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Abstract

We complete our study of non-Abelian gauge theories arbitrary number of Dirac Fermions in the framework of Epstein-Glaser approach to renormalization theory. We consider the consistency of the model in the third order of the perturbation theory and we obtain the so-called axial anomalies. However, we get some discrepancies in comparison to the standard literature. More precisely, we prove that one has to consider two group-covariant tensors instead of the usual one.

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1 Introduction

In some preceding papers [7], [8] we have extended results of Aste, Dütsch and Scharf [1], [5], [2] concerning the uniqueness of the non-Abelian gauge theory describing the consistent interaction Bosons of spin 1. It appeared that the gauge invariance principle is a natural consequence of the description of spin-one particles in a factor Hilbert space: gauge invariance expresses the possibility of factorizing the S-matrix to the physical space, which is usually constructed using the existence of a supercharge $Q$ according to the cohomological-type formula: $\mathcal{H}_{phys} = \text{Ker}(Q) / \text{Im}(Q)$. The obstructions to such a factorisation process are the well-known anomalies. The case when the spin-one Bosons of non-null mass are admitted in the game was studied in [5], [2] for the concrete case of the electroweak interaction i.e. when the gauge group is exactly $SU(2) \times U(1)$.

In [8] we have analyzed the same problem considering that the spin-one Bosons can have non-null masses and we did not impose any restriction on their number and masses and we did not took into account the matter fields. Similar results have been obtained in [10]. We have also considered the effect of including Dirac Fermions and we have proved that the cancelation of the anomaly in the second order of the perturbation theory brings, beside the relations obtained in [7], new relations on the numerical coefficients of the left and right handed components of the interaction Lagrangian. More precisely, a group theoretical property appears, i.e. these coefficients can be organised as two representations of the gauge algebra: $t^+_a$ and $t^-_a$ with $a, b, \ldots = 1, \ldots, r$ group indices; the usual notations are $t^R_a$ and $t^L_a$. Also, a representation property

In this paper we continue the analysis going to the third order of the perturbation. The main result is the following. The cancelation of the anomaly in the third order of the perturbation theory shows that the usual condition of cancelation of the axial anomaly must be amended. In fact, we prove that one has to consider two tensors, one of vectorial and the other of the axial nature

$$V_{abc} \equiv \sum_{\epsilon_1,\epsilon_2,\epsilon_3 = \pm} (1 + \epsilon_1 \epsilon_2 + \epsilon_2 \epsilon_3 + \epsilon_3 \epsilon_1) Tr(t^1_\epsilon [t^2_\epsilon, t^3_\epsilon])$$

$$A_{abc} \equiv \sum_{\epsilon_1,\epsilon_2,\epsilon_3 = \pm} (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_1 \epsilon_2 \epsilon_3) Tr(t^1_\epsilon \{t^2_\epsilon, t^3_\epsilon\})$$

and the anomalies are absent iff we have one of the following two possibilities: (a) $V_{abc} \neq 0$; or (b) $V_{abc} = 0, A_{abc} = 0$.

Also a new condition on the couplings of the Higgs fields appear.

We mention here that the term $A_{abc}$ is similar to the one appearing in the usual treatments of the Adler-Bell-Jackiw anomaly. However, the usual expression is given by (see for instance [12]):

$$A^{ABJ}_{abc} \equiv Tr \left( t^+_a \{t^+_b, t^+_c\} \right) - Tr \left( t^-_a \{t^-_b, t^-_c\} \right).$$

The structure of the paper is the following one. In the next Section we summarize the description of non-null mass spin-one Bosons and construct the the S-matrix up to the second order of the as in [8]. We also consider the coupling of Yang-Mills fields with Dirac Fermions.
summarizing the results of [8]. In Section 4 we go to the third order of the perturbation theory; in [10] the analysis of the pure Boson sector is performed and leads to some restrictions on the coupling of the Higgs Bosons. We investigate the Dirac Fermionic sector and we get the new conditions on the Fermionic representations (a) or (b) from above. We also analyze briefly the Higgs sector with the computational methods developed in [7] and [8] because we will find a supplementary condition on the scalar coupling which does not appear in [10].
2 General Description of the Vector Bosons

2.1 Spin-One Relativistic Free Particles with Positive Mass

As in [7], we take the one-particle space of the problem \( H \) to be the Hilbert space of an unitary irreducible representation of the Poincaré group. We give below the relevant formulae for particles of mass \( m > 0 \) and spin one.

The upper hyperboloid of mass \( m \geq 0 \) is by definition the set of functions \( X^+_m \equiv \{ p \in \mathbb{R}^4 | \| p \|^2 = m^2 \} \) which are square integrable with respect to the Lorentz invariant measure \( d\alpha_m^+(p) \equiv \frac{dp}{p^0} \); (in fact only classes of functions identical up to null-measure sets are considered).

The conventions are the following: \( \| \cdot \| \) is the Minkowski norm defined by \( \| p \|^2 \equiv p \cdot p \) and \( p \cdot q \) is the Minkowski bilinear form \( p \cdot q \equiv p_0 q_0 - p \cdot q \). If \( p \in \mathbb{R}^3 \) we define \( \tau(p) \in X^+_m \) according to \( \tau(p) \equiv (\omega(p), p) \), \( \omega(p) \equiv \sqrt{p^2 + m^2} \).

Let us consider the Hilbert space \( H \equiv L^2(X^+_m, \mathbb{C}^4, d\alpha_m^+) \) with the scalar product

\[
< \phi, \psi > \equiv \int_{X^+_m} d\alpha_m^+(p) \langle \phi(p), \psi(p) \rangle \quad (2.1.1)
\]

where \( \langle u, v >_{\mathbb{C}^4} \equiv \sum_{i=1}^{4} u_i \bar{v}_i \) is the usual scalar product from \( \mathbb{C}^4 \). In this Hilbert space we have the following (non-unitary) representation of the Poincaré group:

\[
(U_{\alpha, \Lambda}) (p) \equiv e^{i\alpha \cdot p} \Lambda \cdot \phi(\Lambda^{-1} \cdot p) \quad \text{for} \quad \Lambda \in \mathcal{L}^1, \\
(U_{I_\mu}) (p) \equiv \bar{\phi}(I_\mu \cdot p) 
\]

(2.1.2) and the following non-degenerate sesquilinear form:

\[
\langle \phi, \psi \rangle \equiv \int_{X^+_m} d\alpha_m^+(p) \ g^{\mu\nu} \bar{\phi}_\mu(p) \psi_\nu(p); 
\]

(2.1.3) the indices \( \mu, \nu \) take the values 0, 1, 2, 3 and the summation convention over the dummy indices is used. Then one has

\[
(U_{\alpha, \Lambda} \phi, U_{\alpha, \Lambda} \psi) = \langle \phi, \psi \rangle, \quad \text{for} \quad \Lambda \in \mathcal{L}^1, \\
(U_{I_\mu} \phi, U_{I_\mu} \psi) = \langle \phi, \psi \rangle. 
\]

(2.1.4)

Let us consider the following subspace of \( H \): \( H_m \equiv \{ \phi \in H | \ p^0 \phi_\mu(p) = 0 \} \). Then one can prove that the sesquilinear form \( (\cdot, \cdot)|_{H_m} \) is strictly positively defined.

As a consequence, the representation (2.1.4) of the Poincaré group leaves invariant the subspace \( H_m \) and the restriction of this representation to this subspace (also denoted by \( U \)) is equivalent to the unitary irreducible representation \( H^{[m,1]} \) of the Poincaré group (describing particles of mass \( m > 0 \) and spin 1 [11]). The couple \( (H_m, U) \) is called a spin-one Boson of mass \( m \).

We turn now to the second quantization procedure applied to such an elementary system. We express the (Bosonic) Fock space of the system \( \mathcal{F}_m \equiv \mathcal{F}^+(H_m) \equiv \oplus_{n \geq 0} \mathcal{H}_n^\prime \), \( \mathcal{H}_0^\prime \equiv \mathbb{C} \) as a subspace of an auxiliary Fock space \( \mathcal{H} \equiv \mathcal{F}^+(H) \equiv \oplus_{n \geq 0} \mathcal{H}_n \), \( \mathcal{H}_0 \equiv \mathbb{C} \). One canonically identifies the \( n \text{th} \)-particle subspace \( \mathcal{H}_n \) with the set of Borel functions : \( \Phi^{(n)}_{\mu_1, \ldots, \mu_n} (k_1, \ldots, k_n) : \)
$(X_m^+)^{\times n} \to \mathbb{C}$ which are square summable and verify convenient symmetry properties with respect to the permutations of the couples: $(\mu_i, k_i), \ i = 1, \ldots, n.$

In $\mathcal{H}$ the expression of the scalar product is naturally induced by (2.1. 1) and we have a representation of the Poincaré group given by: $U_g \equiv \Gamma(U_g), \ \forall g \in \mathcal{P}$; (here $U_g$ is given by (2.1. 2)) which leaves the induced sesquilinear form invariant.

Let us consider the following subspace of $\mathcal{H}$: $\mathcal{H}' \equiv \mathcal{F}'(H') = \oplus_{n \geq 0} \mathcal{H}_n'$. Then $\mathcal{H}_n', \ n \geq 1$ is generated by elements of the form $\phi_1 \vee \cdots \vee \phi_n, \ \phi_1, \ldots, \phi_n \in \mathcal{H}'$ and, in the representation adopted previously for the Hilbert space $\mathcal{H}_n$ we can take them to be formed by those elements of $\mathcal{H}_n'$ which verify the transversality condition $k_1^{\nu \mu_1} \Phi_{\nu_1 \cdots \nu_n}^{(n)}(k_1, \ldots, k_n) = 0$.

Moreover, the sesquilinear form $(\cdot, \cdot)|_{\mathcal{H}'}$ is strictly positively defined and there exists an canonical isomorphism of Hilbert spaces $\mathcal{F}_m \simeq \mathcal{H}'$.

We can define the corresponding field as an operator on the Hilbert space $\mathcal{H}$ in complete analogy to the electromagnetic field; we define for every $p \in X_m^+$ the usual annihilation and creation operators $A_{\nu}(p)$ and $A_{\nu}^\dagger(p)$ and next, the field operators in the point $x$ according to

$$A_{\nu}(x) \equiv A_{\nu}^+(x) + A_{\nu}^-(x) \quad (2.1. 5)$$

where the expressions appearing in the right hand side are the positive (negative) frequency parts and are defined by:

$$A_{\nu}^+(x) = \frac{1}{(2\pi)^{3/2}} \int_{X_m^+} d\alpha_{\nu}(p) e^{ip \cdot x} A_{\nu}^\dagger(p), \quad A_{\nu}^-(x) = \frac{1}{(2\pi)^{3/2}} \int_{X_m^+} d\alpha_{\nu}(p) e^{-ip \cdot x} A_{\nu}(p). \quad (2.1. 6)$$

The explicit expressions are

$$(A_{\nu}^+(x)\Phi)_{\mu_1 \cdots \mu_n}^{(n)}(k_1, \ldots, k_n) = \sqrt{n + 1} \int_{X_m^+} d\alpha_{\nu}(p) e^{ip \cdot x} \Phi_{\nu_1 \cdots \nu_n}^{(n+1)}(p, k_1, \ldots, k_n) \quad (2.1. 7)$$

and

$$(A_{\nu}^-(x)\Phi)_{\mu_1 \cdots \mu_n}^{(n)}(k_1, \ldots, k_n) = -\frac{1}{(2\pi)^{3/2} \sqrt{n}} \sum_{i=1}^n e^{ik_i \cdot x} g_{\nu \mu_i} \Phi_{\nu_1 \cdots \nu_n}^{(n-1)}(k_1, \ldots, \hat{k}_i, \ldots, k_n) \quad (2.1. 8)$$

Some of the properties of the field operators $A_{\nu}(x)$ are given below:

$$(A_{\nu}(x)\Psi, \Phi) = (\Psi, A_{\nu}(x)\Phi), \ \forall \Psi, \Phi \in \mathcal{H}, \quad (2.1. 9)$$

$$\Box + m^2)A_{\nu}(x) = 0 \quad (2.1. 10)$$

and

$$[A_{\mu}(x), A_{\nu}(y)] = -g_{\mu \nu} D_m(x - y) \times 1; \quad (2.1. 11)$$

where

$$D_m(x) = D_m^+(x) + D_m^-(x) \quad (2.1. 12)$$

is the Pauli-Jordan distribution.
We now give an alternative description of the Fock space $\mathcal{F}_m$ using the ghosts fields; we have to introduce beside the Fermionic ghosts some Bosonic ghost.

We consider the Hilbert space $\mathcal{H}^gh \equiv L^2(X_m^+, \mathbb{C}^2, d\alpha_m^+)$ and also $\mathcal{H}^gh_+ \equiv L^2(X_m^+, \mathbb{C}, d\alpha_m^+)$ with the natural scalar products. In this spaces act the usual unitary representations of the Poincaré group. The Fock space $\mathcal{F}^gh \equiv \mathcal{F}^-(\mathcal{H}^gh) \otimes \mathcal{F}^+(\mathcal{H}^gh)$ is called ghost particle Hilbert space. Let us define the auxiliary Fock space $\mathcal{H}^gh \equiv \mathcal{H} \otimes \mathcal{F}^gh$ where $\mathcal{H}$ has been defined previously. We can write $\mathcal{H}^gh = \bigotimes_{n=1}^{\infty} \mathcal{H}_{nwls}$ where one can identify $\mathcal{H}_{nwls}$ with the set of Borel functions $\Phi^{(nwls)}_{\mu_1, \ldots, \mu_n}(K; P; Q; R) : (X_0^+)^{n+w+l+s} \to \mathbb{C}$ which are square integrable with respect to the product measure $(\alpha_m^+) \times (\alpha_m^+) \times (\alpha_m^+) \times (\alpha_m^+)$ and verify convenient (anti) symmetry properties.; here $K \equiv (k_1, \ldots, k_n)$, $P \equiv (p_1, \ldots, p_w)$, $Q \equiv (q_1, \ldots, q_l)$ and $R \equiv (r_1, \ldots, r_l)$.

In this representation we can construct the annihilation and creation operators $A^\#(t), b^\#(t), c^\#(t)$ and $a^\#(t)$ (see [8]). They verify usual canonical (anti)commutation relations and behave naturally with respect to Poincaré transformations.

Then the fields $u(x)$, $\tilde{u}(x)$ and $\Phi(x)$ can be constructed; they are called the Fermionic (resp. Bosonic) ghost fields.

Now we can define the operator:

$$Q \equiv \int_{X_m^+} d\alpha_m^+(q) \left[ k^\# (A_{\mu}(k) c^\#(k) + A_{\mu}^+(k) b(k)) + im (a(k) c^\#(k) - a^\#(k) b(k)) \right] \quad (2.1.13)$$

called supercharge. Its most important property is

$$Q^2 = 0 \implies \text{Im}(Q) \subset \text{Ker}(Q). \quad (2.1.14)$$

An explicit expression of the supercharge in this representation can be found in [8].

As a consequence, we have

**Theorem 2.1** There exists the following vector spaces isomorphism:

$$\text{Ker}(Q)/\text{Im}(Q) \cong \mathcal{H} \quad (2.1.15)$$

where the subspace $\mathcal{H}$ has been defined previously. The isomorphism (2.1. 15) extends to a Hilbert space isomorphism:

$$\frac{\text{Ker}(Q)/\text{Im}(Q)}{\text{Im}(Q)} \cong \mathcal{F}_m$$

(with an appropriate scalar product) and the factorized representation of the Poincaré group coincides with the representation acting into the space $\mathcal{H}'$.

One can easily see that one can take the limit $m \searrow 0$ in the expressions for the various Hilbert spaces and quantum fields and also on the expression of the supercharge $Q$. In this limit we can write $\mathcal{H}^gh \simeq \mathcal{H}_0^gh \otimes \mathcal{H}_q$ where $\mathcal{H}_0^gh$ is the Hilbert space generated by the fields $A_{\mu}(x)$, $u(x)$, $\tilde{u}(x)$ and $\mathcal{H}_q$ is generated by the scalar ghosts. Then the supercharge (2.1. 13) takes the form $Q = Q' \otimes 1$ where $Q'$ coincides formally with the expression of $Q$ for $m \searrow 0$ but acts only in $\mathcal{H}_0^gh$. Moreover, we have: $\frac{\text{Ker}(Q)/\text{Im}(Q)}{\text{Im}(Q)} \simeq \frac{\text{Ker}(Q')}{\text{Im}(Q')} \otimes \mathcal{H}_q$, i.e. we can see that the states from $\mathcal{H}_q$ decouple completely and can be considered physical. Moreover, one can see that, in this case, nothing prevents us to consider that the scalar “ghost” has a non-zero
mass. This observation is essential for the construction of the standard model, because a scalar “ghost” field corresponding to a null mass Boson, if considered a physical field of non-zero mass is nothing else but the Higgs field [2].

One denotes by $\mathcal{W}$ the linear space of all Wick monomials on the Fock space $\mathcal{H}^{gh}$ i.e. containing the fields $A_{\mu}(x)$, $u(x)$, $\tilde{u}(x)$ and $\Phi(x)$. If $M$ is such a Wick monomial, we define by $gh_{\pm}(M)$ the degree in $\tilde{u}$ (resp. in $u$). The ghost number is, by definition, the expression:

$$gh(M) \equiv gh_{+}(M) - gh_{-}(M)$$

The BRST operator is defined by linearity, the action on the elementary fields:

$$d_Q u = 0, \quad d_Q \tilde{u} = -i(\partial_{\mu} A_{\mu} + m\Phi), \quad d_Q A_{\mu} = i\partial_{\mu} u, \quad d_Q \Phi = imu;$$

and the derivation property:

$$d_Q (MN) = (d_Q M) N + (-1)^{gh(M)} M (d_Q N), \quad \forall M, N \in \mathcal{W}.$$  

(2.1. 17)

(2.1. 18)

The class of all observables on the factor space emerges (see theorem 2.1): an operator $O : \mathcal{H}^{gh} \to \mathcal{H}^{gh}$ induces a well defined operator $[O]$ on the factor space $\text{Ker}(Q)/\text{Im}(Q) \simeq \mathcal{F}_m$ if and only if it verifies: $d_Q O|_{\text{Ker}(Q)} = 0$. Because $d_Q^2 = 0$, not all operators verifying the condition (2.1) are interesting. In fact, the operators of the type $d_Q O$ are inducing a null operator on the factor space; explicitly, we have:

$$[d_Q O] = 0.$$  

(2.1. 19)

If the interaction Lagrangian is a Wick monomial $T_1 \in \mathcal{W}$ with $gh(T_1) \neq 0$ then the S-matrix is trivial.

The analysis of the possible interactions between the Bosonic spin-one field and “matter” follows the usual lines (see [8]). Let $\mathcal{H}_{\text{matter}}$ be the corresponding Hilbert space of the matter fields; it is elementary to see that we can realize the total Hilbert space $\mathcal{H}_{\text{total}} \equiv \mathcal{F}_m \otimes \mathcal{H}_{\text{matter}}$ as the factor space $\text{Ker}(Q)/\text{Im}(Q)$ where the supercharge $Q$ is defined on $\mathcal{H}_{gh} \equiv \mathcal{H}_{gh} \otimes \mathcal{H}_{\text{matter}}$ by the obvious substitution $Q \to Q \otimes 1$.  

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2.2 Massive Yang-Mills Fields

As in [8], we first define in an unambiguous way what we mean by Yang-Mills fields. All the fields will carry an additional index $a = 1, \ldots, r$ and this can be realized with an appropriate modification of the Hilbert spaces (auxiliary or physical). So we have the fields: $A_{a\mu}$, $u_a$, $\tilde{u}_a$, $\Phi_a$, $\ a = 1, \ldots, r$ given by obvious expressions in such a way that the one-particle subspace is a direct sum of $r$ copies of elementary heavy Bosons of masses $m_a$, $a = 1, \ldots, r$ and spin 1.

These fields verify the following equations of motion:

$$ (\Box + m_a^2)u_a(x) = 0, \quad (\Box + m_a^2)\tilde{u}_a(x) = 0, \quad (\Box + m_a^2)\Phi_a(x) = 0, \quad a = 1, \ldots, r. \quad (2.2.1) $$

The canonical (anti)commutation relations are:

$$ [A_{a\mu}(x), A_{b\nu}(y)] = -\delta_{ab}\delta_{\mu\nu}D_m(x - y) \times 1, $$

$$ \{u_a(x), \tilde{u}_b(y)\} = \delta_{ab}D_m(x - y) \times 1, \quad [\Phi_a(x), \Phi_b(y)] = \delta_{ab}D_m(x - y) \times 1; \quad (2.2.2) $$

and all other (anti)commutators are null. The supercharge is given by (see (2.1.13)):

$$ Q \equiv \sum_{a=1}^{r} \int_{X_m} d\alpha^+_{m_a}(q) \left[ k^\mu (A_{a\mu}(k)c_a^\dagger(k) + A_{a\mu}^\dagger(k)b_a(k)) + \frac{i}{2}m_a (a_a(k)c_a^\dagger(k) - a_a^\dagger(k)b_a(k)) \right] $$

and verifies all the expected properties.

The Krein operator can be defined and used to construct a sesquilinear form such that we have

$$ A_{a\mu}(x)^\dagger = A_{a\mu}(x), \quad u_a(x)^\dagger = u_a(x), \quad \tilde{u}_a(x)^\dagger = -\tilde{u}_a(x), \quad \Phi_a(x)^\dagger = \Phi_a(x). \quad (2.2.4) $$

The ghost degree is defined in an obvious way and the expression of the BRST operator is similar to the previous one. In particular we have (see (2.1.17)):

$$ d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i(\partial_\mu A_{a\mu}^\dagger + m_a \Phi_a), \quad d_Q A_{a\mu}^\dagger = i\partial^\mu u_a, \quad d_Q \Phi_a = im_a u_a, \quad \forall a = 1, \ldots, r. \quad (2.2.5) $$

If we take into account the last observation from the preceding Subsection, it appears that it is possible to make in the formalism presented above some of the masses null. In this case the corresponding scalar ghosts can be considered as physical fields and they will be called Higgs fields. Moreover, we do not have to assume that they are massless i.e. if some Boson field $A_{a\mu}$ has zero mass $m_a = 0$, we can suppose that the corresponding Higgs field $\Phi_a$ has a non-zero mass: $m_a^H$. If the mass of the vector field $A_{a\mu}$ is non-zero $m_a \neq 0$, then we have $m_a^H = m_a$. Moreover, this process of attributing a non-zero mass to the scalar partners of the zero-mass vector fields should not influence the BRST transformation formula (2.2.5); that’s it, this formula remains unchanged.

We will construct a perturbation theory \textit{à la} Epstein-Glaser for the free fields $A_{a\mu}$, $u_a$, $\tilde{u}_a$ and $\Phi_a$, $a = 1, \ldots, r$ in the auxiliary Hilbert space $H_{\gamma,M}^{gh,r}$ imposing the usual axioms of causality,
unitarity and relativistic invariance. Moreover, we want that the result factorizes to the physical Hilbert space in the adiabatic limit. This amounts to

$$\lim_{\epsilon \searrow 0} dQ \int_{(\mathbb{R}^4)^\times n} dx_1 \cdots dx_n g_c(x_1) \cdots g_c(x_n) T_n(x_1, \ldots, x_n) \bigg|_{\text{Ker}(Q)} = 0, \ \forall n \geq 1. \quad (2.2.6)$$

If this condition if fulfilled, then the chronological and the antichronological products do factorize to the physical Hilbert space and they give a perturbation theory verifying causality, unitarity and relativistic invariance.

If one completely exploits the condition of gauge invariance in the first order of perturbation theory obtaining the generic form of the Yang-Mills interaction of spin-one Bosons. We assume the summation convention of the dummy indices $a, b, \ldots = 1, \ldots, r$.

Theorem 2.2

Let us consider the operator

$$T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) T_1(x) \quad (2.2.7)$$

defined on $\mathcal{H}_{Y_M}^{gh,r}$ with $T_1$ a Lorentz-invariant Wick polynomial in $A_\mu$, $u$, $\bar{u}$ and $\Phi$ verifying also $\omega(T_1) = 4$. If $T_1(g)$ induces an well defined non-trivial S-matrix, in the adiabatic limit, then it necessarily has the following form:

$$T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) [T_{11}(x) + T_{12}(x) + T_{13}(x)] \quad (2.2.8)$$

where we have introduced the following notations:

$$T_{11}(x) \equiv f_{abc} \left[ : A_{\mu a}(x) A_{bc}(x) \partial^\mu A^a_\mu(x) : - : A^a_\mu(x) u_b(x) \partial_\mu \bar{u}_c(x) : \right], \quad (2.2.9)$$

$$T_{12}(x) \equiv f'_{abc} \left[ : \Phi_a(x) \partial_\mu \Phi_b(x) A_{\mu c}(x) : - m_b : \Phi_a(x) A_{b\mu}(x) A^c_\mu(x) : - m_b : \Phi_a(x) \bar{u}_b(x) u_c(x) : \right] \quad (2.2.10)$$

$$T_{13}(x) \equiv f''_{abc} : \Phi_a(x) \Phi_b(x) \Phi_c(x) : \quad (2.2.11)$$

Here the various constants from the preceding expression are constrained by the following conditions:

- the expressions $f_{abc}$ are completely antisymmetric
  $$f_{abc} = -f_{bac} = -f_{acb} \quad (2.2.12)$$

and verify:

$$(m_a - m_b) f_{abc} = 0, \quad \text{iff} \quad m_c = 0, \quad \forall a, b = 1, \ldots, r; \quad (2.2.13)$$

- the expressions $f'_{abc}$ are antisymmetric in the indices $a$ and $b$:
  $$f'_{abc} = -f'_{bac} \quad (2.2.14)$$

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verify the relation:

\[(m_a^H - m_b^H) f'_{abc} = 0, \quad \text{iff} \quad m_a = m_b = m_c = 0, \quad \forall a, b = 1, \ldots, r\]  
(2.2. 15)

and are connected to \(f_{abc}\) by:

\[f_{abc} m_c = f'_{cab} m_a - f'_{cba} m_b, \quad \forall a, b, c = 1, \ldots, r;\]  
(2.2. 16)

- the expressions \(f''_{abc}\) remain undetermined for \(m_a = m_b = m_c = 0\) and for the opposite case are given by:

\[f''_{abc} = \frac{1}{6m_c} f'_{abc} \left[ (m_a^H)^2 - (m_a)^2 + (m_b)^2 - (m_c^H)^2 \right], \quad \text{iff} \quad m_c \neq 0, \quad a, b = 1, \ldots, r; \]  
(2.2. 17)

Moreover, we have:

**Corollary 2.3** In the condition of the preceding theorem, one has:

\[d_Q T_1(x) = i\partial_\mu T_1^\mu(x)\]  
(2.2. 18)

where:

\[T_1^\mu \equiv T_{11}^\mu + T_{12}^\mu\]  
(2.2. 19)

and the expression from this formula are defined as follows:

\[T_{11}^\mu \equiv f_{abc} \left( : u_a A_{ba} F_{c}^{\nu\mu} : - \frac{1}{2} : u_a u_b \partial_\mu \tilde{u}_c : \right)\]  
(2.2. 20)

and

\[T_{12}^\mu \equiv f'_{abc} \left( m_a : A_\mu^a \Phi_b u_c : + : \Phi_a \partial_\mu \Phi_b u_c : \right)\]  
(2.2. 21)

The expression \(T_1\) from the preceding theorem verifies the unitarity condition

\[T_1(x)^\dagger = T_1(x)\]

if and only if the constants \(f_{abc}, f'_{abc}\) and \(f''_{abc}\), have real values; it also verifies the causality condition:

\[[T_1(x), T_3(y)] = 0, \quad \forall x, y \in \mathbb{R}^4 \quad \text{s.t.} \quad (x - y)^2 < 0.\]

One can causally split this commutator and obtain the generic expression of \(T_2(x, y)\); in general, this is not a well-defined operator on the factor space \(\mathcal{H}_Y \otimes M\). One can show that this can happen if and only if some severe restrictions are placed on the constants appearing in the expression of the interaction Lagrangian.
Theorem 2.4  The generic expression of the distribution $T_2$ leads, in the adiabatic limit, to a well defined operator on $\mathcal{H}_{YM}$ if and only if:

(a) The constants $f_{abc}$ verify the Jacobi identities:

$$f_{abc}f_{dec} + f_{bdc}f_{aec} + f_{dac}f_{bec} = 0; \quad (2.2.22)$$

in particular, there exists a compact Lie group $G$ with $f_{abc}$ as structure constants; moreover $G$ is of the form $G = H_1 \times \cdots \times H_k \times U(1) \times \cdots \times U(1)$ with $H_1, \ldots, H_k$ compact simple Lie groups.

(b) The constants $f'_{abc}$ verify the identity:

$$f'_{dca}f'_{ceb} - f'_{dcb}f'_{cea} = -f_{abc}f'_{dec}; \quad (2.2.23)$$

in other words, if we define the $r \times r$ (antisymmetric) matrices $T_a, a = 1, \ldots, r$ according to

$$(T_a)_{bc} = -f'_{bca}, \quad \forall a, b, c = 1, \ldots, r,$$

then they are an $r$-dimensional representation of the Lie algebra $\text{Lie}(G)$ determined by the structure constants $f_{abc}$.

(c) The constants $f''_{abc}$ verify the following identities:

$$f''_{cab}f_{cde} + f'_{cdb}f''_{cae} + f'_{ceb}f_{cda} = 0, \quad \text{for } m_b = 0 \quad (2.2.25)$$

and

$$\frac{1}{m_b} \left[ f'_{cab}f_{cde} + f'_{cdb}f''_{cae} + f'_{ceb}f_{cda} \right] = (a \leftrightarrow b), \quad \text{for } m_a \neq 0, \quad m_b \neq 0. \quad (2.2.26)$$

The representation $T_a$ exhibited in the statement of the theorem is nothing else but the representation of the gauge group $G$ into which the Higgs fields live.

Now we have

Corollary 2.5 Suppose that the constants $f_{abc}$, $f'_{abc}$ and $f''_{abc}$ verify the conditions from the statements of theorems 2.2 and 2.4. Then, the general expression for the chronological product $T_2$ is given by the sum between

$$T_2(x,y) = T_2^0(x,y) + i\delta(x-y) [L(x) + N(x)] \quad (2.2.27)$$

The expression $T_2^0(x,y)$ is obtained according to the canonical causal splitting of the commutator $[T_1(x), T_1(y)]$ and $N(x)$ is an finite normalisation of the type (2.2.8).

Here we have also defined the expression

$$L(x) \equiv \frac{1}{4} f_{abc}f_{cde} : A_{ab}(x)A_{bd}(x)A_{ce}(x)A_{de}(x) : -f'_{cda}f'_{ceb} : A_{ab}(x)A_{bd}(x)\Phi_d(x)\Phi_e(x) :$$

$$+ \sum_{m_a,m_b,m_d,m_e \neq 0} \frac{3}{2m_b} f'_{cab}f''_{cde} : \Phi_a(x)\Phi_b(x)\Phi_d(x)\Phi_e(x) \quad (2.2.28)$$

The theory is renormalisable up to order two. The condition of unitarity can be satisfied if and only if $N(x)^\dagger = N(x)$
2.3 Yang-Mills Fields coupled to Matter

We study here the possibility of coupling Yang-Mills fields to “matter”. We suppose that we are given the Hilbert space of “matter” $\mathcal{H}_{\text{matter}}$ which should also be a Fock space. Then the coupled system is described in the tensor product Hilbert space $\mathcal{F}_{YM} \otimes \mathcal{H}_{\text{matter}}$. One can describe this Fock space considering

$$\tilde{\mathcal{H}}_{YM}^{gh,r} = \mathcal{H}_{YM}^{gh,r} \otimes \mathcal{H}_{\text{matter}}.$$ 

We will consider here that the “matter” is formed from Dirac Fermions only. We summarize the results of [8] with some new material added. First, we have the generalisation of theorem 2.2:

**Theorem 2.6** Let us consider the operator

$$T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) T_1(x)$$  \hspace{1cm} (2.3. 1)

defined on $\tilde{\mathcal{H}}_{YM}^{gh,r}$ with $T_1$ a Lorentz-invariant Wick polynomial in $A_{\mu a}, u_a, \bar{u}_a, \Phi_a$ and the matter fields, verifying also $\omega(T_1) = 4$. Then $T_1(g)$ can induce a well defined non-trivial $S$-matrix, in the adiabatic limit, if and only if it has the following form:

$$T_1(g) = \int_{\mathbb{R}^4} dx \ g(x) [T_1^{YM}(x) + A_{\mu a}^\mu(x) j_{a\mu}(x) + \sum_{m_a \neq 0} \frac{1}{m_a} \Phi_a(x) \partial_\mu j_a^\mu(x)]$$  \hspace{1cm} (2.3. 2)

Here $j_{a\mu}$ and $j_a$ are Lorentz covariant currents build only from the matter fields with $\omega(j_{a\mu}) = 1, 2, 3$ and $T_{1,\text{matter}}$ contains only the matter fields. Moreover the following conservation law should be valid:

$$\partial_\mu j_a^\mu(x) = 0, \hspace{0.5cm} \forall m_a = 0.$$  \hspace{1cm} (2.3. 3)

The expression for $T_1$ verifies the unitarity requirement if and only if we have:

$$j_a^\mu(x)^\dagger = j_a^\mu(x), \hspace{0.5cm} \forall a = 1, \ldots, r, \hspace{0.5cm} j_a(x)^\dagger = j_a(x), \hspace{0.5cm} \forall m_a = 0$$  \hspace{1cm} (2.3. 4)

and verifies the causality condition if and only if:

$$[j_a^\mu(x), j_b^\mu(y)] = 0, \hspace{0.5cm} (x - y)^2 < 0, \hspace{0.5cm} \forall a, b = 1, \ldots, r,$$  \hspace{1cm} (2.3. 5)

$$[j_a(x), j_b(y)] = 0, \hspace{0.5cm} (x - y)^2 < 0, \hspace{0.5cm} \forall m_a = m_b = 0,$$  \hspace{1cm} (2.3. 6)

$$[j_a^\mu(x), j_b(y)] = 0, \hspace{0.5cm} (x - y)^2 < 0, \hspace{0.5cm} \forall m_b = 0.$$  \hspace{1cm} (2.3. 7)

**Corollary 2.7** The following formula is true

$$d_\phi T_1(x) = i \frac{\partial}{\partial x^\mu} T_1^\mu(x)$$  \hspace{1cm} (2.3. 8)

where $T_1^\mu$ is obtained by adding to the corresponding expression from the pure Yang-Mills case (see (2.2. 20) and (2.2. 21)) the following contribution due to the presence of matter:

$$T_{13}^\mu(x) \equiv u_a(x) j_a^\mu(x).$$  \hspace{1cm} (2.3. 9)
The detailed structure of the interaction Lagrangian is given by the following results. We define the mass matrix by:

\[ M_{AB} \equiv \delta_{A,B} M_A, \quad \forall A, B = 1, \ldots, N. \]  

(2.3.10)

Then we have:

**Proposition 2.8** Suppose that the Dirac Fermions generating \( \mathcal{H}_{\text{matter}} \) are \( \psi_A \) of masses \( M_A \geq 0, \quad A = 1, \ldots, N \). Then the generic form of the currents from the preceding theorem are:

\[ j_a^\mu (x) = : \bar{\psi}_A (x) (t_a)_{AB} \gamma^\mu \psi_B (x) : + : \bar{\psi}_A (x) (t'_a)_{AB} \gamma^5 \psi_B (x) : \]  

(2.3.11)

and

\[ j_a (x) = : \bar{\psi}_A (x) (s_a)_{AB} \psi_B (x) : + : \bar{\psi}_A (x) (s'_a)_{AB} \gamma^5 \psi_B (x) : \]  

(2.3.12)

The causality conditions from theorem 2.6 are fulfilled and the hermiticity conditions are equivalent with the fact that the complex \( N \times N \) matrices \( t_a, t'_a, s_a, s'_a, \quad a = 1, \ldots, r \) are hermitian and \( s'_a, \quad a = 1, \ldots, r \) anti-hermitean.

The following mass relations are true:

\[ s_a = \frac{i}{m_a} [M, t_a], \quad s'_a = \frac{i}{m_a} \{M, t'_a\}, \quad \forall m_a \neq 0, \]  

(2.3.13)

\[ [M, t_a] = 0, \quad \{M, t'_a\} = 0, \quad \forall m_a = 0. \]  

(2.3.14)

In particular, the matrices \( t_a, \quad \forall m_a = 0 \) can be exhibited into a block diagonal structure (eventually after a relabeling of the Dirac fields) and the masses corresponding to the same block must be equal.

The generic expression of the second order chronological product \( T_2(x,y) \) can be found in [8] and the conditions of factorization to the physical space are given in:

**Theorem 2.9** The second order chronological product \( T_2(x,y) \) induces, in the adiabatic limit, an well-defined operator on the physical space \( \mathcal{H}_{\text{total}} \) if and only if, beside the conditions from theorem 2.4 we also have for all \( a, b = 1, \ldots, r \):

\[ [t_a, t_b] + [t'_a, t'_b] = i f_{abc} t_c, \]  

(2.3.15)

\[ [t_a, t'_b] + [t'_a, t_b] = i f_{abc} t'_c, \]  

(2.3.16)

\[ [t_a, s_b] - \{t'_a, s'_b\} = -i f'_{cba} s_c, \]  

(2.3.17)

\[ [t_a, s'_b] - \{t'_a, s_b\} = -i f'_{cba} s'_c. \]  

(2.3.18)

Moreover, the expression of the finite renormalization remains the same as in the pure Yang-Mills theorem (see formula (2.2.27)).

We exhibit now some total divergence structures which will be of use in the analysis of the factorisation condition in the third order of the perturbation theory.
Corollary 2.10 Let us define

\[ R^c_2(x, y) \equiv R^0_2(x, y) + i\delta(x - y)L(x), \quad A^c_2(x, y) \equiv A^0_2(x, y) + i\delta(x - y)L(x) \]  

(2.3. 19)

where \( L(x) \) has been defined previously by (2.2. 28). Then the following total divergence structure is true:

\[ d_Q R^c_2(x, y) = i\frac{\partial}{\partial x^\mu} R^\mu_1(x, y) + i\frac{\partial}{\partial y^\mu} R^\mu_2(x, y), \]

\[ d_Q A^c_2(x, y) = i\frac{\partial}{\partial x^\mu} A^\mu_1(x, y) + i\frac{\partial}{\partial y^\mu} A^\mu_2(x, y). \]  

(2.3. 20)

This corollary leads us to a formula of the same nature as (2.3. 8), that’s it a total divergence structure for the second order chronological products.

Proposition 2.11 Let us define the canonical chronological and antichronological products by:

\[ T^c_2(x, y) \equiv R^c_2(x, y) + T_1(x)T_1(y) = A^c_2(x, y) + T_1(y)T_1(x) \]  

(2.3. 21)

and respectively

\[ T^c_2(x, y) \equiv -R^c_2(x, y) + T_1(y)T_1(x) = -A^c_2(x, y) + T_1(x)T_1(y). \]  

(2.3. 22)

Then one has

\[ d_Q T^c_2(x, y) = i\frac{\partial}{\partial x^\mu} T^\mu_1(x, y) + i\frac{\partial}{\partial y^\mu} T^\mu_2(x, y). \]  

(2.3. 23)

A similar statement is valid for the antichronological product:

\[ d_Q \overline{T}^c_2(x, y) = i\frac{\partial}{\partial x^\mu} \overline{T}^\mu_1(x, y) + i\frac{\partial}{\partial y^\mu} \overline{T}^\mu_2(x, y). \]  

(2.3. 24)

The expressions \( T^\mu_i(x, y) \) and \( \overline{T}^\mu_i(x, y) \), \( (i = 1, 2) \) have causal factorisation properties of the same type as the chronological products, that’s it:

\[ T^\mu_1(x, y) = \begin{cases} T^\mu_1(x)T_1(y) & \text{if } x \geq y, \\ T_1(y)T^\mu_1(x) & \text{if } x \leq y, \end{cases} \quad T^\mu_2(x, y) = \begin{cases} T_1(x)T^\mu_1(y) & \text{if } x \geq y, \\ T^\mu_1(y)T_1(x) & \text{if } x \leq y, \end{cases} \]

\[ \overline{T}^\mu_1(x, y) = \begin{cases} \overline{T}_1(y)\overline{T}^\mu_1(x) & \text{if } x \geq y, \\ \overline{T}^\mu_1(x)\overline{T}_1(y) & \text{if } x \leq y, \end{cases} \quad \overline{T}^\mu_2(x, y) = \begin{cases} \overline{T}^\mu_1(y)\overline{T}_1(x) & \text{if } x \geq y, \\ \overline{T}_1(x)\overline{T}^\mu_1(y) & \text{if } x \leq y. \end{cases} \]  

(2.3. 25)

The sign \( \geq \) means causal succession.

The proof is elementary and consists in noticing that we have:

\[ T^\mu_1(x, y) = R^\mu_1(x, y) + T^\mu_1(x)T_1(y) = A^\mu_1(x, y) + T_1(y)T^\mu_1(x), \]
\[ T^\mu_2(x, y) = R^\mu_2(x, y) + T_1(x)T^\mu_1(y) = A^\mu_2(x, y) + T^\mu_1(y)T_1(x). \]  

(2.3. 26)
and
\[
T^\mu_1(x, y) = -A^\mu_1(x, y) + T^\mu_1(y)T_1(y) = -R^\mu_1(x, y) + T_1(y)T^\mu_1(x), \\
T^\mu_2(x, y) = -A^\mu_2(x, y) + T_1(x)T^\mu_1(y) = -R^\mu_2(x, y) + T^\mu_1(y)T_1(x).
\] (2.3. 27)

The factorisation properties are now elementary because we have nice support properties for the expressions \(R^\mu_i(x, y)\) and \(A^\mu_i(x, y)\), for \(i = 1, 2\).

A deeper analysis of the relations from the statement of the preceding theorem is possible:

**Theorem 2.12**  Let us define
\[
t_\alpha^\epsilon = t_\alpha + \epsilon t'_\alpha, \quad s_\alpha^\epsilon = t_\alpha + \epsilon s'_\alpha, \quad \forall \alpha = 1, \ldots, r, \quad \epsilon = \pm.
\] (2.3. 28)

Then the equations from the statement of the preceding theorem are equivalent to:
\[
[t_\alpha^\epsilon, t_b^\epsilon] = i f_{abc} t_c^\epsilon, \quad (2.3. 29)
\]
\[
t_\alpha^{-\epsilon} s_b^\epsilon - s_b^{-\epsilon} t_\alpha^\epsilon = i \epsilon f_{\alpha bc} s_c^\epsilon
\] (2.3. 30)
\[
\forall \alpha, b = 1, \ldots, r, \quad \forall \epsilon = \pm. \text{ Moreover, the relations (2.3. 13) and (2.3. 14) are equivalent to:}
\[
s_a^\epsilon = \frac{i}{m_a} (M t_a^\epsilon - t_a^{-\epsilon} M), \quad \forall m_a \neq 0,
\] (2.3. 31)
\[
M t_a^\epsilon = t_a^{-\epsilon} M, \quad \forall m_a = 0.
\] (2.3. 32)

*Finally, the hermiticity conditions are equivalent to:*
\[
(t_a^\epsilon)^* = t_a^\epsilon, \quad (s_a^\epsilon)^* = s_a^{-\epsilon}, \quad \forall \alpha = 1, \ldots, r, \quad \epsilon = \pm
\] (2.3. 33)

so in the relation (2.3. 30) one should consider only one of the two signs.

The expressions (2.3. 11) and (2.3. 12) for the currents can be written as follows:
\[
j_{\alpha}^{\mu}(x) =: \overline{\psi}_A(x)(t_a^\dagger \gamma^\mu 1 + \gamma^5 \psi_B(x)) + : \overline{\psi}_A(x)(t_a^- \gamma^\mu 1 - \gamma^5 \psi_B(x)):
\] (2.3. 34)
\[
j_{\alpha}(x) =: \overline{\psi}_A(x)(s_a^\dagger \gamma^\mu 1 + \gamma^5 \psi_B(x)) + : \overline{\psi}_A(x)(s_a^- \gamma^\mu 1 - \gamma^5 \psi_B(x)):
\] (2.3. 35)

and the components corresponding to the signs + (resp. −) are called *chiral* components of the currents. For further convenience we denote them as follows: \(j_{\alpha}^{\mu\epsilon}(x)\) and respectively \(j_{\alpha}^{\epsilon}(x)\), \(\epsilon = \pm\).

It will also be convenient to decompose even further these expression according to the presence or absence of the matrix \(\gamma_5\). So, we have the pieces: \(j_{\alpha}^{\mu\epsilon(V)}(x)\), \(j_{\alpha}^{\mu\epsilon(A)}(x)\) and respectively \(j_{\alpha}^{\epsilon(V)}(x)\), \(j_{\alpha}^{\epsilon(A)}(x)\) \(\epsilon = \pm\). For instance: \(j_{\alpha}^{\mu\epsilon}(x) = \frac{1}{2} : \overline{\psi}_A(x)(t_a^\dagger \gamma^\mu \gamma_5 \psi_B(x)) + \gamma^5 \psi_B(x) : \) etc.
3 Third-order Gauge Invariance

3.1 Causal Splitting of Distribution

We remind here the basic facts about distribution splitting, following essentially [9]. We will use, for simplicity formal notations. Let \( d(x) \in \mathcal{S}'(\mathbb{R}^m) \) be a distribution. We say that it has the quasi-asymptotics \( d_0(x) \in \mathcal{S}'(\mathbb{R}^m) \) in \( x = 0 \) with respect to the continuous and positive function \( \rho \) if for any test function \( \phi(x) \) the following limit exists:

\[
\lim_{\delta \downarrow 0} \rho(\delta) \int_{\mathbb{R}^m} d^m x \ d(\delta x) \phi(x)
\]  (3.1.1)

exists and determines a distribution \( d_0(x) \not\equiv 0 \). Equivalently, in momentum space, we say that the distribution \( d(p) \in \mathcal{S}'(\mathbb{R}^m) \) has the quasi-asymptotics \( d_0(p) \in \mathcal{S}'(\mathbb{R}^m) \) in \( p \to \infty \) if the following limit exists:

\[
\lim_{\delta \downarrow 0} \rho(\delta) \int_{\mathbb{R}^m} d^m p \ d(\delta^{-1} p) \phi(p)
\]  (3.1.2)

exists and determines a distribution \( d_0(p) \not\equiv 0 \).

In both cases, one can show that the limit \( \rho_0(a) \equiv \lim_{\delta \to 0} \rho(a \delta) \rho(\delta) \) exists and it is of the form \( \rho_0(a) = a^\omega \). The number \( \omega \) is called the order of singularity of the distribution. We note for further use the following fact:

**Lemma 3.1** If \( d_i \), \( i = 1, 2 \) are two distributions with order of singularity \( \omega_1 \neq \omega_2 \) then the distribution \( d \equiv d_1 + d_2 \) has the order of singularity \( \omega = \max(\omega_1, \omega_2) \).

If \( \omega_1 = \omega_2 \) then the same assertion stays true if the two distributions have different supports.

Let us define the causal cones with apex \( x \) by:

\[
V^+(x) \equiv \{ y \in \mathbb{R}^4 | (y - x)^2 \geq 0, \ y_0 \geq x_0 \}, \quad V^-(x) \equiv \{ y \in \mathbb{R}^4 | (y - x)^2 \geq 0, \ y_0 \leq x_0 \}
\]  (3.1.3)

and the following subsets from \( \mathbb{R}^{4n} \):

\[
\Gamma^+_n(x) \equiv \{ (x_1, \ldots, x_n) \in \mathbb{R}^{4n} | x_j \in V^+(x), \ \forall j = 1, \ldots, n \}.
\]  (3.1.4)

Then we say that the distribution \( d(x) \in \mathcal{S}'(\mathbb{R}^{4n}) \) has causal support if we have \( \text{supp}(d) \subset \Gamma^+_n(0) \cup \Gamma^-_n(0) \). We say that the couple of distributions \( (a, r) \) is a causal decomposition of a distribution \( d \) with causal support if we have

\[
d = a - r, \quad \text{with} \quad \text{supp}(a) \subset \Gamma^+_n(0), \quad \text{supp}(r) \subset \Gamma^-_n(0).
\]  (3.1.5)

It is possible to show that such a decomposition always exists. We can give explicit formulæ for the Fourier transforms of \( a \) and \( r \).
Proposition 3.2 Suppose $\omega < 0$. In this case, the causal splitting (3.1. 5) is unique if we impose that the distributions $a$ and $r$ have the same order of singularity $\omega$ as the distribution $d$. Moreover, we have the following expressions:

$$\tilde{a}(p) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{dt}{t + i0} \tilde{d}(p - tv), \quad \tilde{r}(p) = \frac{i}{2\pi} \int_{\mathbb{R}} \frac{dt}{t - i0} \tilde{d}(p - tv)$$  \hspace{1cm} (3.1. 6)

where $v \in \Gamma^+_n(0)$ is arbitrary. In particular, this means that the right hand sides of the preceding formulæ do not depend on the choice of $v$.

We call the preceding causal splitting the minimal solution.

In the case of positive singularity order, the formulæ are more complicated. We give them for a particular but important case.

Proposition 3.3 Suppose $\omega > 0$ and that there exists an open set $E \subset \mathbb{R}^m$ such that $0 \in E$ and $E \cap \text{supp}(\tilde{d}) = \emptyset$.

Then, for any $\omega' \geq \omega$ there exists a causal distribution given by the following formulæ for $p \in \Gamma^+_n(0) \cup \Gamma^-_n(0)$:

$$\tilde{a}(p) = \frac{1}{2\pi i} \int_{\mathbb{R}} dt \frac{\tilde{d}(tp)}{t^{\omega'+1}(t - 1 - i0)}, \quad \tilde{r}(p) = \frac{1}{2\pi i} \int_{\mathbb{R}} dt \frac{\tilde{d}(tp)}{t^{\omega'+1}(t - 1 + i0)}.$$  \hspace{1cm} (3.1. 7)

For an arbitrary $p$ these distributions can be reconstructed by analytic continuation. The order of singularity of the distributions $a$ and $r$ is $\omega'$.

We call the splittings given above the central solution. For $\omega' = \omega$ we say that we have a minimal solution. There is a difference between the two case, namely, for positive order of singularity, one can find other splittings of the distribution $d$ such that $\omega(a) = \omega(r) = \omega(d)$. In fact, the arbitrariness can be shown to be an arbitrary polynomial of maximal degree $\omega$ in $p$.

We also note that if the distribution $d$ has some supplementary invariance properties, as for instance, Poincaré invariance, the central splitting preserves them.

Now we define in an abstract setting the notion of anomaly. Let $d^\mu$ be a set of distributions with order of singularity $\omega$ and with causal support. Let us define the distribution

$$d \equiv \frac{\partial}{\partial x^1} d^\mu.$$  \hspace{1cm} (3.1. 8)

Suppose that one can find some distribution splittings $d^\mu = a^\mu - r^\mu$ and $d = a - r$ such that

$$a = \frac{\partial}{\partial x^1} a^\mu, \quad r = \frac{\partial}{\partial x^1} r^\mu.$$  \hspace{1cm} (3.1. 9)

Then we say that there is no anomaly. On the contrary, if such a splitting does not exists, we call the expression

$$A \equiv a - \frac{\partial}{\partial x^1} a^\mu$$  \hspace{1cm} (3.1. 10)
an anomaly. The main problem in establishing the gauge invariance of the $S$-matrix (or in our approach, the possibility of factorizing $S$ to the physical Hilbert space) comes from the possible appearance of anomalies in the process distribution splitting.

We can describe rather well the expression of the anomalies in two cases and we will see that these cases are all we need to perform the splittings in the next Subsection. We have:

**Proposition 3.4** In the conditions described above, suppose that we have $\omega(d^\mu) < 0$ and $\omega(d) < 0$. Then, if we apply the distribution splitting described by prop. 3.2 to $d^\mu$ and $d$, there is no anomaly.

The proof consists in simply computing the anomaly according to the definition (3.1. 10) and observe that it is proportional to $v$. But $v$ is arbitrary, so it can be made arbitrary small.

The case of positive order of singularity is a little more complicated. In general, we have

$$\omega(d) = \omega(d^\mu) + 1 \quad (3.1. 11)$$

because the operation of derivation, increases the order of singularity. However, this is not the rule, and there are cases when this do not happen i.e. we have

$$\omega(d) < \omega(d^\mu) + 1. \quad (3.1. 12)$$

This is exactly the case when anomalies do appear and we have their explicit expression given in the following proposition:

**Proposition 3.5** In the conditions described in the statement of the proposition 3.3, suppose that we have $\omega \equiv \omega(d^\mu) > 0$, $\omega_0 \equiv \omega(d) > 0$ and $\omega_0 < \omega + 1$. Then, if we apply the minimal distributions splitting described by prop. 3.3 to $d^\mu$ and $d$, the anomaly is given by the following formula for $p \in \Gamma_+^n(0) \cup \Gamma_-^n(0)$

$$\tilde{A}(p) = \frac{1}{2\pi i} \int_\mathbb{R} dt \left( \frac{1}{t^{\omega + 2}} + \ldots + \frac{1}{t^{\omega_0 + 2}} \right) \tilde{d}(tp) \quad (3.1. 13)$$

(and for arbitrary $p$ by analytic continuation.) The anomaly given above is a polynomial in $p$ having only the terms with the degree of homogeneity between $\omega$ and $\omega_0$. 

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3.2 The Derivation of the Anomaly

In this Section we will analyze the possible obstructions to factorization of the $S$-matrix in the third order of the perturbation theory. In principle, there is no difference with respect to the preceding Section. Nevertheless, the details of distribution splitting are considerably more complicated and the same is true for the whole combinatorial argument. As before, we will investigate the action of the BRST operator $d_Q$ on the third order commutator

$$D_3(x, y; z) = A'(x, y; z) - R'(x, y; z)$$  \hspace{1cm} (3.2. 1)

where

$$A'(x, y; z) = T_1(z)\overline{T}_2(x, y) - T_2(x, z)\overline{T}_1(y) - T_2(y, z)\overline{T}_1(x),$$  \hspace{1cm} (3.2. 2)

$$R'(x, y; z) = \overline{T}_2(x, y)T_1(z) - \overline{T}_1(y)T_2(x, z) - \overline{T}_1(x)T_2(y, z);$$  \hspace{1cm} (3.2. 3)

here we have put, for simplicity: $T_2 \equiv T_2^s, \overline{T}_2 \equiv \overline{T}_2^s$.

As in the case of the second order factorization condition, the effect of applying the BRST operator on the third order commutator is a total divergence expression. This follows easily from the definitions given above and formulae (2.3. 8) and a similar one for the second order chronological products (2.3. 23) + (2.3. 24). In fact, a much more convenient expression can be found:

**Proposition 3.6** There exists a formula of the following type:

$$d_Q D_3(x, z; y) = i \frac{\partial}{\partial x^\mu} D_1^\mu(x, y; z) + i \frac{\partial}{\partial y^\mu} D_2^\mu(x, y; z) + i \frac{\partial}{\partial z^\mu} D_3^\mu(x, y; z)$$  \hspace{1cm} (3.2. 4)

where the distributions $D_i^\mu(x, y; z), i = 1, 2, 3$ have causal support, that’s it supp($D_i^\mu(x, y; z)$) $\subset$ $\Gamma_2^\mu(0) \cup \Gamma_{2}^{-}(0)$.

**Proof:**

We apply to the equations (3.2. 2) and (3.2. 3) the BRST operator and use the formulæ (2.3. 8) and (2.3. 23). We get the following expressions for the distributions $D_i^\mu(x, y; z), i = 1, 2, 3$:

$$D_1^\mu(x, y; z) = [T_1(z), \overline{T}_1^\mu(x, y)] - [T_1^\mu(x, z), \overline{T}_1(y)] - [T_2(y, z)\overline{T}_1^\mu(x)],$$

$$D_2^\mu(x, y; z) = [T_1(z), \overline{T}_2^\mu(x, y)] - [T_2(z, x), \overline{T}_1^\mu(y)] - [T_2^\mu(y, z)\overline{T}_1(x)],$$

$$D_3^\mu(x, y; z) = [T_1^\mu(z), \overline{T}_2(x, y)] - [T_2^\mu(x, z), \overline{T}_1(y)] - [T_2^\mu(y, z)\overline{T}_1(x)].$$  \hspace{1cm} (3.2. 5)

Now the support properties can be proved using the causal support properties exhibited in the proposition 2.11 and applying some ideas from [6].

To proceed further, we need some alternative expressions for the distributions with causal support $D_i^\mu(x, y; z), i = 1, 2, 3$. The idea is that in the preceding expressions there are two types of terms: ones with a factor of the type $\delta(x - y)$ and the others. It is convenient to separate them. It is also convenient to eliminate completely the antichronological products. The result is the following one:
Proposition 3.7 The following formulae are true:

\[ D_1^\mu(x, y; z) = [R_0^\mu(x, y), T_1(z)] - [R_0^\mu(x, z), T_1(y)] + [T_1^\mu(x), A_2^0(y, z)] -
\]

\[ i\delta(x - y)[L^\mu(x), T_1(z)] - i\delta(x - z)[L^\mu(x), T_1(y)] + i\delta(y - z)[T_1^\mu(x), L(y)], \]

\[ D_2^\mu(x, y; z) = D_1^\mu(x, y; z) \]

\[ D_3^\mu(x, y; z) = [R_0^\mu(x, y), T_1^r(z)] - [R_0^\mu(x, z), T_1(y)] - [R_0^\mu(z, y), T_1(x)] -
\]

\[ i\delta(x - y)[T_1^r(z), L(x)] + i\delta(z - x)[L^\mu(z), T_1(y)] + i\delta(y - z)[L^\mu(z), T_1(x)]. \] (3.2. 6)

Now we present the main result.

Theorem 3.8 The distribution \( T_3(x, y, z) \) induces (in the adiabatic limit) an well defined expression on the (physical) factor space if and only if, beside the conditions from the statement of theorem 2.2, we also have the following set of supplementary conditions:

(I) we have either: (a) \( V_{abc} \neq 0 \); or (b) \( V_{abc} = A_{abc} = 0 \). Here the expressions \( V_{abc} \) and \( A_{abc} \) have been defined in the introduction by the formulae:

\[ V_{abc} \equiv \sum_{\epsilon_1, \epsilon_2, \epsilon_3 = \pm} (1 + \epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_3\epsilon_1) \text{Tr} (t_a^1[t_b^2, t_c^3]) \] (3.2. 7)

and respectively

\[ A_{abc} \equiv \sum_{\epsilon_1, \epsilon_2, \epsilon_3 = \pm} (\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_1\epsilon_2\epsilon_3) \text{Tr} (t_a^{\epsilon_1}[t_b^2, t_c^3]). \] (3.2. 8)

(II) we must also have

\[ S_{ab}S_{cd}f_{abc}f_{def}f_{dgh} = 0. \] (3.2. 9)

(III) Moreover, a number of restrictions must be imposed on the scalar coupling:

- for \( m_a, m_e, m_d, m_h \neq 0 \):

\[ S_{abgh} \sum_{m_a, m_d \neq 0} \frac{1}{m_b} f_{abc}^* [(f'_{dcb} + f'_{deh}) f_{deg}^* + 2f'_{deg}f_{dgh}] = 0, \] (3.2. 10)

- for \( m_a, m_g, m_h \neq 0, m_e = 0 \):

\[ S_{agh} \sum_{m_a, m_d \neq 0} \frac{1}{m_b} f_{abc}^* f_{dhe}^* f_{dgh} = 0, \] (3.2. 11)

- for \( m_a, m_e, m_g \neq 0, m_h = 0 \):

\[ S_{abeg} \sum_{m_a, m_d \neq 0} \frac{1}{m_b} f_{abc}^* [(f'_{dcb} + f'_{deh}) f_{deg}^* + 2f'_{deg}f_{dgh}] = 0, \] (3.2. 12)

- for \( m_a, m_g \neq 0, m_e = m_h = 0 \):

\[ S_{ag}S_{ch} \sum_{m_a, m_d \neq 0} \frac{1}{m_b} f_{abc}^* f_{dhe}^* f_{dgh} = 0 \] (3.2. 13)
- for \( m_a \neq 0, \ m_g = m_e = m_h = 0 \):

\[
S_{egh} \sum_{m_b, m_d \neq 0} \frac{1}{m_b} f_{abc} f_{dbe} f_{dgh} = 0. \tag{3.2.14}
\]

In the preceding relations, we have denoted by \( S_{a,b,...} \) the operator of symmetrization in the corresponding indices.

**Proof:**

The proof is extremely long and intricate, although the idea is, basically simple: we have to split causally the equality (3.2.4) and see if there are some obstructions to this process. The justification of this procedure can be found in the previous two papers dedicated to this topic and references quoted there. To gain some simplicity, we divide it in a number of steps.

(i) If we group linear independent Wick monomials, then we can write the first distribution from (3.2.6) as follows:

\[
D_{\mu_1} = D_{1a}^{\mu} + \text{delta terms} \tag{3.2.15}
\]

where

\[
D_{1a}^{\mu}(x,y;z) = \sum_{j,n} d_{j;\nu_1,\nu_n}^{\mu}(x,y;z) W_{j;\nu_1,\nu_n}(x,y;z) + \sum_{i,n} d_{i;\nu_1,\nu_n}^{\mu}(x,y;z) W_{i;\nu_1,\nu_n}(x,y;z); \tag{3.2.16}
\]

here the (numerical) distributions of the type \( d \) must have also causal support. In fact, the general structure of these distributions is:

\[
d(x,y;z) = \sum d_{1;}(x-y)d_{2;}(x-z)d_{3;}(y-z). \tag{3.2.17}
\]

Here the distributions \( d_i(x-y), \ i = 1,2,3 \) can be of the type

\[
d_{1;}(x-y) = \prod_{i}(\Phi_0, \phi_i(x)\psi_i(y)\Phi_0),
\]

\[
d_{2;}(x-z) = \prod_{j}(\Phi_0, \phi_j(y)\chi_j(z)\Phi_0),
\]

\[
d_{1;}(y-z) = \prod_{k}(\Phi_0, \psi_k(x)\chi_k(z)\Phi_0) \tag{3.2.18}
\]

with the fields \( \phi(x) \) as factors in \( T_1^{\mu}(x) \) and \( \psi(y) \) (resp. \( \chi(z) \)) as factors in \( T_1(y) \) (resp. \( T_1(z) \)); by the sign \( # \) we indicate that some of the distributions \( d^{(+)} \) from above must be substituted by the corresponding advanced or retarded parts: \( d^{adv(ret)} \).

In fact, the rules of these substitutions are the following: one applies Wick theorem to the expression

\[
X_1(x,y,z) = T_1^{\mu}(x)T_1(y)T_1(z) \tag{3.2.19}
\]

and keeps only the terms where there are effective Wick contractions. Next, one one has to make the substitutions indicated above in some factors. After inspecting the expression of \( D_1^{\mu} \)
from the preceding proposition, we arrive quite naturally at a formula of the type:

\[
d(x, y; z) = \sum \{ d_{1}^{ct}(\xi_1 - \xi_2)[d_{2}^{(c)}(\xi_1)d_{3}^{(c)}(\xi_2) - d_{2}^{(c)}(-\xi_1)d_{3}^{(c)}(-\xi_2)] - d_{2}^{ct}(\xi_1)[d_{1}^{(c)}(\xi_1 - \xi_2)d_{3}^{(c)}(-\xi_2) - d_{1}^{(c)}(\xi_1)d_{3}^{(c)}(\xi_2)] + d_{3}^{ct}(\xi_2)[d_{1}^{(c)}(\xi_1 - \xi_2)d_{2}^{(c)}(\xi_1) - d_{1}^{(c)}(\xi_2 - \xi_1)d_{2}^{(c)}(-\xi_1)] \}.
\]

(3.2. 20)

where we have used the (translation invariant) variables \( \xi_1 \equiv x - z, \quad \xi_2 \equiv y - z. \)

Let us define the distributions

\[
d_{i}^{\mu_1,\ldots,\mu_n} = \frac{\partial}{\partial x^\mu} d_{i}^{\mu_1,\ldots,\mu_n}, \quad d_{j;\mu}^{\nu_1,\ldots,\nu_n} = \frac{\partial}{\partial x^\mu} d_{j;\nu_1,\ldots,\nu_n}.
\]

(3.2. 21)

If we can find a causal distribution splitting of the four sets of distributions such that we have, for instance, for the advanced parts:

\[
d_{i}^{\mu_1,\ldots,\mu_n} = \frac{\partial}{\partial x^\mu} d_{i}^{\mu_1,\ldots,\mu_n}, \quad d_{j;\mu}^{\nu_1,\ldots,\nu_n} = \frac{\partial}{\partial x^\mu} d_{j;\nu_1,\ldots,\nu_n}.
\]

(3.2. 22)

then it can be easily seen that one can causally decompose the distribution \( d_Q D_{1a}^\mu \) such that the total divergence structure is preserved. More precisely, if we define similarly to (3.2. 16):

\[
A_{1a}^\mu(x, y; z) = \sum_{i, n} d_{i}^{\mu_1,\ldots,\mu_n}(x, y; z)W_{i;\mu_1,\ldots,\mu_n}(x, y; z) + \sum_{j, n} d_{j;\mu}^{\nu_1,\ldots,\nu_n}(x, y; z)W_{j;\nu_1,\ldots,\nu_n}(x, y; z)
\]

(3.2. 23)

then this is the advanced part of \( D_{1a}^\mu(x, y; z) \) and moreover, the expression \( \frac{\partial}{\partial x^\mu} A_{1a}^\mu(x, y; z) \) is the advanced part of \( \frac{\partial}{\partial x^\mu} D_{1a}^\mu(x, y; z) \).

So, if (3.2. 22) can be fulfilled, there will be no anomalies coming from the first piece of \( d_Q D_3 \).

Comparing to the results of the preceding Subsection, we start to investigate, in our particular case, the possibility of appearance of anomalies.

(ii) It is important to establish a standard procedure of splitting of the distributions. We will do this below in such a way that we will be able to apply the propositions 3.2 and 3.3 from the preceding Subsection. We have to circumvent somehow the possibility that the conditions of the proposition 3.3 are not met. This is particularly important because if there are null-mass particles in the theory, some of the distributions appearing in the expression \( d_Q D_3 \) will certainly not fulfill this requirements; more precisely the point 0 will be included, in general, in the spectrum of the Fourier transform of some of these distributions. Although it is possible to modify the formula from prop. 3.3 such that these cases are also covered (see [4]), this modification makes the analysis more complicated that it already is! We will prefer a different trick.

Let us consider a typical distribution (3.2. 20) and exhibit the dependence on the masses: \( d(\mu_1, \ldots, \mu_i) \). The problems are created by those distributions for which some of the parameters \( \mu_1, \ldots, \mu_i \) can be null. We first can prove that the preceding distribution admits a Taylor
expansion with rest:
\[
d(\mu_1, \ldots, \mu_l) = d(\mu, \ldots, \mu) + \sum (\mu_i - \mu) \frac{\partial d}{\partial \mu_i}(\mu, \ldots, \mu) + \ldots + \frac{1}{k!} \sum (\mu_{i_1} - \mu) \cdots (\mu_{i_k} - \mu) \frac{\partial^k d}{\partial \mu_{i_1} \ldots \partial \mu_{i_k}}(\mu, \ldots, \mu) + r_{k+1}(\mu_1, \ldots, \mu_l)
\]
where the last contribution is the Taylor rest and \( \mu > 0 \) is an arbitrary parameter. Now it is not difficult to see that for expressions of the type (3.2. 20) the preceding series has the following nice properties:

- every term in the sum has the order of singularity with at least one unit less than the preceding term. This means that is we take \( k \) sufficiently large we can make the order of singularity of the Taylor rest \( \leq 2 \);
- if the distribution \( d \) has causal support, then every term in the series has also causal support; indeed the operations derivation and of fixing the parameters to the value \( \mu \) cannot destroy the support properties. In fact the structure of the type (3.2. 20) is conserved by this operations;
- all the terms in (3.2. 24), except the last one, have in the momentum space the following structure (we use the notation \( P \equiv p_1 + p_2 \)):
\[
\theta(p_1^2 - \lambda_1^2)g_1 + \theta(p_2^2 - \lambda_2^2)g_2 + \theta(P^2 - \lambda_3^2)g_3
\]
with \( \lambda_i > 0, \quad i = 1, 2, 3 \). In particular, these terms meet the conditions of proposition 3.5.

As a consequence, we adopt the following standard procedure of splitting distributions of the type (3.2. 20): for all terms of the series (3.2. 24) we apply proposition 3.5 and for the Taylor rest we apply proposition 3.4. As a consequence of prop 3.4 the Taylor rest does not produce anomalies. The nice thing about this recipe is that one can compute the anomalies for the first \( k \) terms of (3.2. 24) using the minimal central decomposition formula.

(iii) Now we apply the splitting procedure described above and we have two cases.

(I) The first one is given by the first terms of the formula (3.2. 16) i.e. the distributions do not carry the index \( \mu \); the index appears in the corresponding Wick monomial. It is not hard to see that in this case we have:
\[
\omega \left( \frac{\partial d^\mu}{\partial x^\mu} \right) = \omega(d) + 1;
\]
as a consequence, proposition 3.5 shows that these terms do not produce anomalies.

(II) The second case corresponds to the second term in (3.2. 16) and we have distributions carrying the index \( \mu \). There are two subcases:

(a) the index \( \mu \) comes from a derivative

In this case one investigates a typical term of (3.2. 20); the Fourier transform of such a term has the structure
\[
\tilde{d}^\mu(p_1, p_2) = \text{const.} \int_{\mathbb{R}^4} r^\mu \tilde{d}_1(r) \tilde{d}_2(p_1 - r) \tilde{d}_3(p_2 + r)
\]
or
\[ \hat{d}(p_1, p_2) = \text{const.} \int_{\mathbb{R}^4} (p_1 - r^\mu)\hat{d}_1(r)\hat{d}_2(p_1 - r)\hat{d}_3(p_2 + r). \]

One investigates now if it is possible that \( \omega(\partial_x^\mu) < \omega(d) + 1 \). By applying the derivation, in the preceding formulæ we get \( r^\mu \mapsto p_1 \cdot r \) and respectively \( p_1^\mu - r^\mu \mapsto p_1^2 - p_1 \cdot r \) so the order of singularity increases, except for the case when the variable \( r \) is fixed in the first (resp. the second) case to 0 (resp. to \( p_1 \)) i.e we have \( \hat{d}_1(r) \sim \delta(r) \) (resp. \( \hat{d}_2(s) \sim \delta(s) \)). But it is easy to see that this cannot happen in all terms appearing into formulæ of the type (3.2. 20). So, this case does not produce anomalies.

(b) the index \( \mu \) comes from a matrix \( \gamma^\mu \)

In this case, we can have two subcases:

(b1) \( d^\mu \sim (\gamma^\mu)_{\alpha\beta} f \)

This case does not lead to anomalies, as in case (a) above.

(b2) \( d^\mu = Tr (\gamma^\mu \Gamma) \hat{f} \) with \( \Gamma \) a matrix depending on \( (p_1, p_2) \).

In this case we have
\[ \hat{d} = Tr (\gamma \cdot p_1 \Gamma) \hat{f} \]
so the term of maximal degree in \( p \) can cancel from purely algebraic reasons, that’s it by taking of the trace.

We conclude that there are two type of terms which can produce anomalies:

(A) those of the type (b2) from above in which a trace is present;

(B) those coming from the delta terms appearing in the formula (3.2. 6). These terms have been analyzed previously in [10] and we will mention briefly the outcome using our methods, at the very end of the proof.

(iv) We concentrate now on the anomalies of origin (A). There are four types of terms \( X_1^\mu \) (see (3.2. 19)) which can produce such an anomaly; let us list them:

\[
\begin{align*}
X_1^\mu(x, y; z) &= u_a(x) j_\mu^a(x) A_{bp}(y) j_\mu^p(y) A_{c\lambda}(z) j_\lambda^c(z), \\
X_2^\mu(x, y; z) &= u_a(x) j_\mu^a(x) \Phi_b(y) j_\mu^b(y) \Phi_c(z) j_\lambda^c(z), \\
X_3^\mu(x, y; z) &= u_a(x) j_\mu^a(x) A_{bp}(y) j_\mu^b(y) \Phi_c(z) j_\lambda^c(z), \\
X_4^\mu(x, y; z) &= u_a(x) j_\mu^a(x) \Phi_b(y) j_\mu^b(y) A_{c\lambda}(z) j_\lambda^c(z). 
\end{align*}
\tag{3.2. 27}
\]

In these terms we have to consider Wick contractions leading to traces. It is not hard to see that in this case the generic expression (3.2. 16) becomes:

\[
\begin{align*}
D_1^\mu(x, y; z) &= d_{abc}^{\mu\lambda}(x, y; z) : u_a(x) A_{bp}(y) A_{c\lambda}(z) : + d_{abc}^{\mu\mu}(x, y; z) : u_a(x) \Phi_b(y) \Phi_c(z) : + \\
&\quad d_{abc}^{\mu\lambda}(x, y; z) : u_a(x) \Phi_b(y) A_{c\lambda}(z) : + \tilde{d}_{abc}^{\mu\mu}(x, y; z) : u_a(x) A_{bp}(y) \Phi_c(z) : + \\
&\quad d_{abc}^\mu(x, y; z) u_a(x) + \cdots. \tag{3.2. 28}
\end{align*}
\]

where by \( \cdots \) we mean the terms which cannot produce anomalies.

According to the general strategy developed so far, we have to compute explicitly the five types of distributions appearing in this formula and investigate if, after applying the operator \( \frac{\partial}{\partial x^\mu} \) it is possible that the order singularity does not increase by an unit, as we would normally
expect. If we index the two chiral pieces of the currents from the formulæ (2.3. 34) and (2.3. 35) by appointing an \( \epsilon \) which can take the values + or −, then it is clear that every one of the operators \( X_i^\mu \) listed above are, in fact a sum of eight contributions: \( X_i^\mu = \sum_{(\epsilon)} X_i^{\mu(\epsilon)} \) where we have denoted \( (\epsilon) \equiv (\epsilon_1, \epsilon_2, \epsilon_3) \). Moreover, every operator of the type \( X_i^{\mu(\epsilon)} \) is a sum of eight terms because we have in every chiral component we have a vector piece coming from the factor \( \frac{1}{2} \gamma^\mu \) and an axial piece coming from the piece \( \frac{1}{2} \gamma^\mu \gamma_5 \). So, in fact we have a further decomposition

\[
X_i^{\mu(\epsilon)} = X_i^{\mu(\epsilon)}_{(VVV)} + X_i^{\mu(\epsilon)}_{(VAA)} + \cdots X_i^{\mu(\epsilon)}_{(AAA)}.
\]

So, all in all, every distribution appearing into the formula (3.2. 28) is formed of \( 4 \times 4 \times 4 \) pieces. We investigate a typical piece. ¿From the generic formula (3.2. 20) one can find, paying attention to the signs appearing in Wick theorem for Fermions:

\[
d_{abc;VVV}^{\mu}(\xi_1, \xi_2) = \frac{1}{8} (f_{a}^{\mu})_{AB}(f_{b}^{\mu})_{BC}(f_{c}^{\mu})_{CA}
\]

\[
Tr[S_A^{(-)}(-\xi_1)\gamma^\mu S_B^{(+)}(\xi_1 - \xi_2)\gamma^\rho S_C^{(+)\gamma}(\xi_2)\gamma^\lambda - S_A^{(+)}(-\xi_1)\gamma^\mu S_B^{(+)}^\dagger(\xi_1 - \xi_2)\gamma^\rho S_C^{(-)}(\xi_2)\gamma^\lambda
\]

\[
- S_A^{(-)}(-\xi_1)\gamma^\mu S_B^{(+)}(\xi_1 - \xi_2)\gamma^\rho S_C^{(+)}(\xi_2)\gamma^\lambda + S_A^{(+)}(-\xi_1)\gamma^\mu S_B^{(+)}^\dagger(\xi_1 - \xi_2)\gamma^\rho S_C^{(-)}(\xi_2)\gamma^\lambda
\]

\[
+ S_A^{(-)}(-\xi_1)\gamma^\mu S_B^{(+)}(\xi_1 - \xi_2)\gamma^\rho S_C^{(++)\alpha}(\xi_2)\gamma^\lambda - S_A^{(+)}(-\xi_1)\gamma^\mu S_B^{(+)\alpha}(\xi_1 - \xi_2)\gamma^\rho S_C^{(++)\alpha}(\xi_2)\gamma^\lambda]
\]

The piece \( d_{abc;VAA}^{\mu(\epsilon)} \) is obtained by making \( \gamma^\rho \rightarrow \epsilon_2 \gamma^\rho \gamma_5 \) and \( \gamma^\lambda \rightarrow \epsilon_3 \gamma^\lambda \gamma_5 \), etc. It is convenient to group all the terms in which there remains no \( \gamma_5 \) factor into a vector part \( d_{abc;V}^{\mu(\epsilon)} \) and the rest into an axial part \( d_{abc;A}^{\mu(\epsilon)} \).

Next, one should perform a Fourier transform of the distribution given above and after that construct the the Taylor series (3.2. 24). We consider the first term of the series

\[
f_{abc}^{\mu(\epsilon)} \equiv d_{abc;V}^{\mu(\epsilon)}(M, \ldots, M)
\]

i.e. we put all the Fermion masses equal to a certain positive value \( M \).

After some very tedious computations the following result is obtained:

\[
\tilde{f}_{abc}^{\mu(\epsilon)}(p_1, p_2) = \sum_{i=1}^{3} f_{abc(i)}^{\mu(\epsilon)}(p_1, p_2)
\]

(3.2. 31)

where the three terms correspond to the three pieces of the generic expression (3.2. 25). The structure of these components is given below:

\[
f_{abc(i)}^{\mu(\epsilon)}(p_1, p_2) = \frac{i}{8(2\pi)^6} \int_{\mathbb{R}^4} dr \left[ \theta(r_0 - p_{10})\theta(-r_0 + P_0) - \theta(-r_0 + p_{10})\theta(r_0 - P_0) \right]
\]

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by the dots we understand contributions with lower order of singularity. Variables \( r, p \) and by the highest power part of the polynomials. So, we have after some computations that:

\[ \text{lemma 3.1. Next, we can easily establish that the order of singularity of these terms is given} \]

\[ \text{by studying the three terms individually and applying} \]

\[ \epsilon(p_0)\theta(p_0^2 - 4M^2)g^{\mu\rho\lambda}_{ab}(p_1, p_2), \quad (3.2.32) \]

\[ \tilde{f}^{\mu\rho\lambda}_{abc(2)}(p_1, p_2) = \frac{i}{8(2\pi)^6} \int_{\mathbb{R}^4} dr \left\{ \right. \theta(r_0 + P_0)\theta(-r_0 - p_{20}) - \theta(-r_0 - P_0)\theta(r_0 + p_{20}) \left. \right\} \]

\[ \delta((r + P)^2 - M^2)\delta((r + p_2)^2 - M^2) \frac{P^{\mu\rho\lambda}_{abc(2)}}{r^2 - M^2 - i\epsilon} = \]

\[ \epsilon(p_2)\theta(p_2^2 - 4M^2)g^{\mu\rho\lambda}_{ab}(p_1, p_2), \quad (3.2.33) \]

\[ \tilde{f}^{\mu\rho\lambda}_{abc(3)}(p_1, p_2) = \frac{i}{8(2\pi)^6} \int_{\mathbb{R}^4} dr \left\{ \right. \theta(r_0 - p_{10})\theta(-r_0 - p_{20}) - \theta(-r_0 + p_{10})\theta(r_0 + p_{20}) \left. \right\} \]

\[ \delta((r - p_1)^2 - M^2)\delta((r + p_2)^2 - M^2) \frac{P^{\mu\rho\lambda}_{abc(3)}}{r^2 - M^2 - i\epsilon} = \]

\[ \epsilon(P_0)\theta(P^2 - 4M^2)g^{\mu\rho\lambda}_{ab}(p_1, p_2), \quad (3.2.34) \]

where \( P_i \) are some polynomials having terms of third order and of first order degree in the variables \( r, p_1 \) and \( p_2 \).

In particular, they have different support properties, so one can compute the order of singularity of the preceding distribution by studying the three terms individually and applying lemma 3.1. Next, we can easily establish that the order of singularity of these terms is given by the highest power part of the polynomials. So, we have after some computations that:

\[ \tilde{f}^{\mu\rho\lambda}_{abc(i)}(p_1, p_2) = V^{abc}_{abc(i)(V)}(p_1, p_2) + A^{abc}_{abc(i)(A)}(p_1, p_2) + \cdots \quad (3.2.35) \]

Here the expression \( V^{abc}_{abc} \) and \( A^{abc}_{abc} \) have been defined by the formulæ (3.2.7) and (3.2.8); by the dots we understand contributions with lower order of singularity.

The vector and axial parts from this formula are induced by the corresponding decomposition of the polynomials from (3.2.32)-(3.2.34) into a vector and a axial part:

\[ P^{\mu\rho\lambda}_{abc(i)} = P^{\mu\rho\lambda}_{abc(i)(V)} + P^{\mu\rho\lambda}_{abc(i)(A)}, \quad i = 1, 2, 3. \quad (3.2.36) \]

We give only the expression for \( i = 3 \).

\[ P^{\mu\rho\lambda}_{abc(3)(V)} = V^{\alpha\gamma\beta}(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\rho \gamma^\gamma \gamma^\lambda)(r - p_1)\alpha r_\beta(r + p_2) \quad (3.2.37) \]

and

\[ P^{\mu\rho\lambda}_{abc(3)(A)} = A^{\alpha\gamma\beta}(\gamma^\alpha \gamma^\mu \gamma^\beta \gamma^\rho \gamma^\gamma \gamma^\lambda)(r - p_1)\alpha r_\beta(r + p_2) \quad (3.2.38) \]

Now it is clear that we must study the order of singularity of distributions of the following type:

\[ \tilde{f}^{\mu\rho\lambda}_{3(V)}(p_1, p_2) = \int_{\mathbb{R}^4} dr \left\{ \right. \theta(-r_0 + p_{10})\theta(r_0 + p_{20}) - \theta(r_0 - p_{10})\theta(-r_0 - p_{20}) \left. \right\} \]

\[ \delta((r - p_1)^2 - M^2)\delta((r + p_2)^2 - M^2) \frac{1}{r^2 - M^2 - i\epsilon} = \quad (3.2.39) \]

\[ \frac{1}{r^2 - M^2 - i\epsilon} \]
and
\[ \tilde{f}_{3(A)}^{\mu\rho\lambda}(p_1, p_2) = \int_{\mathbb{R}^4} dr \{ \theta(-r_0 + p_{10})\theta(r_0 + p_{20}) - \theta(r_0 - p_{10})\theta(-r_0 - p_{20}) \} \\
\delta((r - p_1)^2 - M^2)\delta((r + p_2)^2 - M^2) Tr(\gamma_5\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\lambda)(r - p_1)_\alpha r_\beta (r + p_2)_r \frac{1}{r^2 - M^2 - i\epsilon}. \] (3.2. 40)

Then we should contract them with \( p_{1\mu} \) and see what happens with the order of singularity. The results are obtained by elementary computations. We have:
\[ \omega(\tilde{f}_{3(A)}^{\mu\rho\lambda}) = 1. \] (3.2. 41)

Now we turn to the distributions
\[ \tilde{f}_{3(V)}^{\rho\lambda}(p_1, p_2) \equiv p_{1\mu}\tilde{f}_{3(V)}^{\mu\rho\lambda}(p_1, p_2), \quad \tilde{f}_{3(A)}^{\rho\lambda}(p_1, p_2) \equiv p_{1\mu}\tilde{f}_{3(A)}^{\mu\rho\lambda}(p_1, p_2) \] (3.2. 42)

and find that something interesting, finally happens. Indeed we have:
\[ \omega(\tilde{f}_{3(V)}^{\rho\lambda}) = \omega(\tilde{f}_{3(A)}^{\rho\lambda}) + 1 \] (3.2. 43)

but
\[ \omega(\tilde{f}_{3(A)}^{\rho\lambda}) = \omega(\tilde{f}_{3(A)}^{\rho\mu\lambda}). \] (3.2. 44)

The same analysis goes through for the other two terms from the formula (3.2. 31). So, finally we obtain two distinct cases:

(a) \( V_{abc} \neq 0 \)

In this case we have
\[ \omega(\tilde{f}_{abc}) = \omega(\tilde{f}_{abc}) + 1 \] (3.2. 45)

and the distributions do not produce anomalies when we split them causally.

(b) \( V_{abc} = 0 \)

We have two subcases:

(b1) \( A_{abc} \neq 0 \)

In this case an anomaly can appear. The explicit computation uses the proposition 3.5 and goes as in [9]. In particular, only the last term of the expression (3.2. 31) gives something non-trivial and one obtains:
\[ A_{abc}^{\rho\lambda} = \frac{1}{6i} A_{abc}\epsilon^{\alpha\beta\gamma\delta} p_1_{\alpha} p_2_{\beta}. \] (3.2. 46)

(b2) \( A_{abc} = 0 \)

In this case the distribution \( \tilde{f}_{abc}^{\mu\rho\lambda} \) vanishes and from the proposition 3.5 it follows that the anomaly must be a constant (independent of \( p_1 \) and \( p_2 \)). We start again from the generic expression (3.2. 29) for the distribution \( d_{abc}^{\mu\rho\lambda} \) and observe from considerations of Lorentz invariance the generic expression (3.5) of the anomaly that we must have
\[ A_{abc}^{\rho\lambda} = K_0\epsilon^{\rho\lambda\alpha\beta} p_1_{\alpha} p_2_{\beta} + K_1 p_1^{\rho} p_1^{\lambda} + K_2 p_2^{\rho} p_2^{\lambda} + K_3 p_1^{\rho} p_2^{\lambda} + K_4 p_2^{\rho} p_1^{\lambda} \] (3.2. 47)
with $K_i$ functions of the invariants $p^2_1$, $p^2_2$ and $P^2$. It is clear now that these functions has to be in fact equal to 0.

We must consider now the next term in the Taylor series (3.2. 24) for the distribution $d_{\mu\lambda}^{\rho\gamma\delta}$. It is easy to prove that this term can have the order of singularity at most 0 so it can produce an anomaly which is a constant. Then, the same reasoning as before applies and there are no anomalies. The third term of the Taylor series can be also be analyzed and, again, it does not produce anomalies. In conclusion, the distribution $d_{\mu\nu\lambda}^{\rho\gamma\delta}$ can produce anomalies only in the case $V_{abc} = 0$, $A_{abc} \not\equiv 0$.

We now analyze the other distributions from the expression (3.2. 28). We begin with $d_{\mu\nu\lambda}^{\rho\gamma\delta}$ and observe that this distribution can be obtained from $d_{\mu\nu\lambda}^{\rho\gamma\delta}$ by making some simple transformations: $\gamma^\rho, \gamma^\lambda \rightarrow 1, t^\alpha_1 \rightarrow s^\alpha_1$ and $t^\alpha_3 \rightarrow s^\alpha_3$. But in this case the axial part of this distribution will be null because $Tr(\gamma^5 \gamma^\alpha \gamma^\beta) = 0$ so, repeating the same argument as before, we conclude that the second term from (3.2. 28) does not produces anomalies.

The next two terms of the formula (3.2. 28) have the structure (3.2. 32)-(3.2. 34) but the polynomials are of degree 2. This means that these distributions can have the order of singularity at most 0. But in this case, the anomaly must be a constant. On the other hand, a direct inspection of the anomaly, starting from the proposition 3.5 and using consideration of Lorentz covariance, leads to a generic form of the anomaly:

$$A^\lambda = p^\lambda_1 L_1 + p^\lambda_2 L_2$$

where $L_i$ are functions of the invariants $p^2_1$, $p^2_2$ and $P^2$. As before, it is clear that these functions has to be in fact equal to 0.

Finally, we analyze the last contribution from (3.2. 28) i.e. the distribution $d_{a}^{\mu\nu\lambda}$. First, we prove as before that the axial part is 0. Next, we observe from Lorentz covariance arguments that we must have

$$d_{a}^{\rho}(p_1, p_2) = p^\lambda_1 A_1 + p^\lambda_2 A_2$$

so

$$\tilde{d}_a(p_1, p_2) = p^\mu_1 \tilde{d}_a^{\mu}(p_1, p_2) = p^\lambda_1 A_1 + p_1 \cdot p_2 A_2.$$ 

This implies that we have $\omega(d_a) = \omega(d_{a}^{\mu}) + 1$ and, according to prop. 3.5, there are no anomalies.

So, we can summarize the result of the analysis for the first term $D_{A}^{\mu}$ from the formula (3.2. 6) as follows: the anomalies can appear only in the case $V_{abc} = 0$, $A_{abc} \not\equiv 0$ and in this case, the explicit expression in the coordinates space is

$$A = -\frac{i}{6} A_{abc} e^{\alpha\lambda\beta\gamma} \delta(x - z) \delta(y - z) : \frac{\partial}{\partial x^\alpha} u_a(x) \frac{\partial}{\partial x^\beta} A_{b\gamma}(x) A_{c\lambda}(x) :$$

(3.2. 48)

which is not a coboundary, as can be easily proved.

It is easy to prove that the second term from the formula (3.2. 6) produces the same anomaly.

An important difference comes in the analysis of the last term of (3.2. 6). We follow the same line as before, and get distributions of the type (3.2. 39) and (3.2. 40). But now we must
contract these distributions with $P_{\mu}$ because the distribution $D_{\mu}^a$ is differentiated with respect to $z$. It is easy to prove that in this case the axial part is null, so there are no anomalies.

In conclusion, we can have only the anomaly (3.2.48) in the conditions $V_{abc} = 0, \quad A_{abc} \neq 0$ and we obtain the first condition from the statement of the theorem.

(v) We still have to investigate the possible anomalies coming from the delta terms from the expressions (3.2.6) $D_{\mu}^a$. As we have said before, these terms have been analyzed in [10]. We present here briefly the analysis of these terms using our technology and we find a new condition of consistency which seems to be missing in this references. For $i$ we mean the rest of the commutator which cannot produce anomalies. Now, as $\delta(y-z) \{ T_{\mu}^a(x), L(y) \}$ can produce anomalies. One can compute the commutator and select the terms which will lead, in principle, to an anomaly. We get:

$$[T_{\mu}^a(x), L(y)] = f_{abc} f_{dcb} f_{dgh} \frac{\partial}{\partial x_{\mu}} D_{mc}(x - y) : u_a(x) A_{\mu b}(x) A_{\lambda c}(y) A_{\rho d}(y) A_{\lambda e}(y) :$$

$$- \frac{1}{2} f_{abc} f_{dcb} f_{dgh} \frac{\partial}{\partial x_{\mu}} D_{mc}(x - y) : u_a(x) A_{\rho b}(x) A_{\lambda c}(y) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$

$$+ 2 f_{abc} f_{dfb} f_{dgh} \frac{\partial}{\partial x_{\mu}} D_{mc}(x - y) : \Phi_a(x) u_c(x) A_{\lambda}^{\rho}(x) A_{\rho e}(y) \Phi_{\lambda}(y) :$$

$$+ \frac{3}{2m_b} \frac{\partial}{\partial x_{\mu}} D_{mc}(x - y) [f_{abc} f_{dfb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$

$$+ f_{abc} f_{dcb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$

$$+ f_{abc} f_{dcb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\lambda}(y) :] + \cdots (3.2.49)$$

where by $\cdots$ we mean the rest of the commutator which cannot produce anomalies. Now, as in [7] and [8] we get from this commutator a possible anomaly:

$$A = A_1 + A_2 + A_3 + A_4$$  \hspace{1cm} (3.2.50)

where

$$A_1 = \delta(y-z) \delta(x-z) f_{abc} f_{dcb} f_{dgh} : u_a(x) A_{\mu b}(x) A_{\lambda c}(y) A_{\rho d}(y) A_{\lambda e}(y) :$$  \hspace{1cm} (3.2.51)

$$A_2 = - \frac{1}{2} \delta(y-z) \delta(x-z) f_{abc} f_{dcb} f_{dgh} : u_a(x) A_{\rho b}(x) A_{\lambda c}(y) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$  \hspace{1cm} (3.2.52)

$$A_3 = 2 \delta(y-z) \delta(x-z) f_{abc} f_{dfb} f_{dgh} : \Phi_a(x) u_c(x) A_{\lambda}^{\rho}(x) A_{\rho e}(y) \Phi_{\lambda}(y) :$$  \hspace{1cm} (3.2.53)

and

$$A_4 = \delta(y-z) \delta(x-z) \sum_{m_a, m_b, m_d, m_c \neq 0} \frac{3}{2m_b} f_{abc} f_{dcb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$

$$+ \sum_{m_a, m_b, m_d, m_c \neq 0} \frac{3}{2m_b} f_{abc} f_{dcb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\lambda}(y) :$$

$$+ \sum_{m_a, m_b, m_d, m_c \neq 0} \frac{3}{m_b} f_{abc} f_{dcb} f_{dgh} : \Phi_a(x) u_c(x) \Phi_{\rho}(y) \Phi_{\rho}(y) \Phi_{\lambda}(y) :]$$  \hspace{1cm} (3.2.54)
But in [3] it is proved that the first expression is, in fact, null due to the Jacobi identity. Also the third expression can be shown to be null. We are left with the other terms which are not coboundaries. The other anomalies coming from the other delta terms from $D_2^\mu$ and $D_3^\mu$ can be analyzed similarly. In the end, they lead to the conditions from the statement.

We have analyzed in full generality the possibilities of coupling non-trivially heavy Bosons of spin one with Dirac Fermions and determined some restrictions of the type of the ABJ anomaly which are new in the literature. These restrictions might put severe restrictions on the models and deserve further investigations.

We end with the following proposition

Proposition 3.9 The expression $V_{abc}$ has the alternative expression

$$V_{abc} = i f_{bcd} Tr(t_a t_d + t'_a t'_d).$$

(3.2. 55)

In particular, there exist Lie groups for which this tensor is not identically null.
References


