Comment on “Consistency, amplitudes and probabilities in quantum theory” by A. Caticha

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Abstract

A carefully written paper by A. Caticha [Phys. Rev. A57, 1572 (1998)] applies consistency arguments to derive the quantum mechanical rules for compounding probability amplitudes in much the same way as earlier work by the present author [J. Math. Phys. 29, 398 (1988) and Int. J. Theor. Phys. 27, 543 (1998)]. These works are examined together to find the minimal assumptions needed to obtain the most general results.

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In a recent article [1], A. Caticha uses consistency arguments to derive the quantum mechanical rules for combining probability amplitudes. Caticha’s work bears close resemblance, both in approach and execution, to earlier work by the present author [2,3]. With hindsight, it seems a good time to take stock and see what has been achieved by this approach and what are the minimal assumptions needed to obtain the most general results. Our point of departure is the recognition that in quantum mechanics one cannot directly assign probabilities to processes. In contrast to the classical situation, not every proposition can be answered by yes or no (which slit did the particle go through?). Therefore Boolean algebra does not apply and the road opened by R.T. Cox [4] to introduce probabilities is not open to us. Probability must, therefore, be introduced indirectly as a function of the corresponding probability amplitude [3]. Let us review briefly how this is done.

The basic entities of concern are the transition amplitudes \( \langle B \mid A \rangle \) between experimentally determined initial and final states A and B. Both time dependent transitions \( \langle B(t_2) \mid (A(t_1)) \rangle \) and transitions at a given time \( t_2 = t_1 \) are of interest. To each transition one assigns a complex number - the probability amplitude for the process. This number is assumed to depend only on the given process \( A \rightarrow B \) and to be independent of the past history (Markovian property).

Among the possible processes there are two kinds of special interest: processes in series, an example of which will be \( C \rightarrow B \rightarrow A \) with amplitude \( \langle A \mid C \rangle_{\text{via } B} \), where \( C \) is made to pass through a filter \( B \) before \( A \) is verified, and processes in parallel, the simplest of which will be

\[
\begin{array}{c}
B \rightarrow C_1 \rightarrow A \\
\downarrow \quad \uparrow \\
C_2
\end{array}
\]

, where \( B \) can proceed to \( A \) only through two orthogonal filters \( C_1, C_2 \). The amplitude for this process will be denoted by \( \langle A \mid B \rangle_{\text{via } C_1,C_2} \). Very special cases of these “in series” or “in parallel” processes are referred to as AND, OR setups in [1]. Our only assumption regarding these processes is that the amplitudes for the processes are given analytic functions of the partial complex amplitudes \( x \) and \( y \), namely,
\[ \langle A \mid C \rangle_{\text{via } B} = f(x, y) \]  

where

\[ x = \langle A \mid B \rangle \quad \text{and} \quad y = \langle B \mid C \rangle, \]  

and

\[ \langle A \mid B \rangle_{\text{via } C_1 C_2} = g(x, y) \]

where

\[ x = \langle A \mid B \rangle_{\text{via } C_1} \quad \text{and} \quad y = \langle A \mid B \rangle_{\text{via } C_2}. \]

Our task is to find the possible form of these functions, subject to consistency demands.

Consider now the process \( D \rightarrow C \rightarrow B \rightarrow A \) with amplitudes

\[ z = \langle C \mid D \rangle, \quad y = \langle B \mid C \rangle \quad \text{and} \quad x = \langle A \mid B \rangle. \]

We could calculate the amplitude for the process in two different ways: (a) combine first \( C \rightarrow B \rightarrow A \) to obtain \( \langle A \mid C \rangle_{\text{via } B} = f(x, y) \) and then calculate \( f(f(x, y), z) \) or, (b) combine first \( D \rightarrow C \rightarrow B \) to obtain \( \langle B \mid D \rangle_{\text{via } C} = f(y, z) \) and then calculate \( f(x, f(y, z)) \). Consistency then demands that the two calculations give the same result, namely, the function \( f(x, y) \) should obey the associative law

\[ f(x, f(y, z)) = f(f(x, y), z). \]  

Similarly, for processes in parallel, consistency entails

\[ g(x, g(y, z)) = g(g(x, y), z). \]

Finally, consider the combined process

\[ C \rightarrow B \ \downarrow^{C_1} \ \uparrow_{C_2} \rightarrow A. \]

One way to calculate the amplitude is to consider it as a process in series, with amplitude \( f(g(x, y), z) \), where
Another way to look at the same process is to consider it as a process in parallel, with an amplitude \( g(f(x, z), f(y, z)) \). Demanding that the two representations agree, we have the distributive law

\[
f(g(x, y), z) = g(f(x, z), f(y, z)).
\]  

(9)

This is all that is needed. From here on the rest is mathematics. In particular, there is no need to assume commutativity for processes in parallel, as was done in [1] and [2]. The equality \( g(x, y) = g(y, x) \) follows automatically from Cox’s solution [5] of the associative law (7), as recounted for example, in [1]. Disposing with commutativity renders the use of artificial time-dependent filters [1] unnecessary, and allows general formulation in terms of arbitrary filters and states. As shown in [2] and [1], given the functions \( g(x, y) \) and \( f(x, y) \) it is always possible to find a transformation \( x' = H(x) \) which will bring \( g \) and \( f \) to the canonical form

\[
[g(x, y)]' = x' + y', \quad [f(x, y)]' = x'y'.
\]  

(10)

Conversely, starting with Eq. (10), one can make a transformation \( x'' = K(x') \) such that, in terms of the new variables \( x'' \) and \( y'' \), the addition and multiplication laws (10) change their form, without changing their contents.

From here on we shall restrict our discussion to transitions at a given time. Our aim is to amend the general proof of Born’s law

\[
\Pr(A | B) = |x|^2, \quad x = \langle A | B \rangle
\]  

(11)

given in [3]. Here \( \Pr(A | B) \) stands for the probability of transition, at a given time, from \( B \) to \( A \). To achieve this, neither dynamics nor the introduction of dubious multi-particle filters [1] are needed. As shown in [3], the amplitude for the inverse transition \( A \to B \) satisfies \( \langle B | A \rangle = \langle A | B \rangle^* \). Furthermore, the probability for the process \( B \to A \) was shown there to be of the form
\[ \Pr(A \mid B) = \mid x \mid ^\alpha, \quad \alpha > 0. \] (12)

Consider now all the orthogonal states \( A_i \), which can be reached from \( B \), with an amplitude \( x_i = \langle A_i \mid B \rangle \). Since

\[ \langle B \mid B \rangle = \sum_i \langle B \mid A_i \rangle \langle A_i \mid B \rangle = \sum_i x_i^* x_i, \] (13)

and since the probability of the certain event satisfies \( \Pr(B \mid B) = 1 \), we have by (12) and (13)

\[ \Pr(B \mid B) = (\sum_i |x_i|^2)^\alpha = 1. \] (14)

Hence, taking the logarithm of both sides, we obtain

\[ \sum |x_i|^2 = 1. \] (15)

But the totality of processes \( B \to A_i \) form an exhaustive and mutually exclusive set of alternatives, satisfying (see Eq. (12))

\[ \sum \Pr(A_i \mid B) = \sum |x_i|^\alpha = 1. \] (16)

Comparing (15) and (16) we find \( \alpha = 2 \) and

\[ \Pr(A \mid B) = |x|^2. \] (17)

In summary, the assumptions (a) that amplitudes for processes in series or in parallel are represented by analytic functions of the complex partial amplitudes, and (b) that the probability of a process is a function of the amplitude for the process, are enough to derive the known quantum mechanical rules for combining amplitudes and for calculating the corresponding probabilities. This is achieved using general states and filters. That these assumptions are all that is needed, was not fully realised either in [2], [3] or in [1]. The work of Caticha certainly helped to put things in sharper focus. In particular, as shown by Caticha, assumption (a) is enough to establish the linearity of the Schrödinger equation.

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REFERENCES


