poles are truncated rectangular hyperbolae and alternate in polarity. The field shape (Fig. 4) is such that it is zero on the axis of the device but its strength rises linearly with distance from the axis. This can be seen from a superficial examination of Fig. 4, remembering the product of field and length of a field line joining the poles is a constant. Symmetry tells us that the field is vertical in the median plane (and purely horizontal in the vertical plane of asymmetry). The field must be downwards on the left of the axis if it is upwards on the right.

![Fig. 4 Components of field and force in a magnetic quadrupole. Positive ions approach the reader on paths parallel to the y axis.](image)

This last observation ensures that the horizontal focusing force, $e v B_z$, has an inward direction on both sides and, like the restoring force of a spring, rises linearly with displacement, $x$. The strength of the quadrupole is characterised by its gradient $dB_z/dx$ normalised with respect to magnetic rigidity:

$$k = \frac{1}{(B\rho)} \frac{dB_z}{dx}.$$

The angular deflection given to a particle passing through a short quadrupole of length $Q$ and strength $k$ at a displacement $x$ is therefore:

$$\Delta x = \ell k x.$$
Compare this with a converging lens:

\[ \Delta x' = -x / f \]

and we see that the focal length of such a horizontally focusing quadrupole is

\[ f = -1/(k\ell) \, . \]

Note that conventionally \( k \) is negative for a horizontally focusing quadrupole.

The particular quadrupole shown in Fig. 4 would focus positive particles coming out of the paper or negative particles going into the paper in the horizontal plane. A closer examination reveals that such a quadrupole is defocusing in the vertical plane and deflects particles with a vertical displacement away from the axis – vertical displacements are defocused. You can immediately see this if you turn the figure 90°. In spite of this seemingly damning characteristic, a FODO pattern of alternating polarity quadrupoles has a net focusing effect in both planes.

4. THE GUTTER ANALOGY

In order to understand focusing we ignore vertical defocusing for a moment and consider an infinitely long quadrupole. A particle oscillates within it exactly like a small sphere rolling down a slightly inclined gutter with constant speed. Figure 5 shows two views of this motion and from the right hand view we recognise the motion as a sine wave: note too that the sphere makes four complete oscillations along the gutter. In accelerator terms its motion has a wave number \( Q = 4 \). In a ring \( Q \) is the number of oscillations per turn.

Now extend this analogy by bending the gutter into a circle rather like the brim of a hat. Suppose we provide the necessary instrumentation to measure the displacement of the sphere from the centre of the gutter each time it passes a given mark on the brim. We also suppose we have a means to measure its transverse velocity which, with the aid of a computer, we might convert into its divergence angle defined in Fig. 6:

\[ x' = \frac{dx}{dt} = \frac{v_{t}}{v_{||}} \]

Suppose we make the hat out of a slightly different length of gutter than shown so that \( Q \) is not an integer. If we plot a diagram of \( x' \) against \( x \) we can plot a point for each arrival of the sphere. This is called a phase space diagram. The sphere may have a large transverse velocity as it crosses the axis of the gutter or it might have almost zero transverse velocity as it reaches its maximum displacement.

The locus will be an ellipse (Fig. 6) and the phase will advance by \( Q \) revolutions each time the particle returns.
Fig. 6 The elliptical locus of a particle's history in phase space as it circulates in a synchrotron

The time has come to define some of the beam dynamical quantities more rigorously. The area of the ellipse is a measure of how much the particle departs from the ideal trajectory represented by the origin:

\[ \text{Area} = \pi \varepsilon \text{ [mm.mrad]} \]

The maximum amplitude, or major axis, of the ellipse is defined:

\[ \hat{x} = \sqrt{\pi \beta} \]

so that to satisfy Eq. (15)

\[ \hat{x} = \sqrt{\pi / \beta} \]

The quantity \( \beta \) is a property of the focusing system, not the beam, it varies around the ring and is the same function we have plotted in Fig. 1. In an alternating gradient focusing system such as Fig. 1, the brim of the hat will vary its width and curvature around the crown and \( \beta \) will follow this variation in some way. Note that the aspect ratio of the ellipse is just \( \beta \).

5. **TRANSVERSE EQUATION OF MOTION**

In Section 3 we derived an expression for the change in divergence of a particle passing through the quadrupole. The angular deflection given to a particle passing through a short quadrupole of length \( ds \) and strength \( k \) at a displacement \( x \):

\[ dx' = k x ds \]

We can immediately rearrange this to form the differential equation for the motion

\[ x'' - k(s)x = 0 \]

This is the famous Hill's Equation, a second-order linear equation with periodic coefficient, \( k(s) \) which is the distribution of focusing strength around the ring. In the horizontal plane we must strictly include an extra focusing term for the curvature which can be significant in small rings:

\[ x'' + \frac{1}{\rho(s)^2} - k(s) x = 0 \]
In the vertical plane the equation has the opposite sign of \( k \)
\[
z'' + k(s)z = 0
\]

The equation is reminiscent of simple harmonic motion but with a restoring constant \( k(s) \) which varies around the accelerator. In order to arrive at a solution we first assume that \( k(s) \) is periodic on the scale of one turn of the ring or some smaller unit, a cell or period, from which the ring is built. The solution of Hill's Equation is not unlike simple harmonic motion:
\[
x = \sqrt{\beta(s) e} \sin[\phi(s) + \phi_0]
\]

In simple harmonic motion the amplitude is a constant but now we see that in addition to \( \sqrt{e} \), a property of the beam, there is another amplitude component, a function, \( \sqrt{\beta(s)} \) which is defined by the lattice pattern. Another difference with harmonic motion is that phase does not advance linearly with time and with distance \( s \) around the ring but is a seemingly arbitrary function of \( s \). It will perhaps not be difficult to believe that these functions of \( s \) must have the same periodicity as the lattice. (All this is to be seen in Fig. 1.)

6. CIRCLE DIAGRAM

When we plot the locus of the motion of a particle obeying Hill's equation we obtain ellipses in phase space similar to Fig. 6. The aspect ratio of the ellipse depends on the position of the observer in the ring. Close to an F quadrupole where \( \beta(s) \) is large the horizontal ellipse will be very wide and not very high in divergence angle and it is in such positions that a small angular kick due to an imperfection has the greatest effect on the beam. Imagine, for example, how little angular displacement is needed to move the ellipse by its own height and increase the beam dimensions by a factor 2. The effect of errors is most serious near F quadrupoles.

![Phase-space diagram at \( \beta \) minimum and \( \beta \) maximum](image)

So predominant is the effect of perturbations near \( \hat{\beta} \) positions that you can often do quite good "back of the envelope" calculations by closing your eyes to what happens to the protons in between F quadrupoles. At F quadrupoles the ellipse always looks the same, i.e., upright, with semi-axes in displacement and divergence
\[
\sqrt{\beta e}, \quad \sqrt{e / \beta}
\]

This can be reduced to a circle radius by using the new coordinates
\[
y = y
\]
\[
p = \beta y'
\]
If the machine has 108 periods and a $Q$ of 27.6, the proton advances in phase by $2\pi Q/108$ from one period to the next; this is just the angle subtended at the centre of the circle multiplied by $Q$. After one turn of the machine, it has made 27 revolutions of the circle plus an angle of $2\pi$ multiplied by the fractional part of $Q$, see Fig. 8.

7. CLOSED ORBIT DISTORTION

In designing a synchrotron, the bending field is matched to some ideal synchronous momentum, $p_0$. A particle of this momentum and of zero emittance will pass down the centre of each quadrupole, be bent by exactly $2\pi$ by the bending magnets in one turn of the ring and remain synchronous with the r.f. frequency. Its path is called the central (or synchronous) momentum closed orbit, (Fig. 2). This orbit closes on itself so that $x$ and $x'$ remain zero. Imperfections in the guide field can distort this orbit as in Fig. 9 where a dipole error is progressively increased.

![Dipole](image)

Fig. 9 Closed orbit distortion as a dipole is slowly switched on

Even the best synchrotron magnets cannot be made absolutely identical. Each magnet differs from the mean by some small error in integrated strength:

$$\delta(Bt) = \int B \, dt - \left( \int B \, dt \right)_{\text{ideal}} .$$

These and other machine imperfections, such as survey errors which can be expressed as equivalent to field errors, are randomly spread around the ring.

One of the most important considerations in designing a machine is to keep this closed orbit distortion to a minimum because it eats up available machine aperture. Also, once we have succeeded in getting a few turns round the machine, we want to reduce this distortion with correcting dipole magnets whose strength must be estimated before construction. As a first step let us consider the effect on the orbit of such a correcting dipole located at a position where $\beta = \beta_K$ and observed at another position.

We turn on a short dipole (we shall assume it is a delta function in $s$) which makes a constant angular kick in divergence

$$\delta\psi = \delta(Bt) / (Bp) ,$$
8. CLOSED ORBIT IN THE CIRCLE DIAGRAM

In the unperturbed part of the lattice the distorted closed orbit must of necessity follow the same equations of motion as a particle. Its locus in phase space will therefore be an ellipse. To illustrate the physics of closed orbit distortion let us examine this ellipse only at all points of equal $\beta(s)$, for example at the middle of the F quadrupoles. By multiplying the divergence by $\beta(s)$ the locus becomes a circle. Consider the special case where dipole error and observation point are at the same value of $\beta(s)$. We see quite from Fig. 10 how the equation for the amplitude of the distortion appears. The "kick" due to the dipole just completes the circular trajectory after a non-integer number of turns.

9. SOURCES OF DISTORTION

Table 1

<table>
<thead>
<tr>
<th>Type of element</th>
<th>Source of kick</th>
<th>r.m.s. value</th>
<th>$\left&lt; \Delta \ell / (B \rho) \right&gt;_{\text{rms}}$</th>
<th>Plane</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gradient magnet</td>
<td>Displacement</td>
<td>$&lt;\Delta y&gt;$</td>
<td>$k_d &lt;\Delta y&gt;$</td>
<td>$x, z$</td>
</tr>
<tr>
<td>Bending magnet</td>
<td>Tilt</td>
<td>$&lt;\Delta&gt;$</td>
<td>$\theta_i &lt;\Delta&gt;$</td>
<td>$z$</td>
</tr>
<tr>
<td>Bending magnet</td>
<td>Field error</td>
<td>$&lt;\Delta B / B&gt;$</td>
<td>$\theta_i &lt;\Delta B / B&gt;$</td>
<td>$x$</td>
</tr>
<tr>
<td>Straight sections</td>
<td>Stray field</td>
<td>$&lt;\Delta B_s&gt;$</td>
<td>$d_i &lt;\Delta B_s&gt; / (B \rho)_{\text{inj}}$</td>
<td>$x, z$</td>
</tr>
</tbody>
</table>

The principal imperfections in a synchrotron causing orbit distortion are shown in Table 1. These include errors in magnetic field and in the alignment of the ring. The first line in the table represents the random variations in the position of quadrupole magnets with respect to their ideal location. A small displacement of a quadrupole gives an effective dipole perturbation, $k l \Delta y$. The tilt of bending magnets causes a small resultant dipole in the horizontal direction which deflects vertically. Obviously there may also be random errors in magnet gap, length or in the coercivity of the steel yoke which determines remanent field contributing to the third line. Both remanent and stray fields in straight sections tend to be constant and their effect scales as $l / B$ as the machine pulses. Their effect should therefore be evaluated where it is worst, at injection. In a modern superconducting machine the persistent current fields play the role of remanent effects.

10. UNCORRELATED ERRORS

In estimating the effect of a random distribution of dipole errors we must take the r.m.s. average, weighted according to the $\beta_i$ values over all of the kicks $\delta y_i$ from the $N$ magnets in
the ring. If we observe the distortion at $\beta(s)$ and its source is at $\beta_i$, the effect is scaled as $\sqrt{\beta(s)/\beta_i}$ and the expectation value of the amplitude is:

$$
\langle y(s) \rangle = \sqrt{\beta(s) \over 2 \sqrt{2 \sin \pi Q}} \sqrt{\sum_i \beta_i \delta y_i^2} = \sqrt{\beta(s) \beta \over 2 \sqrt{2 \sin \pi Q}} \sqrt{N} \left( \Delta B \ell \right)_{\text{rms}} \over B \rho .
$$

The factor $\sqrt{2}$ comes from averaging over all the phases of distortion produced. In designing a machine it used to be conventional wisdom to make sure that the vacuum chamber will accommodate twice this expectation value. The probability of no particles making the first turn is thus reduced to a mere 2%. More modern designs rely on closed-orbit steering to thread the first turn and thereafter assume that orbit correction to a millimetre or so will be feasible.

11. MAGNIFICATION OF ERRORS

A more rigorous renormalisation of phase space which does not imply any approximation but which simplifies the problem is the $(\eta, \psi)$ transformation to convert Hill's equation into that of a harmonic oscillator:

$$
\frac{d^2 \eta}{d\psi^2} + Q^2 \eta = g(\psi)
$$

where $g(\psi)$ is the azimuthal pattern of some perturbation of the guide field related to

$$
F(s) = \frac{\Delta B(s)}{B \rho} .
$$

In the ideal case $g(\psi)$ is everywhere zero.

I will not bother you with how this transformation is found, but just state it. The new coordinates are related to the old:

$$
\eta = \beta^{-1/2} \psi
$$

$$
\psi = \int \frac{ds}{Q \beta} , \quad g(\psi) = Q^2 \beta^{3/2} F(s) ,
$$

where $\psi$ advances by $2\pi$ every revolution. It coincides with $\theta$ at each location and does not depart very much from $\theta$ in between.

One of the advantages of reducing the problem to that of a harmonic oscillator in $(\eta, \psi)$ coordinates is that perturbations can be treated as the driving term of the oscillator. They may be broken down into their Fourier components, and the whole problem solved like the forced oscillations of a pendulum. The driving term is put on the right hand side of Hill's equation:

$$
\frac{d^2 \eta}{d\psi^2} + Q^2 \eta = Q^2 \sum_{n=1}^{\infty} f_n e^{i k \psi} = Q^2 \beta^{3/2} F(s) ,
$$

where $F(s)$ is the azimuthal pattern of the perturbation $\Delta B/(B \rho)$; and $Q^2 \beta^{3/2}$ comes from the transformation from physical coordinates to $(\eta, \psi)$.

The Fourier amplitudes are defined:
\[ f(\psi) = \beta^{3/2} F(s) = \sum_{k} f_k e^{ik\psi}, \]

where

\[ f_k = \frac{1}{2\pi} \int_{0}^{2\pi} f(\psi) e^{-ik\psi} d\psi = \frac{1}{2\pi\beta^{1/2}} \int f(s) e^{-ik\psi} ds. \]

We can then solve Hill's equation as

\[ \eta = \sum_{n=1}^{\infty} \frac{Q^2 f_k}{Q^2 - k^2} e^{ik\psi} \quad \text{(or its real part)}. \]

In fact this solution is a closed orbit, a particular solution of Hill's differential equation, to which we must add the general solutions which describe betatron oscillations about this orbit.

The function \( Q^2/(Q^2 - k^2) \) is sometimes called the magnification factor for a particular Fourier component of \( A \). It rises steeply when the wave number \( k \) is close to \( Q \), and the effect of the two Fourier components in the random error pattern with \( k \) values adjacent to \( Q \) accounts for about 60% of the total distortion due to all random errors. Figure 11 shows a closed orbit pattern from electrostatic pick-ups in the FNAL ring, whose \( Q \) is between 19 and 20. The pattern shows strong components with these wave numbers.

![Figure 11: FNAL main ring electrostatic pick-ups show closed orbit around the ring (Q ≈ 19.2)](image)

**12. MULTIPOLe FIELD EXPANSION**

Before we come to discuss the non-linear terms in the dynamics, we shall need to describe the field errors which drive them. The magnetic vector potential of a magnet with \( 2n \) poles in Cartesian coordinates is:

\[ A = \sum_{n} A_n f_n(x,z), \]

where \( f_n \) is a homogeneous function in \( x \) and \( z \) of order \( n \).

**Table 2**

Cartesian Solutions of Magnetic Vector Potential

<table>
<thead>
<tr>
<th>Multipole</th>
<th>( n )</th>
<th>Regular ( f_n )</th>
<th>Skew ( f_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrupole</td>
<td>2</td>
<td>( x^2 - z^2 )</td>
<td>2xz</td>
</tr>
<tr>
<td>Sextupole</td>
<td>3</td>
<td>( x^3 - 3xz^2 )</td>
<td>3x^2z - z^3</td>
</tr>
<tr>
<td>Octupole</td>
<td>4</td>
<td>( x^4 - 6x^2z^2 + z^4 )</td>
<td>4x^3z - 4xz^3</td>
</tr>
<tr>
<td>Decapole</td>
<td>5</td>
<td>( x^5 - 10x^3z^2 + 5xz^4 )</td>
<td>5x^4z - 10x^2z^3 + z^5</td>
</tr>
</tbody>
</table>
Table 2 gives $f_n(x, z)$ for low-order multipoles, both regular and skew. Figure 12 shows the distinction. We can obtain the function for other multipoles from the binomial expansion of

$$f_n(x, z) = (x + iz)^n.$$  

The real terms correspond to regular multipoles, the imaginary ones to skew multipoles. The difference between regular and skew is illustrated in Fig. 12.

![Normal 6 Pole vs Skew 6 Pole](image)

**Fig. 12** Pole configurations for a regular sextupole and a skew sextupole

For numerical calculations it is useful to relate $A_n$ and the corresponding derivative of field and compare the resulting series with a Taylor expansion of the field expressed as a series of derivatives. For regular magnets:

$$B_z(z = 0) = \frac{\partial A_x}{\partial x} = \sum_{n=1}^\infty n A_n x^{n-1} = \sum_{n=1}^\infty \frac{1}{(n-1)!} \left( \frac{d^{(n-1)} B_z}{dx^{n-1}} \right)_0 x^{n-1},$$

so that by inspection

$$A_n = \frac{1}{n!} \left( \frac{d^{(n-1)} B_z}{dx^{n-1}} \right)_0.$$

A more modern convention (in Europe) is to speak of multipole coefficients, $b_i$, for normal components and $a_i$ for skew components, where $R_0$ is some reference radius (10 mm for the LHC), $B_i$ is the magnitude of the nominal dipole field, $B_y$, and $Z = x + iy$.

$$B_y + iB_z = B_i \sum_n (b_n + ia_n) (Z/R_0)^{n-1}.$$  

The suffix, $n = 1$ for the dipole, 2 for the quadrupole and 3 for sextupole etc. Note that US notation may start with $n = 0$ for the dipole and different laboratories use other reference radius. In spite of the possibilities for confusion this has the advantage that the coefficients are a measure of the tolerated fraction of field error inside the reference circle, where beam is supposed to be stable.

As a practical example of how one may identify the multipole components of a magnet by inspecting its symmetry, we digress a little to discuss the sextupole errors in the main dipoles of a large synchrotron.

Let us look at a simple dipole (Fig. 13). It is symmetric about the vertical axis and its field distribution will contain mainly even exponents of $x$, corresponding to odd $n$ values: dipole, sextupole, decapole, etc. We can see, too, that cutting off the poles at a finite width can produce a virtual sextupole. Moreover, the remanent field pattern is frozen in at high field where the flux lines leading to the pole edges are shorter than those leading to the centre. The remanent magneto-motive force $\int \mathcal{H}_c d\ell$ is weaker at the pole edges, and the field tends to sag into a parabolic or sextupole configuration. This too produces a sextupole.
Fig. 13 The field in a simple dipole. The $\delta N$ and $\delta S$ poles superimposed on the magnet poles give the effect of cutting off the poles to a finite width.

These three sources of sextupole error are the principle non-linearities in a large machine like the SPS. Note that these sextupole fields have no skew component. Similar sextupole fields which are even stronger and which vary with time are a principle error component in the injection field shape of a modern superconducting ring such as LHC. These are caused by persistent circulating currents set up in the individual turns of the superconducting coils rather as eddy currents are set up during magnet pulsing in the vacuum chamber of a warm machine. In the LHC dipoles the field shape is determined by the positional tolerances on the position of the windings which are very difficult tolerances to achieve. However, before launching into non-linearities let us examine a simple linear resonance.

13. WORKING DIAGRAM

Apart from the obvious need to minimise closed orbit distortion, measures must be taken to reduce the influence of non-linear resonances on the beam. A glance at the working diagram (Fig. 14) shows why this is so. The $Q_H, Q_V$ plot is traversed by a mesh of non-linear resonance lines or stopbands of first, second, third, and fourth order. The order, $n$, determines the spacing in the $Q$ diagram; third-order stopbands, for instance, converge on a point which occurs at every 1/3 integer $Q$-value (including the integer itself). The order, $n$, is related to the order of the multipole which drives the resonance. For example, fourth-order resonances are driven by multipoles with $2n$ poles, i.e. octupoles. Multipoles can drive resonances of lower-order; octupoles drive fourth- and second-order; sextupoles third- and first-order, etc., but here we simply consider the highest order driven.

The non-linear resonances are those of third-order and above, driven by non-linear multipoles. Their strength is amplitude-dependent so that they become more important as we seek to use more and more of the machine aperture. Theory used to discount resonances of fifth- and higher-order as harmless (self-stabilised), but experience in the ISR, FNAL and SPS suggests this is not to be relied upon when we want beams to be stored for more than a second or so.

Each resonance line is driven by a particular pattern of multipole field error which can be present in the guide field. The lines have a finite width depending directly on the strength of the error. In the case of those driven by non-linear fields, the width increases as we seek to exploit a larger fraction of the magnet aperture. We must ensure that the errors are small enough to leave some clear space between the stopbands to tune the machine, otherwise particles will fall within the stopbands and be rapidly ejected before they have even been accelerated. In general, the line width is influenced by the random fluctuations in multipole error around the ring rather than the mean multipole strength.

Systematic or average non-linear field errors also make life difficult. They cause $Q$ to be different for the different particles in the beam depending on their betatron amplitude or momentum defect. Such a $Q$-spread implies that the beam will need a large resonance-free window in the $Q$ diagram.
14. PHASE SPACE TRAJECTORY FOR 1/3 RESONANCE

The third-integer stopbands are driven by sextupole field errors and are therefore non-linear. First take the phase space of Fig. 8 and imagine the perturbation is a single short sextupole of length \( \ell \), near a horizontal maximum beta location. Its field is

\[
\Delta B = \frac{d^2 B}{dx^2} x^2 = \frac{B''}{2} x^2,
\]

and it kicks a particle with betatron phase \( Q\theta \) by

\[
\Delta p = \frac{\beta B''}{2B\rho} x^2 = \frac{\beta B'' a^2}{2B\rho} \cos^2 Q\theta
\]

inducing increments in phase and amplitude,

\[
\frac{\Delta a}{a} = \frac{\Delta p}{a} \sin Q\theta = \frac{\beta B'' a}{2B\rho} \cos^2 Q\theta \sin Q\theta
\]

\[
\Delta \phi = \frac{\Delta p}{a} \cos Q\theta = \frac{\beta B'' a}{2B\rho} \cos^3 Q\theta
\]

\[
= \frac{\beta B'' a}{8B\rho} (\cos 3Q\theta + 3 \cos Q\theta).
\]
Suppose $Q$ is close to a third integer, then the second term in the above equation averages to zero over three turns and we are left with a phase shift:

$$2\pi \Delta Q = \Delta \phi = \frac{\beta \ell B'' a \cos 3Q \theta}{8B\rho}.$$  

Suppose that Fourier analysis of $\beta(s)B''(s)$ results in a term with an azimuthal dependence $\cos p \theta$. Together with $3Q \theta$ in the above equation this produces a slowly varying term, $\cos(3Q - p) \theta$. Hence, close to $Q = p/3$, where $p$ is an integer, $\cos(3Q - p) \theta$ varies slowly, wandering within a band about the unperturbed $Q_0$ within the limits

$$Q_0 - \frac{\beta \ell B'' a}{16\pi B\rho} < Q < Q_0 + \frac{\beta \ell B'' a}{16\pi B\rho}.$$  

This is the stopband width but in reality is a perturbation in the motion of the particle itself.

We can write the expression for amplitude perturbation

$$\frac{\Delta a}{a} = \frac{\beta \ell B'' a}{8B\rho} \sin 3Q \theta.$$  

Suppose the third integer $Q$-value is somewhere in the band. Then, after a sufficient number of turns, the perturbed $Q$ of the machine will be modulated to coincide with $3p$. On each subsequent revolution this increment in amplitude builds up until the particle is lost. Growth is rapid and the modulation of $Q$ away from the resonant line is comparatively slow.

Before leaving these expressions, we should note that the resonant condition, $3Q = \text{integer}$, arises because of the $\cos^3 Q \theta$ which occurs early in the above analysis, which in turn stems from the $x^2$ dependence of the sextupole field. This reveals the link between the order of the multipole and that of the resonance. We also see that the $a^2$ leads to a linear dependence of width upon amplitude. The reader may care to speculate how other higher-order multipoles link to other lines in the working diagram and their degree of non-linearity expressed as the exponent of $a$.

But returning to the subject of stopbands, if $Q_0$ is a distance $\Delta Q$ from the third integer resonance, particles with amplitudes less than

$$a < \frac{16\pi(B\rho)\Delta Q}{\beta \ell B''}$$

will never reach a one third integer $Q$ and are in a central region of stability. All of this is perhaps best visualised in transverse phase space shown in Fig. 15 where the circular orbit of small amplitude becomes triangular for particles of larger amplitude experiencing more perturbation from the non-linear field. At a certain amplitude, obtained by replacing the inequality above by an equality, the effect of the non-linear force just brings the $Q$ to the $1/3$ integer value and we find metastable fixed points in phase space where there is resonant condition but infinitely slow growth.

Fig. 15 Third-order separatrix
For a one third integer resonance there are three fixed points at $\theta = 0, 2\pi/3, \text{and } 4\pi/3$. For a resonance of order, $n$, there will be $n$ such points. The fixed points are joined by a separatrix, which is the bound of stable motion. This stable area is a rudimentary example of dynamic aperture. A more rigorous theory, which takes into account the perturbation in amplitude, would tell us that the separatrix is triangular in shape with three arms to which particles cling on their way out of the machine.

15. INJECTION STUDIES AT FNAL

As a cautionary tale to complete this motivation towards precision in magnet measurement, we should examine the contour model of beam survival during the one second it takes to inject into the FNAL main ring. The peaks of the mountains in Fig. 17 show the small regions in the $Q (\nu \text{ in US parlance})$ - diagram where only 20 or 30% of the beam is lost due to the steep valleys of non-linear resonances in the working diagram. The driving term was a remanent sextupole randomly distributed among the dipoles of the ring. This machine broke new ground while its predecessors, much smaller machines without an injection dwell time, had hardly experienced the problem of non-linear resonances. Its constructors may therefore be forgiven for not taking stringent precautions against non-linear resonances. Measurements which might have predicted such a driving term and prompted a remedy were preserved for only one sample magnet and there was no attempt to smooth out the effect by shuffling laminations. Magnet ripple, another effect which scales adversely with ring size, ensured that the resonances were broadened to make it virtually impossible to obtain full survival. Fortunately the resonances were subsequently compensated and the machine went on to break all records, at the cost of a year's delay in understanding and correcting the problem. Naturally the machines which followed took good care to learn from this experience — a practice we should not forget to emulate!

Fig. 17 Beam survival impaired by resonances in the FNAL main ring