Enhancing Mechanisms of Neutrino Transitions in a Medium of Nonperiodic Constant - Density Layers and in the Earth

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Abstract

We perform a systematic study of the enhancement of the transitions $\nu_\mu (\nu_e) \rightarrow \nu_e (\nu_\mu, \tau), \nu_2 \rightarrow \nu_e, \bar{\nu}_\mu \rightarrow \bar{\nu}_s, \nu_e \rightarrow \nu_s$, etc. of neutrinos passing through a medium of nonperiodic density distribution, consisting of i) two layers of different constant densities (e.g., Earth core and mantle), or ii) three layers of constant density with the first and the third layers having identical densities and widths which differ from those of the second layer (mantle-core-mantle in the Earth). The extrema of the two-neutrino transition probabilities in both cases of a medium with two or three constant density layers are analyzed. We find that, in addition to the MSW and neutrino oscillation length resonance (NOLR) local maxima previously discussed, there exist new resonance-like absolute maxima of interference nature. The latter correspond to a new effect of total neutrino conversion and are absolute maxima in any independent variable: the neutrino energy, the widths of one of the layers, etc. We find the complete set of such maxima. We show that for any layers’ densities there are values of the layers’ widths and the neutrino oscillation parameters for which the conditions for existence of the new maxima are fulfilled. They are fulfilled, in particular, for the Earth-core-crossing solar and atmospheric neutrinos. We prove that the NOLR and the new resonance-like enhancement are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. Thus, they are interference (local and absolute) maxima which have nothing to do with the parametric resonance in the neutrino transitions, possible in a medium with periodically varying density. We show that the strong resonance-like enhancement of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos is due to the new effect of total neutrino conversion. This enhancement was previously associated with the NOLR.

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1 Introduction

It was pointed out in [1] that the $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_e$ ($\nu_\tau \rightarrow \nu_\mu(\tau)$) oscillations of solar and atmospheric neutrinos in the Earth, caused by neutrino mixing (with nonzero mass neutrinos) in vacuum, can be strongly amplified by new type of resonance-like mechanism which differs from the MSW one and takes place when the neutrinos traverse the Earth core on the way to the detector (see also [2, 3, 4]). As numerical calculations have shown, at small mixing angles ($\sin^2 2\theta \lesssim 0.05$), the maxima due to this new enhancing mechanism in the corresponding transition probabilities, $P(\nu_2 \rightarrow \nu_e) \equiv P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e) (P(\nu_\tau \rightarrow \nu_\mu(\tau)))$, are absolute maxima and dominate in $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$: the values of the probabilities at these maxima in the simplest case of two-neutrino mixing are considerably larger - by a factor of $\sim (2.5 - 4.0)$ ($\sim (3.0 - 7.0)$), than the values of $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_\mu(\tau))$ at the local maxima associated with the MSW effect taking place in the Earth core (mantle). The effect of the new enhancement is less dramatic at large mixing angles. The enhancement is present and dominates also in the $\nu_2 \rightarrow \nu_e$ transitions in the case of $\nu_e - \nu_\mu$ mixing and in the $\nu_e \rightarrow \nu_\tau$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ transitions at small mixing angles [1, 2, 3, 4]. Even at small mixing angles the enhancement is relatively wide [1, 2, 4] in the Nadir angle, $h$, and in the neutrino energy - it is somewhat wider than the MSW resonance. It also exhibits rather strong energy dependence.

The new resonance-like enhancement of the probability $P_{e2}$ has important implications for the tests of the MSW $\nu_e \rightarrow \nu_\mu(\tau)$ and $\nu_e \rightarrow \nu_\tau$ transition solutions of the solar neutrino problem [1, 3, 5, 6] (see also [7]). For values of $\Delta m^2$ from the small mixing angle (SMA) MSW solution region and the geographical latitudes at which the Super-Kamiokande, SNO and ICARUS detectors are located, the enhancement takes place in the $\nu_e \rightarrow \nu_\mu(\tau)$ case for values of the $^8\text{B}$ neutrino energy lying in the interval $\sim (5 - 12)$ MeV to which these detectors are sensitive. Accordingly, at small mixing angles the new resonance is predicted [5] to produce a much bigger - by a factor of up to $\sim 6$, day-night (D-N) asymmetry in the Super-Kamiokande sample of solar neutrino events, whose night fraction is due to the core-crossing neutrinos, in comparison with the asymmetry determined by using the whole night event sample. As a consequence, it can be possible to test a substantial part of the MSW $\nu_e \rightarrow \nu_\mu(\tau)$ SMA solution region in the $\Delta m^2 - \sin^2 2\theta$ plane (see, e.g., [8]) by performing core D-N asymmetry measurements [5, 7]. The Super-Kamiokande collaboration has already successfully applied this approach to the analysis of their solar neutrino data [8]: the limit the collaboration has obtained on the D-N asymmetry utilizing only the core event sample permitted to exclude a small part of the MSW SMA solution region. In contrast, the current Super-Kamiokande upper limit on the whole night D-N asymmetry [8] does not allow to probe the SMA solution region: the predicted asymmetry is too small (see, e.g., [5]).

The same new enhancement mechanism can and should be operative also [1] in

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2 The $\nu_2 \rightarrow \nu_e$ transition probability accounts, as is well-known, for the Earth effect in the solar neutrino survival probability in the case of the MSW two-neutrino $\nu_e \rightarrow \nu_\mu(\tau)$ and $\nu_e \rightarrow \nu_\tau$ transition solutions of the solar neutrino problem, $\nu_s$ being a sterile neutrino.

3 The Nadir angle determines uniquely the neutrino trajectory through the Earth.
the $\nu_\mu \rightarrow \nu_e$ ($\nu_e \rightarrow \nu_{\mu(\tau)}$) small mixing angle transitions of the atmospheric neutrinos crossing the Earth core if the atmospheric $\nu_\mu$ and $\bar{\nu}_\mu$ indeed take part in large mixing vacuum $\nu_\mu(\bar{\nu}_\mu) \leftrightarrow \nu_e(\bar{\nu}_e)$, oscillations with $\Delta m^2 \sim (10^{-3} - 8 \times 10^{-3})$ eV$^2$, as is strongly suggested by the Super-Kamiokande data [9], and if all three flavour neutrinos are mixed in vacuum. The new resonance can take place practically for all neutrino trajectories through the core (e.g., for the trajectories with $h = (0^\circ - 23^\circ)$). It can produce an excess of e-like events at $-1 \leq \cos \theta_z \leq -0.8$, $\theta_z$ being the Zenith angle, in the Super-Kamiokande multi-GeV atmospheric neutrino data and can be responsible for at least part of the strong Zenith angle dependence of the Super-Kamiokande multi-GeV and sub-GeV $\mu$-like data [1, 2, 4].

The Earth enhancement of the two-neutrino transitions of interest of the solar and atmospheric neutrinos at relatively small mixing angles has been discussed rather extensively, see, e.g., refs. [7, 10, 11, 14]. Some of the articles contain plots of the probabilities $P_{e2}$ and/or $P(\nu_\mu \rightarrow \nu_e)$ or $P(\nu_e \rightarrow \nu_{\mu(\tau)})$ on which one can clearly recognize now the dominating maximum due to the new enhancement effect. However, this maximum was invariably interpreted to be due to the MSW effect in the Earth core (see, e.g., [10, 11]) before the appearance of [1].

In view of the important role the new mechanism of enhancement of the neutrino transitions in the Earth can play in the interpretation of the results of the experiments with solar and atmospheric neutrinos and in obtaining information about possible small mixing angle $\nu_\mu$ ($\bar{\nu}_\mu$) and $\nu_e$ ($\bar{\nu}_e$) oscillations, it would be desirable to have an unambiguous understanding of the underlying physics of the mechanism. In [1] it was interpreted as an effect of constructive interference between the various probability amplitudes (notably of the neutrino transitions in the Earth mantle and in the Earth core), entering into the sum representing the probability amplitude of the transition in the Earth.

The exact conditions for the enhancement of the probabilities $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e)$ ($P(\nu_e \rightarrow \nu_{\mu(\tau)})$, etc. by the new mechanism can be studied analytically [1] owing to the results of a detailed numerical analysis [13] (see also [14]) which showed that for the calculation of the probabilities of interest the two-layer model of the Earth density distribution provides an excellent approximation to the more complicated density distributions predicted by the existing models of the Earth [15, 16]. In the two-layer model, the neutrinos crossing the Earth mantle and the core traverse effectively two layers with constant but different densities, $\bar{\rho}_{\text{man}}$ and $\bar{\rho}_{\text{c}}$, and chemical composition or electron fraction numbers, $Y_{\text{e}}^{\text{man}}$ and $Y_{\text{e}}^{\text{c}}$. The neutrino transitions of interest result from two-neutrino oscillations taking place i) first in the mantle over a distance $X'$ with a mixing angle $\theta_m'$ and oscillation length $L_{\text{man}}$, ii) then in the core over a distance $X''$ with mixing angle $\theta_m'' \neq \theta_m'$ and oscillation length $L_c \neq L_{\text{man}}$, and iii) again in the mantle over a distance $X'$ with $\theta_m''$ and $L_{\text{man}}$. Utilizing the above prescription, analytic expressions for the probabilities of neutrino transitions in the Earth $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e) = P(\nu_e \rightarrow \nu_{\mu(\tau)})$, etc. were derived and were used to study their extrema [1]. Assuming that the neutrino oscillation parameters $\Delta m^2 > 0$ and $\cos 2\theta > 0$ are fixed and treating the neutrino paths $X'$ and $X''$ as independent

\footnote{The densities $\bar{\rho}_{\text{man,c}}$ should be considered as mean densities along the neutrino trajectories.}
variables, it was found in [1] that in addition to the maxima corresponding to the MSW effect in the Earth mantle or in the Earth core, there exists a new maximum in $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$. The latter takes place when the relative phases the states of the two energy-eigenstate neutrinos acquire after the neutrinos have crossed the mantle, $\Delta E'X' = 2\pi X'/L_{man}$, and the core, $\Delta E''X'' = 2\pi X''/L_c$, obey the constraints,

$$\Delta E'X' = \pi(2k + 1), \quad \Delta E''X'' = \pi(2k' + 1), \quad k, k' = 0, 1, 2, \ldots,$$  \hspace{1cm} (1)

and if the inequality $[1, 2, 3]$

$$\cos(2\theta'_m - 4\theta'_m + \theta) < 0,$$  \hspace{1cm} (2)

is fulfilled. Condition (2) is valid for the probability $P_{e2}$. When equalities (1) hold, (2) ensures that $P_{e2}$ has a maximum. At the maximum $P_{e2}$ takes the form $[1]$

$$P_{e2}^{max} = \sin^2(2\theta'_m - 4\theta'_m + \theta).$$  \hspace{1cm} (3)

The analogs of eqs. (1) - (3) for the probability $P(\nu_\mu(\nu_e) \rightarrow \nu_e(\nu_{\mu(\tau)}))$ can be obtained by formally setting $\theta = 0$ while keeping $\theta''_m \neq 0$ and $\theta''_m \neq 0$ in (1) - (3) $^5$. The term “neutrino oscillation length resonance” (NOLR) was used in [1] to denote the resonance-type enhancement of $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e) (P(\nu_e \rightarrow \nu_\mu(\tau)))$, etc. associated with the conditions (1) - (2).

The new mechanism of strong enhancement of the probabilities $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e)$, etc. for the Earth core crossing neutrinos, was identified in [1] with the neutrino oscillation length resonance. This interpretation was based on the observation that for the several test values of $\sin^2 2\theta \lesssim 0.02$ and Nadir angle $h$ considered in [1], conditions (1) were approximately satisfied at the corresponding non-MSW absolute maxima of $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$, and the numerically calculated values of $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$ at these maxima were reproduced by eq. (3) with a rather good accuracy. Doubts about the correctness of such an interpretation remained since i) for the parameters of the Earth [15] (core radius, mantle width, etc.), used in the calculations, the set of conditions (1) - (2), as was shown in [1], cannot be exactly satisfied at small mixing angles, and ii) although the phase $\Delta E'X'$ was close to $\pi$ at the relevant absolute maxima of $P_{e2}$ and $P(\nu_\mu \rightarrow \nu_e)$ in the test cases studied, the values of the phase $\Delta E''X''$ differed quite substantially from those required by eq. (1). Moreover, the same (or similar) enhancement mechanism was found to be operative in the $\nu_2 \rightarrow \nu_e$ transitions in the case of $\nu_e - \nu_s$ mixing and in the $\nu_e \rightarrow \nu_s$ transitions, at small mixing angles [1]. However, as was noticed in [1], the conditions (1) - (2) are not even approximately fulfilled for the $\nu_2 \rightarrow \nu_e \cong \nu_s \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_s$ transitions at small mixing angles; similar conclusions were reached for the $\tilde{\nu}_\mu \rightarrow \tilde{\nu}_s$ transitions [2].

The indicated results stimulated the systematic study of the extrema of the probabilities $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e)$, $P(\nu_e \rightarrow \nu_s)$, etc. of the neutrino transitions in the Earth and more generally, in a medium of non-periodic constant density layers, which is

\footnote{The conditions for the corresponding maximum in $P(\nu_e \rightarrow \nu_s)$ and $P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$ coincide in form with those for $P(\nu_\mu(e) \rightarrow \nu_e(\mu,\tau))$.}
presented here. We consider transitions of neutrinos in a medium consisting of i) two layers of different constant densities, and ii) three layers of constant density with the first and the third layers having identical densities and widths which differ from those of the second layer. Neutrinos pass through such systems of layers on the way to the neutrino detectors, e.g., when they i) are produced in the central region of the Earth and traverse both the Earth core and mantle, ii) travel first in vacuum and then in the Earth mantle, iii) cross the Earth mantle, the core and the mantle again. We derive and analyze the complete set of extrema of the two-neutrino transition probabilities in both cases of a medium with two and three constant density layers, assuming that \[ \Delta m^2 / E \text{ and } \sin^2 2 \theta \] are fixed and treating the widths of the layers, \( X' \) and \( X'' \), as independent variables. For both media considered we find that in addition to the local maxima associated with the MSW effect and the NOLR, there exist absolute maxima caused by a new effect of enhancement and corresponding to a total neutrino conversion. We use a specific but universal form of the two-neutrino transition probabilities of interest to find the complete set of such absolute maxima. The latter are absolute maxima in any possible independent variables: the neutrino energy, the widths of one of the layers, etc. The conditions for existence of the new maxima are derived and it is shown that they are fulfilled, in particular, for the transitions of the Earth-core-crossing solar and atmospheric neutrinos. We show that the strong resonance-like enhancement of these transitions is due to the new effect of total neutrino conversion. At small mixing angles and in the case of the transitions \( \nu_2 \to \nu_e \) and \( \nu_\mu \to \nu_e \), for instance, the values of the parameters at which the total neutrino conversion takes place for neutrinos traversing the Earth core are rather close to the values of the parameters for which the NOLR could occur. In both transitions, however, only the maximal neutrino conversion mechanism is operative.

We demonstrate that the NOLR and the newly found resonance-like enhancement effect are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. Thus, the local and absolute maxima they produce in the neutrino transition probabilities are of interference nature and have nothing to do with the parametric resonance, possible in a medium with periodic change of density [17]. Therefore we confirm the general physical interpretation of the strong resonance-like enhancement of the transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos, given in [1].

Most of the results of our analysis are illustrated by the realistic examples of the transitions of neutrinos passing through the Earth. These examples and the results regarding the transitions of Earth-core-crossing solar and atmospheric neutrinos are derived using the Stacey model from 1977 [15] as a reference Earth model. The density distribution in the Stacey model is spherically symmetric and in addition to the two major density structures - the core and the mantle, there are a certain number of substructures (shells or layers). The core has a radius \( R_c = 3485.7 \) km, the Earth mantle depth is approximately \( R_{man} = 2885.3 \) km, and the Earth radius in the Stacey model is \( R_\oplus = 6371 \) km. Therefore, when the nadir angle \( h \) is less than 33.17°, neutrinos arriving at the detector pass through three layers: the mantle, the core and the mantle again. The mean values of the matter densities of the core and of the mantle read, respectively: \( \bar{\rho}_c \approx 11.5 \) g/cm³ and \( \bar{\rho}_{man} \approx 4.5 \) g/cm³. The
density distribution in the 1977 Stacey model practically coincides with that in the more recent PREM model [16]. For $Y_e$ we have used the standard values [15, 16, 18] (see also [5]) $Y_{eman}^e = 0.49$ and $Y_e^c = 0.467$.

A brief description of the results of the present study was given in ref. [19].

2 Preliminary Remarks

We will consider the simple case of mixing of two weak-eigenstate neutrinos $\nu_\alpha$ and $\nu_\beta$, $\alpha \neq \beta = e, \mu, \tau, s$, $\nu_s$ being a sterile neutrino, in vacuum. The mixing matrix relating the states of the neutrinos $\nu_{\alpha,\beta}$ and of the neutrinos $\nu_{1,2}$ having definite masses $m_{1,2}$ in vacuum is chosen in the form:

$$
\begin{pmatrix}
\nu_\alpha \\
\nu_\beta
\end{pmatrix} =
\begin{pmatrix}
\cos \theta & \sin \theta \\
-\sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
\nu_1 \\
\nu_2
\end{pmatrix},
$$

(4)

where $\theta$ is the vacuum mixing angle. In the case of relativistic and stable neutrinos $\nu_{1,2}$, the evolution equation for neutrino propagation in constant-density medium reads

$$
i \frac{d}{dt} \begin{pmatrix}
\nu_\alpha \\
\nu_\beta
\end{pmatrix} = \frac{\Delta E}{2} \begin{pmatrix}
-\cos(2\theta_m) & \sin(2\theta_m) \\
\sin(2\theta_m) & \cos(2\theta_m)
\end{pmatrix}
\begin{pmatrix}
\nu_\alpha \\
\nu_\beta
\end{pmatrix} = \mathcal{M}
\begin{pmatrix}
\nu_\alpha \\
\nu_\beta
\end{pmatrix}.
$$

(5)

Here $\theta_m$ is the mixing angle in matter,

$$
\cos(2\theta_m) = \frac{1}{\Delta E} ((\Delta m^2 / 2E) \cos(2\theta) - V_{\alpha\beta}),
$$

(6)

$E$ being the neutrino energy, $\Delta E$ is the difference between the energies of the two energy-eigenstate neutrinos in matter,

$$
\Delta E = \frac{\Delta m^2}{2E} \sqrt{\left(\cos(2\theta) - \frac{2EV_{\alpha\beta}}{\Delta m^2}\right)^2 + \sin^2(2\theta)},
$$

(7)

$\Delta m^2 = m_2^2 - m_1^2$, and $V_{\alpha\beta}$ is the difference between the effective potentials of $\nu_\alpha$ and $\nu_\beta$ in the medium. We shall always assume in what follows that

$$
\cos(2\theta) > 0, \quad \Delta m^2 > 0.
$$

(8)

In the case of neutrino propagation in an electrically neutral unpolarized cold medium, like the Earth, one has:

$$
V_{e\mu} = \sqrt{2} G_F N_e, \quad V_{es} = \sqrt{2} G_F (N_e - \frac{1}{2} N_n), \quad V_{\mu s} = -\sqrt{2} G_F N_n / 2,
$$

(9)

where $N_e$ and $N_n$ are the electron and the neutron number densities in the medium, respectively. The antineutrino states $\bar{\nu}_\alpha$ and $\bar{\nu}_\beta$ satisfy the same equations (2) with $V_{\alpha\beta}$ replaced by $V_{\bar{\alpha}\bar{\beta}} = -V_{\alpha\beta}$ in eqs. (3) and (4). For the Earth $N_e \approx N_n$ and in addition to $V_{e\mu} > 0$ we have $V_{es} > 0$ and $V_{\mu s} > 0$. If (5) holds, the neutrino
mixing can be enhanced by the Earth matter only in the cases of \( \nu_\mu \rightarrow \nu_e \) \((\nu_2 \rightarrow \nu_e)\), \( \nu_e \rightarrow \nu_s \) and \( \bar{\nu}_\mu \rightarrow \bar{\nu}_s \) transitions, while for, e.g., \( \cos(2\theta) > 0 \) and \( \Delta m^2 < 0 \) this will be true for the \( \bar{\nu}_\mu \rightarrow \bar{\nu}_e \) \((\nu_2 \rightarrow \nu_e), \nu_e \rightarrow \nu_s \) and \( \nu_\mu \rightarrow \nu_s \) transitions. Although our general results will be formulated for the generic weak-eigenstate neutrino transitions, \( \nu_\alpha \rightarrow \nu_\beta \), for neutrinos crossing the Earth they will be valid either for the former or for the latter set of transitions, depending on whether eq. (5) holds or \( \Delta m^2 \cos(2\theta) < 0 \). The case of (solar) \( \nu_2 \rightarrow \nu_e \) transitions in a three-layer medium (the Earth) will be considered separately.

The evolution of the neutrino system is given by the unitary matrix

\[
U = \exp(-iMt) = \cos \phi - i(\sigma n) \sin \phi,
\]

where

\[
\phi = \frac{1}{2} \Delta E t
\]

and

\[
n = (\sin(2\theta_m), 0, -\cos(2\theta_m))
\]

is a real unit vector. The probabilities of the \( \nu_\alpha \rightarrow \nu_\beta \) \((\nu_\beta \rightarrow \nu_\alpha)\) transition, \( P_{\alpha\beta}(\beta\alpha) = |A_{\alpha\beta}|^2 \), and of the \( \nu_\alpha \) \((\nu_\beta)\) survival, \( P_{\alpha\alpha}(\beta\beta) = |A_{\alpha\alpha}|^2 \), are determined by the elements of the evolution matrix \( U \), which coincide with the four different probability amplitudes \( A_{\alpha\beta} = U_{\beta\alpha} \), etc.:

\[
P_{\alpha\beta} = |U_{\beta\alpha}|^2 = (n_1 \sin \phi)^2 + (n_2 \sin \phi)^2 = P_{\beta\alpha}
\]

and

\[
P_{\alpha\alpha} = |U_{\alpha\alpha}|^2 = \cos^2 \phi + (n_3 \sin \phi)^2 = 1 - P_{\alpha\beta} = P_{\beta\beta}.
\]

We are interested in the extrema of the transition probability \( P_{\alpha\beta} \) when neutrinos propagate through a medium with (nonperiodic sequence of) layers of different constant density and chemical composition. In this case the evolution matrix, as is well-known, represents a product of the evolution matrices for the different layers and can always be written in the same form as in eq. (10). It follows from (13) and (14) that the conditions for an absolute maximum of the transition probability, \( P_{\alpha\beta} = 1 \), read

\[
\max P_{\alpha\beta} = 1 : \begin{cases} 
\cos \phi = 0 \\
n_3 \sin \phi = 0,
\end{cases}
\]

while the conditions for an absolute minimum of \( P_{\alpha\beta} \) have the form:

\[
\min P_{\alpha\beta} = 0 : \begin{cases} 
n_1 \sin \phi = 0 \\
n_2 \sin \phi = 0,
\end{cases}
\]

When neutrinos propagate in a constant-density medium the transition probability

\[
P_{\alpha\beta} = \frac{1}{2}(1 - \cos 2\phi) \sin^2(2\theta_m) = (\sin \phi \sin(2\theta_m))^2
\]
can be non-negligible even in the case of small vacuum mixing angle $\theta$ if the neutrino mixing in matter is maximal, $\theta_m = \pi/4$. This is the well-known MSW effect which takes place when the resonance condition
\[
\cos(2\theta_m) = 0
\]
(18)
is fulfilled. In order to get total neutrino conversion, $P_{\alpha\beta} = 1$, the additional requirement
\[
\cos \phi = 0
\]
(19)
has to be satisfied. For a fixed time of propagation $t$, or a distance $X \equiv t$ traveled by the neutrino, there always exists a solution of the system of equations (15) The absolute minima $P_{\alpha\beta} = 0$ correspond, as it follows from (16), to the curves
\[
\sin \phi = 0.
\]
(20)

In the next sections we discuss the case of neutrino propagation through a number of constant-density layers. A new phenomenon of maximal constructive interference between the probability amplitudes of the transitions in the different layers takes place in this case, leading to a substantial enhancement of the transition probability. Even when the oscillation parameters have values very different from those required by the resonance conditions (18) for the individual layers, the neutrino transition probability can reach its absolute maximum, $P_{\alpha\beta} = 1$, due to this effect.

3 Medium with Two Constant - Density Layers

Suppose neutrinos $\nu_\alpha$ or $\nu_\beta$ have crossed a medium consisting of two layers having constant but different density and chemical composition. These could be the Earth core and mantle (for neutrinos born in the Earth central region), the vacuum and the Earth mantle, etc. Let us denote the effective potential differences and the lengths of the paths of the neutrinos (antineutrinos) in the two layers by $V'_{\alpha\beta} (V'_{\alpha\bar{\beta}}), V''_{\alpha\beta} (V''_{\alpha\bar{\beta}})$ and $X', X''$, respectively. We shall assume without loss of generality that
\[
0 \leq |V'_{\alpha\beta(\bar{\alpha}\bar{\beta})}| < |V''_{\alpha\beta(\bar{\alpha}\bar{\beta})}|.
\]
(21)
The transitions of $\nu_\alpha$ and $\nu_\beta$ crossing the two layers will be determined by the two sets of two parameters - the mixing angle in matter (3) or the unit vector (9), and the phase difference (8) with $t = X'$ or $X''$, which characterize the transitions in each of the layers: $\theta'_m$ or $n'$ and $\phi'$, and $\theta''_m$ or $n''$ and $\phi''$. Since the evolution matrix of the system is given by $U = U''U'$, $U'$ and $U''$ being the evolution matrices in the first and in the second layers, the parameters of $U$, $\phi$ and $n$ (eq. (10)), can be expressed in terms of $\phi'$, $n'$ and $\phi''$, $n''$ as follows:
\[
\begin{align*}
\cos \phi &= \cos \phi' \cos \phi'' - (n' \cdot n'') \sin \phi' \sin \phi'', \\
n \sin \phi &= n' \sin \phi' \cos \phi'' + n'' \cos \phi' \sin \phi'' - [n' \times n''] \sin \phi' \sin \phi''.
\end{align*}
\]
(22)
Using, e.g., eqs. (13) and (22) it is not difficult to derive the transition probability \( P_{\alpha \beta} = P_{\beta \alpha} \) in this case:

\[
P_{\alpha \beta} = \sin(2\theta'') \cos \phi' \sin \phi'' + \sin(2\theta''') \cos \phi' \sin \phi''' = \frac{1}{2} [1 - \cos 2\phi'] \sin^2 \theta' + \frac{1}{2} [1 - \cos 2\phi''] \sin^2 \theta'' + \frac{1}{4} [1 - \cos 2\phi'] [1 - \cos 2\phi''] \times
\]
\[
\times \left[ \sin^2(2\theta'' - 2\theta') - \sin^2 \theta' - \sin^2 \theta'' \right] + \frac{1}{2} \sin 2\phi' \sin 2\phi'' \sin 2\theta' \sin 2\theta''. \tag{23}
\]

Due to T-invariance and unitarity, the probability (23) is symmetric with respect to the interchange of the parameters of the two layers, i.e., does not depend on the order in which the neutrinos traverse them.

For fixed \( \sin^2(2\theta) \) and \( \Delta m^2/E \) and given density and chemical composition in the two layers, the unit vectors \( \mathbf{n}' \) and \( \mathbf{n}'' \) are also fixed for each layer. The phases \( \phi' \) and \( \phi'' \), which depend on the widths of the layers \( X' \) and \( X'' \), can be independent variables of the system if the two widths \( X' \) and \( X'' \) can be varied independently. This is an interesting case with a clear physical meaning and we are going to investigate it in the present and the next Sections. In this way the NOLR resonance in the \( \nu_2 \to \nu_e, \nu_\mu (\nu_e) \to \nu_e (\nu_{\mu;\tau}) \) and \( \nu_e \to \nu_s \) transitions was discovered [1]. However, let us just note here that it does not correspond to the real case of the Earth, which will be considered in Section 5.

Varying the phases \( \phi' \) and \( \phi'' \) we can investigate the number and the structure of the extrema of the transition probability \( P_{\alpha \beta} \). These are determined by the system of two equations

\[
\frac{dP_{\alpha \beta}}{d\phi'} = \sin 2\theta'_m F(2\theta'_m - 2\theta''', 2\theta''', 2\phi', 2\phi'') = 0, \tag{24}
\]
\[
\frac{dP_{\alpha \beta}}{d\phi''} = \sin 2\theta''_m F(2\theta'''', 2\theta'''', 2\phi'', 2\phi') = 0, \tag{25}
\]

where

\[
F(Y, Z; \varphi, \psi) = \sin Y \cos Z \sin \varphi + \sin Z \left[ \sin \varphi \cos \psi \cos Y + \cos \varphi \sin \psi \right], \tag{26}
\]

and by two known supplementary conditions on the values of the second derivatives of \( P_{\alpha \beta} \) at the points where eqs. (24) and (25) hold.

We are interested first of all in the \textit{absolute} maxima of the neutrino conversion,

\[
\text{case A:} \quad \max P_{\alpha \beta} = 1. \tag{27}
\]

They are determined by the equations (15) and (22)

\[
\max P_{\alpha \beta} = 1 : \quad \begin{cases} 
\cos \phi' \cos \phi'' - \cos(2\theta'' - 2\theta') \sin \phi' \sin \phi'' = 0 \\
\cos(2\theta') \sin \phi' \cos \phi'' + \cos(2\theta'') \cos \phi' \sin \phi'' = 0.
\end{cases} \tag{28}
\]
It is not difficult to check that conditions (24) and (25) are satisfied if relations (28) hold. The solutions of (28) can be readily found:

\[
\begin{align*}
\tan \phi' &= \pm \sqrt{\frac{-\cos(2\theta'')}{\cos(2\theta''_m) \cos(2\theta' - 2\theta'')}}, \\
\tan \phi'' &= \pm \sqrt{\frac{-\cos(2\theta'')}{\cos(2\theta''_m) \cos(2\theta' - 2\theta'')}},
\end{align*}
\]

(29)

where the signs are correlated. Obviously, the solutions (29) do not exist for any pairs of values of the parameters \(\sin^2(2\theta)\) and \(\Delta m^2/E\). Under the assumptions (8), (21) and for \(V_{\alpha\beta} \geq 0\) (e.g., for the \(\nu_\mu (\nu_e) \rightarrow \nu_e (\nu_{\mu\tau}), \nu_e \rightarrow \nu_s\) and \(\bar{\nu}_\mu \rightarrow \bar{\nu}_s\) transitions of the Earth-crossing neutrinos), they can take place only in the region limited by the three conditions \(^6\) (Fig. 1)

\[
\begin{align*}
\cos 2\theta'_m &\geq 0 \\
\cos 2\theta''_m &\leq 0 \\
\cos(2\theta''_m - 2\theta'') &\geq 0.
\end{align*}
\]

(30)

The first boundary, \(\cos 2\theta'_m = 0\), corresponds to a total neutrino conversion in the first layer. From (29) we get the correct result for the relevant conditions on the phases (see (19) and (20)),

\[
\begin{align*}
\cos \phi' &= 0, \text{ or } 2\phi' = \pi(2k' + 1), \ k' = 0, 1, ..., \\
\sin \phi'' &= 0, \text{ or } 2\phi'' = 2\pi k'', \ k'' = 0, 1, ...,
\end{align*}
\]

(31)

guaranteeing that i) neutrino conversion takes place in the first layer, and ii) no neutrino conversion occurs in the second layer, so that when \(\cos 2\theta'_m = 0\) we have maximal transition probability, \(P_{\alpha\beta} = 1\). It is easy to show by direct calculations using eqs. (23) - (25) that the phase conditions (31) correspond to the local maxima

\[
\text{case B: } \max P_{\alpha\beta} = \sin^2 2\theta'_m,
\]

(32)

provided the oscillation parameters \(\sin^2(2\theta)\) and \(\Delta m^2/E\) belong to the region B,

\[
\text{region B: } \cos 2\theta'_m \leq 0.
\]

(33)

It proves useful to interpret this result in terms of probability amplitudes. If neutrinos \(\nu_\alpha\) have crossed the two layers, the amplitude of the \(\nu_\alpha \rightarrow \nu_\beta\) transition represents a sum of two terms:

\[
A_{\alpha\beta} = U_{\beta\alpha}' = U''_{\beta\alpha} U'_{\alpha\alpha} + U''_{\beta\beta} U'_{\beta\alpha},
\]

(34)

where

\[
\begin{align*}
U'_{\alpha\alpha} &= \cos \phi' + i \cos(2\theta'_m) \sin \phi', \\
U''_{\beta\alpha} &= -i \sin(2\theta''_m) \sin \phi'', \\
U''_{\beta\beta} &= \cos \phi'' - i \cos(2\theta''_m) \sin \phi''.
\end{align*}
\]

(35)

\(^6\)These conditions can also be derived from the two supplementary inequalities ensuring that the solutions (29) of (24) and (25) indeed correspond to maxima of \(P_{\alpha\beta}\).
In the general case the transition probability

\[ P_{\alpha\beta} = |U_{\beta\alpha}''|^2 |U_{\alpha\alpha}'|^2 + |U_{\beta\alpha}'|^2 |U_{\beta\beta}'|^2 + 2 \text{Re} \left( \left( U_{\beta\alpha}'' U_{\alpha\alpha}' \right)^* U_{\beta\beta}' U_{\beta\alpha}' \right) \]  

(36)
is a sum of two products of the probabilities of neutrino oscillations in the different layers and of the interference term. The latter can play a crucial role in the resonance enhancement of the neutrino transitions.

The phase requirements (31) reduce the expression in eq. (36) to only one term, \( P_{\alpha\beta} = |U_{\beta\alpha}'|^2 \), with no contribution from the interference term because \( U_{\beta\alpha}'' = 0 \).

Thus, the maxima in region \( B \) (33) can be ascribed to the MSW effect in the first layer.

The second boundary \( \cos(2\theta''_m) = 0 \) in (30) corresponds to a total neutrino conversion in the second layer.

**Solution C:**

\[
\begin{align*}
\sin \phi' = 0, \quad & \text{or} \quad 2\phi' = 2\pi k', \quad k' = 0, 1, ..., \\
\cos \phi'' = 0, \quad & \text{or} \quad 2\phi'' = \pi(2k'' + 1), \quad k'' = 0, 1, ..., \end{align*}
\]

(37)

This case is completely analogous to the previously considered case \( B \), with the two layers interchanged. Using eqs. (23) - (25) (or (29)) we get solution \( C \) which realizes the local maxima

**Case C:** \( \max P_{\alpha\beta} = \sin^2 2\theta''_m \),

(38)
in the region

**Region C:** \( \cos 2\theta''_m \geq 0 \).

(39)

This can be associated with the MSW effect taking place in the second layer.

We get a very different mechanism of enhancement of the neutrino transition probability \( P_{\alpha\beta} \) when relations (29) hold in the intermediate region

\[
\begin{align*}
\cos 2\theta'_m > 0, \\
\cos 2\theta'_m < 0,
\end{align*}
\]

(40)

located between the regions \( B \) (33) and \( C \) (39). The maxima of \( P_{\alpha\beta} \) in this region are caused by a maximal contribution of the interference term in eq. (36) for \( P_{\alpha\beta} \). Indeed, using eq. (36) one can write the probability \( P_{\alpha\beta} \) in the form:

\[ P_{\alpha\beta} = |z_1 + z_2|^2, \]

(41)

where \( z_1 = (\text{Re}(U''_{\beta\alpha} U'_{\alpha\alpha}), \text{Im}(U''_{\beta\alpha} U'_{\alpha\alpha})) \) and \( z_2 = (\text{Re}(U''_{\beta\beta} U'_{\beta\alpha}), \text{Im}(U''_{\beta\beta} U'_{\beta\alpha})) \) are two vectors in the complex plane. For fixed vectors \( z_1 \) and \( z_2 \) the interference is maximal and constructive, as it follows from (41), when \( z_1 \) and \( z_2 \) are collinear and point in the same direction, i.e., when

\[
\frac{\text{Im}(U''_{\beta\alpha} U'_{\alpha\alpha})}{\text{Im}(U''_{\beta\beta} U'_{\beta\alpha})} = \frac{\text{Re}(U''_{\beta\alpha} U'_{\alpha\alpha})}{\text{Re}(U''_{\beta\beta} U'_{\beta\alpha})} > 0. \]

(42)

The second equation in (15) (or (28)) ensure that the two vectors are collinear, while the constraints (40) guarantee that they point in the same direction. Actually, the
equation in (42) coincides with the second equation in (15) for the resonance condition (18)
\[
\cos(2\theta_m) = \frac{1}{\sin\phi} [\cos(2\theta') \sin\phi' \cos\phi'' + \cos(2\theta'') \cos\phi' \sin\phi''] = 0,
\]
with the additional constraints (40), corresponding to the intermediate region where the new type of enhancement takes place. The transition probability of interest (23) can be represented in the two layer case under discussion also in the form (17),
\[
P_{\alpha\beta} = \sin^2(2\theta_m) \sin^2 \phi,
\]
where the first and the second multipliers are defined by eq. (43) and the first equation in (22). To get a total neutrino conversion, \( P_{\alpha\beta} = 1 \), not only eq. (43) has to hold, but also condition (19) must be satisfied. This is possible in region \( A \) (30), but not in the whole intermediate region (40).

Indeed, the boundary \( \cos(2\theta''_m - 2\theta'_m) = 0 \) in (30) corresponds to a maximum of \( P_{\alpha\beta} \) associated with the phase constraints

\[
\text{solution } D : \left\{ \begin{array}{l}
\cos \phi' = 0, \text{ or } 2\phi' = \pi(2k' + 1), \quad k' = 0, 1, ..., \\
\cos \phi'' = 0, \text{ or } 2\phi'' = \pi(2k'' + 1), \quad k'' = 0, 1, ...
\end{array} \right.
\]

The above conditions are necessary for a maximal neutrino conversion in each layer (see (19) and (35)). As can be shown exploiting eqs. (23) - (25), they lead to local maxima

\[
\text{case } D : \max P_{\alpha\beta} = \sin^2(2\theta''_m - 2\theta'_m),
\]
if the oscillation parameters belong to the finite region \( D \),

\[
\text{region } D : \cos(2\theta''_m - 2\theta'_m) \leq 0.
\]

Our results show a very interesting feature of the neutrino transitions in a medium consisting of two constant-density layers. One can have a total neutrino conversion, \( P_{\alpha\beta} = 1 \), even if the MSW resonance does not take place in any of the two layers. As we have seen, this is possible in region \( A \), excluding the two boundaries \( \cos(2\theta''_m) = 0 \) and \( \cos(2\theta'_m) = 0 \), corresponding to the MSW effect. Therefore, in order to have a large transition probability, \( P_{\alpha\beta} \approx 1 \), a periodic density profile is not required even when the MSW resonance is not realized in any of the two layers. Thus, the strong enhancement of \( P_{\alpha\beta} \), corresponding to the solutions \( A \) and \( D \), eqs. (29) and (45), located in the region (40), has nothing to do with the parametric resonances in \( P_{\alpha\beta} \), possible in a medium with periodically varying density and discussed in refs. [17]. These conclusions remain valid, as we shall see, for transitions of neutrinos traversing three layers of constant density, e.g., for the transitions of the solar and atmospheric neutrinos crossing the Earth core on the way to the detectors. It should be emphasized that solution \( A \), eq. (29), providing the absolute maxima of \( P_{\alpha\beta} \), can be and was derived directly from eqs. (12), (19) and (25) without utilizing the standard extrema equations (21) - (22), i.e., its derivation does not depend on the chosen set of variables of \( P_{\alpha\beta} \), which are varied to obtain the extrema conditions.
Our results apply to the case when the first layer is vacuum \( V' = 0 \), because there is no principal difference between the neutrino oscillations in constant-density medium and in vacuum (Fig. 2).

Let us add finally that the absolute minima of \( P_{\alpha\beta} \) in the case under discussion are located on the intersections of the curves

\[
\min P_{\alpha\beta} = 0 : \left\{ \begin{array}{l}
\sin \phi' = 0, \text{ or } 2\phi' = 2\pi k', \ k' = 0, 1, \ldots, \\
\sin \phi'' = 0, \text{ or } 2\phi'' = 2\pi k'', \ k'' = 0, 1, \ldots .
\end{array} \right.
\]

(48)

4 The Earth Type Density Profile

Similar analysis can be performed for the transitions of neutrinos traversing three layers of constant density and chemical composition when the first and the third layers have the same density, chemical composition and width which differ, however, from those of the second layer \(^7\). This corresponds to the physically important case of solar and atmospheric neutrinos crossing the Earth core on the way to detectors located on the Earth surface, the first and third layers being the Earth mantle, and the Earth core playing the role of the second layer.

We will use the same notations for the parameters of the first (third) and the second layers as in Section 2, and will assume that eq. (18) holds. The evolution matrix in the case of interest is given by:

\[
U = U'U''U'.
\]

Accordingly, the parameters of \( U \), eq. (10), can be expressed in terms of the parameters of the first (third) and the second layers as follows:

\[
\left\{ \begin{array}{l}
\cos \phi = \cos(2\phi') \cos \phi'' - (n' \cdot n'') \sin(2\phi') \sin \phi'', \\
n \sin \phi = n' [\sin(2\phi') \cos \phi'' - (n' \cdot n'') (1 - \cos(2\phi')) \sin \phi''] + n'' \sin \phi''.
\end{array} \right.
\]

(49)

We have \( n_2 = 0 \) and the probability \( P_{\alpha\beta} \) can be cast in the form

\[
P_{\alpha\beta} = [F(2\theta''_m - 2\theta'_m; \phi''; 2\phi')]^2,
\]

(50)

where the function \( F(Y, Z; \varphi, \psi) \) is defined by eq. (26). The above expression is simpler to analyze than the analogous one in the two-layer case, eq. (23). The necessary conditions for a maximum of the probability (50) read

\[
\max P_{\alpha\beta} : \left\{ \begin{array}{l}
\sin(2\theta'_m) F(2\theta''_m - 2\theta'_m; \pi/2; 2\phi' + \pi/2) = 0, \\
F(2\theta''_m - 2\theta'_m; 2\phi' + \pi/2; 2\phi') = 0.
\end{array} \right.
\]

(51)

while the requirements for an absolute maximum have the form

\[
\max P_{\alpha\beta} = 1 : \left\{ \begin{array}{l}
F(2\theta''_m - 2\theta'_m; \pi/2; \phi''; 2\phi' + \pi/2) = 0, \\
F(2\theta''_m - 2\theta'_m; \phi''; 2\phi') = 0.
\end{array} \right.
\]

(52)

The absolute maxima of \( P_{\alpha\beta} \),

\[
\text{case A : } \max P_{\alpha\beta} = 1,
\]

(53)

\(^7\)Our results will be valid also if the widths of the first (third) and second layers coincide.
are provided by the solutions of eq. (52), which can be found explicitly:

\[
\text{solution A: } \begin{align*}
\tan \phi' &= \pm \sqrt{\frac{-\cos 2\theta''_m}{\cos(2\theta''_m - 4\theta'_m)}}, \\
\tan \phi'' &= \pm \sqrt{\frac{-\cos(2\theta''_m \cos(2\theta''_m - 4\theta'_m))}{\cos 2\theta'_m}}.
\end{align*}
\]  

(54)

where the signs are correlated.

The probability \( P_{\alpha\beta} (P_{\bar{\alpha}\beta}) \) exhibits a system of of maxima which is similar to that in the two layer case. Under the conditions (5), (18) and if \( V'_{\alpha\beta} > 0 \) (i.e., for the \( \nu_\mu (\nu_e) \to \nu_e (\nu_\mu \tau) \), \( \nu_e \to \nu_s \) and \( \bar{\nu}_\mu \to \bar{\nu}_s \) transitions in the Earth), solutions (54) are realized in the region \( A \) (Figs. 3 - 5),

\[
\text{region A: } \begin{cases} 
\cos(2\theta''_m) \leq 0, \\
\cos(2\theta''_m - 4\theta'_m) \geq 0.
\end{cases}
\]  

(55)

On the (border) line belonging to region \( A \),

\[
\text{region B: } \cos 2\theta'_m = 0,
\]  

(56)

we have

\[
\text{case B: } \max P_{\alpha\beta} = \sin^2 2\theta'_m = 1,
\]  

(57)

provided

\[
\text{solution B: } \begin{cases} 
\cos 2\phi' = 0, \text{ or } 2\phi' = \frac{\pi}{2} (2k' + 1), \ k' = 0, 1, ..., \\
\sin \phi'' = 0, \text{ or } 2\phi'' = 2\pi k'', \ k'' = 0, 1, ...
\end{cases}
\]  

(58)

Besides these absolute maxima, there exist two regions,

\[
\text{region C: } \cos(2\theta''_m) \geq 0,
\]  

(59)

and

\[
\text{region D: } \cos(2\theta''_m - 4\theta'_m) \leq 0,
\]  

(60)

with maxima

\[
\text{case C: } \max P_{\alpha\beta} = \sin^2 2\theta''_m,
\]  

(61)

and

\[
\text{case D: } \max P_{\alpha\beta} = \sin^2(2\theta''_m - 4\theta'_m),
\]  

(62)

which correspond to the solutions

\[
\text{solution C: } \begin{cases} 
\sin \phi' = 0, \text{ or } 2\phi' = 2\pi k', \ k' = 0, 1, ..., \\
\cos \phi'' = 0, \text{ or } 2\phi'' = \pi (2k'' + 1), \ k'' = 0, 1, ...
\end{cases}
\]  

(63)

and

\[
\text{solution D: } \begin{cases} 
\cos \phi' = 0, \text{ or } 2\phi' = \pi (2k' + 1), \ k' = 0, 1, ..., \\
\cos \phi'' = 0, \text{ or } 2\phi'' = \pi (2k'' + 1), \ k'' = 0, 1, ...
\end{cases}
\]  

(64)
respectively. In contrast to the two-layer case, the region $B$ is just a line, belonging actually to the region $A$, and on it only absolute maxima can be realized due to the MSW effect in the first (third) layer (the mantle in the case of the Earth). The case $C$ corresponds to the MSW effect in the second layer (Earth core), while the cases $A$ and $D$ correspond to the new resonance-like effect of constructive interference between transition amplitudes in the first and second layers. Due to this effect the transition probability can reach its maximal value, $P_{\alpha\beta} = 1$, if the oscillation parameters $\sin^2(2\theta)$ and $\Delta m^2/E$ belong to the region $A$, eq. (55).

The absolute minima of the probability (50) are determined by the equation:

$$\min P_{\alpha\beta} = 0 : F(2\theta'_{m} - \theta'_{m}, 2\theta'_{m} : \phi''_{m}, 2\phi') = 0. \quad (65)$$

We get the same system of maxima also in the $\nu_{2} \rightarrow \nu_{e}$ transition probability, $P(\nu_{2} \rightarrow \nu_{e}) \equiv P_{e2}$, $\nu_{2}$ being the heavier of the two mass eigenstate neutrinos in vacuum, which can be used to account for the the Earth matter effects in the transitions of solar neutrinos traversing the Earth. The probability $P_{e2}$ is given in the case of $\nu_{e} - \nu_{\mu(\tau)}$ mixing by the $U_{e\mu}$ element, $P_{e2} = |U_{e\mu}|^2$, of the evolution matrix

$$U^\odot \equiv \begin{pmatrix} U_{e1} & U_{e2} \\ U_{\mu1} & U_{\mu2} \end{pmatrix} = \begin{pmatrix} U_{ee} & U_{e\mu} \\ U_{\mu e} & U_{\mu\mu} \end{pmatrix} \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (66)$$

If $\nu_{e} - \nu_{e}$ mixing takes place, the index $\mu$ appearing in the above equation has to be replaced by $s$. The matrix $U^\odot$ can be written in the general form (10):

$$U^\odot = \cos \phi^\odot - i(\sigma n^\odot) \sin \phi^\odot. \quad (67)$$

Thus, the parameters of the evolution matrix (67), $\phi^\odot$ and $n^\odot$, are related to the parameters $\phi$ and $n$ defined in (49) as follows:

$$\left\{ \begin{array}{l}
\cos \phi^\odot = \cos \theta \cos \phi + n_2 \sin \theta \sin \phi, \\
n_1^\odot \sin \phi^\odot = (n_1 \cos \theta + n_3 \sin \theta) \sin \phi, \\
n_2^\odot \sin \phi^\odot = -\sin \theta \cos \phi + n_2 \cos \theta \sin \phi, \\
n_3^\odot \sin \phi^\odot = (-n_1 \sin \theta + n_3 \cos \theta) \sin \phi.
\end{array} \right. \quad (68)$$

After some transformations the transition probability $P_{e2}$ can be written in the form

$$P_{e2} = \left[ F(2\theta''_{m} - 2\theta'_{m}, 2\theta'_{m} : \phi", 2\phi') \right]^2 + \sin^2 \theta \left[ F(2\theta''_{m} - 2\theta'_{m}, \pi/2 ; \phi", 2\phi' + \pi/2) \right]^2$$

$$= \sin^2 \theta + F(2\theta''_{m} - 2\theta'_{m}, 2\theta'_{m} : \phi", 2\phi') F(2\theta''_{m} - 2\theta'_{m}, 2\theta'_{m} - 2\theta ; \phi", 2\phi'). \quad (69)$$

Using eqs. (69) and (26) we get the necessary conditions for extrema of the probability $P_{e2}$,

$$\max P_{\alpha\beta}, \min P_{\alpha\beta} : \left\{ \begin{array}{l}
\sin(2\theta'_{m}) F(2\theta''_{m} - 2\theta'_{m}, \pi/2 ; \phi", 2\phi' + \pi/2) = 0, \\
F(2\theta''_{m} - 2\theta'_{m}, 2\theta'_{m} - \theta ; \phi" + \pi/2, 2\phi') = 0.
\end{array} \right. \quad (70)$$
while from eqs. (15) and (16), utilizing eqs. (49), (68) and (69), one obtains the requirements for absolute extrema of the probability $P_{e2}$:

$$\begin{align*}
\max P_{e2} = 1 : \quad & \begin{cases}
F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0 \\
F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta + \pi/2; \phi'', 2\phi') = 0,
\end{cases} \\
\min P_{e2} = 0 : \quad & \begin{cases}
F(2\theta''_m - 2\theta'_m, \pi/2; \phi'', 2\phi' + \pi/2) = 0 \\
F(2\theta''_m - 2\theta'_m, 2\theta'_m - \theta; \phi'', 2\phi') = 0.
\end{cases}
\end{align*}$$

(71)

(72)

These are the same type of equations as (52) and we can easily find their solutions.

The absolute maxima of $P_{e2}$,

$$\text{case } A^\circ : \max P_{e2} = 1,$$

(73)

are reached for

$$\begin{align*}
solution A^\circ : \quad & \begin{cases}
\tan \phi' = \pm \sqrt{-\frac{\cos(2\theta''_m - \theta)}{\cos(2\theta''_m - 4\theta'_m + \theta)}}, \\
\tan \phi'' = \pm \sqrt{-\frac{\cos(2\theta''_m - \theta)}{\cos(2\theta''_m - 4\theta'_m + \theta)}},
\end{cases}
\end{align*}$$

(74)

where the signs are correlated, in the region $A^\circ$ (Fig. 4),

$$\text{region } A^\circ : \begin{cases}
\cos(2\theta''_m - \theta) \leq 0 \\
\cos(2\theta''_m - 4\theta'_m + \theta) \geq 0.
\end{cases}$$

(75)

There are the line

$$\text{region } B : \cos(2\theta'_m - \theta) = 0,$$

(76)

belonging to $A$, and two bordering regions,

$$\text{region } C^\circ : \cos(2\theta''_m - \theta) \geq 0,$$

(77)

and

$$\text{region } D^\circ : \cos(2\theta''_m - 4\theta'_m + \theta) \leq 0,$$

(78)

where the solutions $B$, eq. (58); $C$, eq. (63); and $D$, eq. (64), are realized. These solutions correspond to the following maxima of $P_{e2}$:

$$\text{case } B^\circ : \max P_{e2} = \sin^2(2\theta'_m - \theta) = 1,$$

(79)

$$\text{case } C^\circ : \max P_{e2} = \sin^2(2\theta''_m - \theta),$$

(80)

and

$$\text{case } D^\circ : \max P_{e2} = \sin^2(2\theta''_m - 4\theta'_m + \theta).$$

(81)

Solutions $D^\circ$ and $D$ (eqs. (64), (47), (78), etc.) correspond to the NOLR discussed in [1].

For the absolute minima of $P_{e2}$ we get the following solutions:

$$\begin{align*}
\min P_{e2} = 0 : \quad & \begin{cases}
\tan \phi' = \pm \sqrt{-\frac{\sin(2\theta''_m - \theta)}{\sin(2\theta''_m - 4\theta'_m + \theta)}}, \\
\tan \phi'' = \pm \sqrt{-\frac{\sin(2\theta''_m - \theta)}{\sin(2\theta''_m - 4\theta'_m + \theta)}}.
\end{cases}
\end{align*}$$

(82)

where again the signs are correlated.

---

8Solutions $B$ and $C$ (eqs. (58) and (63)) were considered briefly in [1] as well.
5 Transitions of Neutrinos Traversing the Earth Core

In the case of transitions in the Earth of (solar and atmospheric) neutrinos which pass through the Earth mantle, the core and the mantle again, the two lengths $X'$ and $X''$ are not independent due to the spherical symmetry of the Earth. We can choose as independent variables, for example, As it follows from eq. (47), the corresponding extrema conditions read:

\[
\frac{dF}{d\cos h} = \frac{2\Delta E'' R_{\oplus}}{X''} \left( \frac{dF}{d\phi''} R_{\oplus} \cos h \right) = 0, \quad (83)
\]

\[
\frac{dF}{dy} = \frac{dF}{d(2\phi'')} X' y - 2 V_{\alpha\beta} \cos(2\theta) \frac{4\Delta E^2}{4\Delta E'^2} + \frac{dF}{d\phi''} \frac{X'' y - 2 V_{\alpha\beta} \cos(2\theta)}{4\Delta E'^2} \\
+ \frac{V_{\alpha\beta}'}{\Delta E'^2} \sin(2\theta) \cos(\phi'') \left[ \cos(4\theta_m - 2\theta_m') \sin\phi' \sin\phi'' - \cos(2\theta_m') \cos\phi' \cos\phi'' \right] \\
- \frac{V_{\alpha\beta}''}{\Delta E'^2} \sin(2\theta) \sin\phi'' \left[ \cos(4\theta_m - 2\theta_m') \sin^2\phi' + \cos(2\theta_m') \cos^2\phi' \right] = 0, \quad (84)
\]

where $F \equiv F(2\theta_m'' - 2\theta_m'; 2\phi''; 2\phi')$, $y \equiv \Delta m^2/E$ and $R_{\oplus}$ is the radius of the Earth. It is clear, that, in general, the extrema of $P_{\alpha\beta}$ in the variables $\phi'$ and $\phi''$ do not correspond to extrema in the variables $\cos h$ and $\Delta m^2/E$. It is not difficult to check, however, that the solutions (54) for the absolute maxima, corresponding to a total neutrino conversion, $P_{\alpha\beta} = 1$, are solutions of the system of equations (83) - (84) as well. They give the absolute maxima of the neutrino transition probability $P_{\alpha\beta}$ in any variable. Equation (54) gives also the complete set of such solutions. The solutions (63) and (64) for the local maxima with $P_{\alpha\beta} < 1$ no longer correspond to extrema in the variables $\cos h$ and $\Delta m^2/E$. New solutions for the local maxima of $P_{\alpha\beta}$, which do not coincide with the solutions associated with phases equal to multiples of $\pi$, are possible. The same conclusions are valid for the solutions (74) and (63), (64) (giving (80), (81)) for the absolute and local maxima of the probability $P_{e2}$. Our numerical studies confirm these conclusions.

The solutions (54) ((74)), providing a total neutrino conversion, $P_{\alpha\beta} = 1 \ (P_{e2} = 1)$, exist for the Earth density profile and for all neutrino transitions of interest, $\nu_2 \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_\mu$, $\nu_e \rightarrow \nu_\mu$, and at small, intermediate and large mixing angles. They are realized for very different sets of values of the phases $2\phi'$ and $2\phi''$, which are not multiples of $\pi$. In Table 1 - 3 we give a rather complete list of these solutions for the transitions $\nu_\mu \rightarrow \nu_\mu$, $\nu_\mu \rightarrow \nu_e$, $\nu_e \rightarrow \nu_\mu$, etc. transitions in the Earth of the Earth-core-crossing solar and atmospheric neutrinos, discussed in [1, 2, 3, 4]. In [1] this enhancement was interpreted to be due to the neutrino oscillation length resonance - the solutions (64) in the case of the Earth-core-crossing neutrinos. At small mixing angles the values of the parameters at which the maximal
neutrino conversion takes place for the $\nu_2 \rightarrow \nu_e \cong \nu_\mu \rightarrow \nu_e$ and $\nu_e \rightarrow \nu_\mu$ transitions are rather close to the values of the parameters for which the NOLR could occur (Table 1), while for the $\nu_e \rightarrow \nu_\mu$ and $\nu_\mu \rightarrow \nu_\mu$ transitions these values are very different $[1, 2]$ (Table 2). In both cases of transitions, however, only the maximal neutrino conversion mechanism is operative for the Earth-core-crossing neutrinos.

In some cases the total neutrino conversion solutions $A$ and $A^0$ of interest occur for values of the parameters which are close to those for the solutions $B$, eq. (58), or $C$, eq. (63) (Tables 1 - 3). In all such cases the corresponding absolute maxima of the neutrino transition probabilities lie in the regions of the solutions $A$ and $A^0$, eqs. (55) and (75).

As Tables 1 - 3 and Figs. 6 - 8 indicate, for the Earth-core-crossing neutrinos, the new enhancement mechanism produces one relatively broad (in $\sin^2 \theta$, $\Delta m^2/E$ and $h$) resonance-like interference peak of total neutrino conversion in each of the probabilities $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e)$, $P(\nu_e \rightarrow \nu_\mu)$ and $P(\nu_\mu \rightarrow \nu_\mu)$, at $\sin^2 \theta \lesssim 0.10$. The absolute maxima of the different probabilities are located in the interval $0.03 \lesssim \sin^2 \theta \lesssim 0.10$. For all the neutrino transitions considered, the corresponding interference peak is sufficiently wide in all variables, which makes the transitions observable in the region of the enhancement $^9$: the relative width of the peak in the neutrino energy is $\Delta E/E_{\text{max}} \cong (0.3 - 0.5)$ and practically does not vary with $\sin^2 \theta$, while in $\sin^2 \theta$ it is $\sim (2.0 - 2.3)$; the absolute width in $h$ for the peaks located at $h \lesssim 25^\circ$ is $\sim (25 - 30)^\circ$. The points at the “ridges” leading to these absolute maxima peaks at values of $\sin^2 \theta \lesssim 0.03$ in the, e.g., $\sin^2 \theta - \Delta m^2/E$ plane, represent the dominating local maxima of the indicated probabilities (Figs. 6 - 8), seen, e.g., in the variable $\Delta m^2/E$ at fixed $h$ and $\sin^2 \theta$ (see, e.g., Figs. 1 - 2 in [1] and Figs. 5, 10 - 15 in [2]).

As Tables 1 - 3 and Figs. 6 - 8 show, the absolute maxima of total neutrino conversion are present in the probabilities $P_{e2}$, $P(\nu_\mu \rightarrow \nu_e)$, $P(\nu_e \rightarrow \nu_\mu)$ and $P(\nu_\mu \rightarrow \nu_\mu)$, at large values of $\sin^2 \theta \cong (0.50 - 1.0)$ as well. They are also present in certain transitions at $\sin^2 \theta \cong (0.15 - 0.50)$. These resonance-like interference maxima are sufficiently wide and can have observables effects both at $\sin^2 \theta \cong (0.50 - 1.0)$ and $\sin^2 \theta \cong (0.15 - 0.50)$.

5.1 Transitions of Atmospheric Neutrinos

$\nu_\mu \rightarrow \nu_e$, $\nu_\mu \rightarrow \nu_\mu(\tau)$. These can be sub-dominant transitions of the atmospheric $\nu_\mu$, driven by the values of $\Delta m^2$ suggested by the Super-Kamiokande atmospheric neutrino data $[9]$,

$$\Delta m^2 \cong (10^{-3} - 8 \times 10^{-3}) \text{eV}^2,$$

and by a relatively small mixing $[1, 2, 3, 4]$. Such transitions should exist if three-flavour-neutrino (or four-neutrino) mixing takes place in vacuum, which is very natural possibility in view of the present experimental evidences for oscillations of the flavour neutrinos. For the Earth-center-crossing neutrinos there are two solutions of

$^9$Let us note that the peak is not symmetric in any of the variables.
the type $A$, providing a total neutrino conversion, $P_{\mu e} = P_{e\mu} = 1$, at small mixing angles (Table 1): at $\sin^2 2\theta = 0.034; 0.15$ and $\Delta m^2/E = 7.2; 4.8 \times 10^{-7}$ eV$^2$/MeV. This implies that for $\Delta m^2 = 10^{-3}$ eV$^2$, the total neutrino conversion occurs at $E = 1.4; 2.1$ GeV, while if $\Delta m^2 = 5 \times 10^{-3}$ eV$^2$, it takes place at $E = 7.0; 10.5$ GeV. Thus, if the value of $\Delta m^2$ lies in the region (85), the new effect of total neutrino conversion occurs for values of the energy $E$ of the atmospheric $\nu_e$ and $\nu_\mu$ which contribute either to the sub-GeV or to the multi-GeV samples of $e$–like and $\mu$–like events in the Super-Kamiokande experiment. The implications are analogous to the one discussed in [1, 2, 3] in connection with the NOLR interpretation of the enhancement of $P_{\mu e}$ ($P_{e\mu}$). The new effect can produce an excess of $e$–like events in the region $-1 \leq \cos \theta_z \leq -0.8$, $\theta_z$ being the Zenith angle, either in the sub-GeV or in the multi-GeV sample of atmospheric neutrino events, and should be responsible for at least part of the strong Zenith angle dependence, exhibited by the $\mu$–like multi-GeV (sub-GeV) Super-Kamiokande data.

The total neutrino conversion can take place at several values of $\Delta m^2/E$ at large mixing angles (Table 1), $\sin^2 2\theta \gtrsim 0.8$. This can have implications, in particular, for the interpretation of the Super-Kamiokande data on the sub-GeV $e$–like events [20, 21]. The total neutrino conversion due to the new effect (solution $A$, eq. (54)) takes place both at small and large mixing angles. For the Earth-center-crossing neutrinos the absolute maxima, $P_{\mu s} = 1$, are realized at (Table 2) $\sin^2 2\theta = 0.07; 0.59; 0.78; 0.93; 0.999$ for $\Delta m^2/E = 4.1; 2.8; 9.7; 6.7; 3.4 \times 10^{-7}$ eV$^2$/MeV. For the statistically preferred value of $\Delta m^2 \approx 4 \times 10^{-3}$ eV$^2$ [9], this corresponds to $E = 9.8; 14.3; 4.12; 6.0; 11.8$ GeV, which is the range of the multi-GeV $\mu$–like events, studied by the Super-Kamiokande experiment.

5.2 Transitions of Solar Neutrinos

In the case of the transitions of the Earth-core-crossing solar neutrinos the relevant probability is $P_{e2}$. The transitions can be generated by $\nu_e - \nu_{\mu(\tau)}$ or by $\nu_e - \nu_s$ mixing. In the first case the interference maxima of $P_{e2}$, corresponding to solution $A^e$ (eq. (74)), $P_{e2} = 1$, take place both at small and large mixing angles (Table 3). The implications of the new enhancement effect have been discussed in [1] and in much greater detail in [5, 6]. Let us just mention here that due to this enhancement it would be possible to probe at least part of the $\Delta m^2 - \sin^2 2\theta$ region of the SMA MSW $\nu_e \rightarrow \nu_{\mu(\tau)}$ transition solution of the solar neutrino problem by performing selective D-N effect measurements.

6 Conclusions

We have investigated the extrema of probabilities of the two-neutrino transitions $\nu_\mu (\nu_e) \rightarrow \nu_e (\nu_{\mu(\tau)}$, $\nu_2 \rightarrow \nu_e$, $\nu_e \rightarrow \nu_s$, etc. in a medium of nonperiodic density distribution, consisting of i) two layers of different constant densities, and ii) three layers of constant density with the first and the third layers having identical den-
sities and widths which differ from those of the second layer. The first case would correspond, e.g., to neutrinos produced in the central region of the Earth and traversing both the Earth core and mantle on the way to the Earth surface. The second corresponds to e.g., the Earth-core-crossing solar and atmospheric neutrinos which pass through the Earth mantle, the core and the mantle again on the way to the detectors. For both media considered we have found that in addition to the local maxima corresponding to the MSW effect and the NOLR (neutrino oscillation length resonance), there exist absolute maxima caused by a new effect of enhancement and corresponding to a total neutrino conversion, $P_{\alpha\beta} = 1$. The conditions for existence and the complete set of the new absolute maxima were derived. The latter are absolute maxima corresponding to $P_{\alpha\beta} = 1$ in any independent variable characterizing the neutrino transitions: the neutrino energy, the width of one of the layers, etc. It was shown that the new effect of total neutrino conversion takes place, in particular, in the transitions $\nu_\mu (\nu_e) \rightarrow \nu_e (\nu_{\mu,e})$, $\nu_2 \rightarrow \nu_e$, $\nu_e \rightarrow \nu_s$ and $\bar{\nu}_\mu \rightarrow \bar{\nu}_s$ in the Earth of the Earth-core-crossing solar and atmospheric neutrinos. The strong resonance-like enhancement of these transitions discussed recently in [1, 2] (see also [10, 11]), is due to the new effect. This enhancement was previously associated in [1] with the NOLR. A “catalog” of the most relevant absolute maxima corresponding to a total neutrino conversion, was given for the transitions indicated above. We have shown that the NOLR and the newly found enhancement effect are caused by a maximal constructive interference between the amplitudes of the neutrino transitions in the different density layers. Thus, the maxima they produce in the neutrino transition probabilities are of interference nature and have nothing to do with the parametric resonance, possible in a medium with periodic change of density [17]. We have discussed also briefly the phenomenological implications of the new effect of total neutrino conversion in the transitions of the solar and atmospheric neutrinos traversing the Earth core. A more detailed investigation of these implications lies outside the scope of the present article and will be published elsewhere.

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References


Table 1: The values of $\Delta m^2/E \leq 20 \times 10^{-7}$ eV$^2$/MeV and of $\sin^2 2\theta$, at which the absolute maxima of $P_{e\mu} = P_{\mu e}$, corresponding to a total neutrino conversion, $P_{e\mu} = 1$, take place for neutrinos crossing the Earth along trajectories passing through the Earth core: $h = 0^\circ; 13^\circ; 23^\circ; 30^\circ$. The values of $2\phi', 2\phi''$ (in units of $\pi$), $\sin^2 2\theta_m'$, $\sin^2 2\theta''_m$ and $\sin^2(2\theta''_m - 4\theta'_m)$ at the absolute maxima are also given.

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Table 2: The same as in Table 1 for the probabilities $P_{\nu e}$ and $P_{\bar{\nu}_\mu}$ and for $\Delta m^2/E \leq 10 \times 10^{-7}$ eV$^2$/MeV.

\[
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
h^2 & \frac{\Delta m^2}{E} \left[ 10^{-7} \text{ eV}^2 \text{ MeV}^{-1} \right] & \sin^2 2\theta & 2\phi'/\pi & 2\phi''/\pi & \sin^2 2\theta' & \sin^2 2\theta'' & \sin^2 (2\theta'' - 4\theta'_m) \\
\hline
0. & 3.202 & .101 & .409 & .626 & .334 & .834 & .473 \\
0. & 2.715 & .784 & .565 & 1.843 & .981 & .538 & .797 \\
0. & 9.334 & .863 & 2.064 & 4.876 & .956 & 1.000 & .830 \\
0. & 6.997 & .920 & 1.564 & 3.867 & .997 & .952 & .892 \\
0. & 4.101 & .999 & 1.013 & 2.976 & .887 & .599 & 1.000 \\
13. & 4.324 & .998 & 1.111 & 2.789 & .903 & .630 & 1.000 \\
23. & 2.965 & .139 & .441 & .523 & .480 & .691 & .346 \\
23. & 8.045 & .817 & 2.072 & 2.861 & .940 & 1.000 & .768 \\
23. & 2.605 & .931 & .739 & 1.482 & .883 & .445 & .980 \\
23. & 5.160 & .958 & 1.403 & 2.206 & .989 & .812 & .946 \\
30. & 2.611 & .196 & .454 & .372 & .710 & .303 & .117 \\
30. & 2.835 & .985 & 1.020 & .964 & .834 & .444 & 1.000 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|c|}
\hline
h^2 & \frac{\Delta m^2}{E} \left[ 10^{-7} \text{ eV}^2 \text{ MeV}^{-1} \right] & \sin^2 2\theta & 2\phi'/\pi & 2\phi''/\pi & \sin^2 2\theta' & \sin^2 2\theta'' & \sin^2 (2\theta'' - 4\theta'_m) \\
\hline
0. & 4.098 & .070 & .574 & .719 & .192 & .713 & .887 \\
0. & 2.771 & .593 & .497 & 2.011 & 1.000 & .356 & .329 \\
0. & 9.686 & .752 & 2.096 & 4.817 & .905 & 1.000 & .641 \\
0. & 6.748 & .926 & 1.511 & 3.976 & 1.000 & .843 & .820 \\
0. & 3.428 & .999 & .886 & 3.205 & .811 & .361 & .999 \\
13. & 4.000 & .077 & .582 & .701 & .218 & .662 & .881 \\
13. & 2.635 & .656 & .524 & 1.926 & .992 & .323 & .497 \\
23. & 8.415 & .715 & 2.107 & 2.800 & .869 & 1.000 & .525 \\
23. & 2.382 & .825 & .632 & 1.667 & .895 & .261 & .865 \\
23. & 4.528 & .992 & 1.299 & 2.436 & .919 & .531 & .969 \\
30. & 3.114 & .150 & .548 & .485 & .530 & .317 & .625 \\
30. & 6.798 & .551 & 1.913 & 1.152 & .762 & 1.000 & .260 \\
30. & 2.450 & .981 & .927 & 1.127 & .751 & .242 & 1.000 \\
30. & 9.742 & 1.000 & 3.274 & 2.463 & .969 & .815 & .992 \\
\hline
\end{array}
\]
Table 3: The same as in Table 1 for the probability $P_{e2}$ in the cases of $\nu_e \to \nu_\mu(\tau)$ and $\nu_e \to \nu_s$ transitions, and for $\Delta m^2/E \leq 10 \times 10^{-7}$ eV$^2$/MeV.

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<th>$h^0$</th>
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24
FIGURE CAPTIONS

Figure 1. The regions of the four different solutions $A$, $B$, $C$ and $D$ (eqs. (29), (31), (37) and (45)) for the maxima of the transition probability $P_{e\mu} = P_{\mu e}$ in a two-layer medium. The two different layers correspond to the core and the mantle of the Earth [15, 16].

Figure 2. The same as in figure 1 for the case vacuum - Earth mantle.

Figure 3. The regions of the three different solutions $A$, $C$ and $D$ (eqs. (54), (63) and (64)) for the maxima of the transition probability $P_{e\mu} = P_{\mu e}$ in a three-layer medium. The three different layers correspond to the mantle-core-mantle of the Earth.

Figure 4. The same as in figure 3 for the transition probability $P_{e2}$ in the case of $\nu_e - \nu_{\mu(\tau)}$ mixing.

Figure 5. The same as in figure 3 for the transition probability $P_{\bar{\mu}\bar{s}} \equiv P(\bar{\nu}_\mu \rightarrow \bar{\nu}_s)$.

Figure 6. The probability $P_{e\mu} = P_{\mu e}$ for the Earth-center-crossing (atmospheric) neutrinos ($h = 0^\circ$), as a function of $\sin^2 2\theta$ (horizontal axis) and $\Delta m^2/E [10^{-7} \text{eV}^2/\text{MeV}]$ (vertical axis). The ten different colors correspond to values of $P_{e\mu}$ in the intervals: 0.0 - 0.1 (violet); 0.1 - 0.2 (dark blue); ...; 0.9 - 1.0 (dark red). The points of total neutrino conversion (in the dark red regions), $P_{e\mu} = 1$, correspond to solution $A$, eq. (54), for the Earth-core-crossing neutrinos.

Figure 7. The same as in figure 6 for the probability $P_{e2}$ in the case of $\nu_e - \nu_{\mu(\tau)}$ mixing, and for (solar) neutrinos crossing the Earth (core) along the trajectory with $h = 13^\circ$. The points of total neutrino conversion (in the dark red regions), $P_{e2} = 1$, correspond to solution $A$, eq. (74).

Figure 8. The same as in figure 6 for the probability $P_{\bar{\mu}\bar{s}}$ and for neutrinos crossing the Earth (core) along the trajectory with $h = 23^\circ$. The points of total neutrino conversion (in the dark red regions), $P_{\bar{\mu}\bar{s}} = 1$, correspond to solution $A$, eq. (54).