Quantum corrections to the geodesic equation

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In this talk we will argue that, when gravitons are taken into account, the solution to the semiclassical Einstein equations (SEE) is not physical. The reason is simple: any classical device used to measure the spacetime geometry will also feel the graviton fluctuations. As the coupling between the classical device and the metric is non linear, the device will not measure the 'background geometry' (i.e. the geometry that solves the SEE). As a particular example we will show that a classical particle does not follow a geodesic of the background metric. Instead its motion is determined by a quantum corrected geodesic equation that takes into account its coupling to the gravitons. This analysis will also lead us to find a solution to the so-called gauge fixing problem: the quantum corrected geodesic equation is explicitly independent of any gauge fixing parameter.

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I. INTRODUCTION

In quantum field theory there are many physical situations where one is interested in the dynamical evolution of fields rather than in S-matrix elements. The effective action (EA) is a useful tool to obtain the equations that govern such dynamics including the backreaction effects due to quantum fluctuations. However, there are two important problems that should be solved before one can get meaningful equations.

On the one hand, when the usual effective action is used to derive evolution equations, these turn out to be neither real nor causal. The cause is that the EA gives evolution equations for “in-out” matrix elements of the background fields. In order to obtain real and causal equations for expectation values, a different EA (“in-in” EA) has been introduced, which permits a correct approach to initial value problems [1]. On the other hand, both the in-out and the in-in effective actions are not physical quantities off-shell. This is most easily seen in the context of gauge theories, where the EA depends on the gauge fixing condition. The scattering matrix is constructed going on-shell, and therefore it does not suffer from this problem. The equations of motion, on the contrary, are obtained from the off-shell EA, and are thus gauge fixing dependent. The standard approach to tackle this problem is to consider the Vilkovisky-DeWitt effective action [2], which is specifically built to give a reparametrization, gauge invariant action. However, this action suffers from another type of arbitrariness, namely the dependence on the supermetric in the space of fields that is introduced in its definition [3,4].

Backreaction effects on the spacetime metric are relevant in different physical situations like gravitational collapse and black hole evaporation. Any discussion of the backreaction problem should include the effect of gravitons which contribute to the one loop effective stress tensor with terms of the same order as those coming from ordinary matter fields [5]. When graviton loops are included, the metric $g_{\mu\nu}$ that solves the semiclassical Einstein equations depends on the gauge fixing, and as such it is not physical. As an example we can mention calculations of compactification radii in Kaluza-Klein theories [6]. In view of the dependence of the results on the gauge fixing, people turned to the Vilkovisky-DeWitt EA as a way to overcome this setback [7]. However it was eventually shown that this approach was also incomplete because the results depend on the supermetric for the fields manifold [3].

In this talk we put forward a solution different from that advocated in the Vilkovisky-DeWitt EA. Our point is that, due to its interaction with the quantum fluctuations of the gravitational field, a test particle will not follow a $g_{\mu\nu}$-geodesic. Instead its motion is governed by a quantum corrected geodesic equation, which must be gauge fixing independent. Therefore, the solution of the backreaction problem consists of two steps: to solve the semiclassical

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Einstein equations and to extract the physical quantities from the solution. It is this second point that, to our knowledge, has never been considered before.

In order to illustrate these facts we will consider the calculation of the leading quantum corrections to the Newtonian potential. As has been pointed out in [8,9], when General Relativity is looked upon as an effective field theory, low energy quantum effects can be studied without the knowledge of the (unknown) high energy physics. The leading long distance quantum corrections to the gravitational interactions are due to massless particles and only involve their coupling at energies low compared to the Planck mass. Using this idea, many authors have calculated the leading quantum corrections to the Newtonian potential computing different sets of Feynman diagrams [8–11]. Instead of evaluating diagrams and S-matrix elements, we are here concerned with a covariant calculation based on the effective action and effective field equations. This covariant approach is more adequate to study many interesting problems in which one considers fluctuations around non-flat backgrounds, like black hole evaporation, gravitational collapse and backreaction in cosmological settings, among others. We shall first compute the semiclassical Einstein equations for the backreaction problem starting from the standard EA and show how they depend on the gauge fixing. Using a corrected geodesic equation we will deduce a physical quantum corrected Newtonian potential, which does not depend on any gauge fixing parameter. We will also discuss briefly the quantum corrections to the geodesic equation in a cosmological context.

II. THE SEMICLASSICAL EINSTEIN EQUATIONS

The action for gravity coupled to a heavy particle (a classical source) has the form [12]

$$S_G + S_M = \frac{2}{\kappa^2} \int d^4x \sqrt{-g} R - M \int \sqrt{-g} g_{\mu\nu} dx^\mu dx^\nu,$$

where $\hat{R}$ is the curvature scalar, $\bar{g}_{\mu\nu}$ is the metric tensor, $\bar{g} = \det \bar{g}_{\mu\nu}$, and $\kappa^2 = 32\pi G$, with $G$ being Newton’s constant.

In the background field method quantum fluctuations of the gravitational field may be expanded around a background metric, $\bar{g}_{\mu\nu} = g_{\mu\nu} + \kappa s_{\mu\nu}$, and a function $\chi^\mu$ is chosen to fix the gauge, which is implemented through a gauge fixing action

$$S_{gf} = -\frac{1}{2\kappa^2} \int d^4x \sqrt{-g} g_{\mu\nu} \chi^\mu \chi^\nu. \quad (2)$$

We shall consider the one-parameter family of gauge fixing functions, the so-called $\lambda$ family,

$$\chi^\mu(\lambda) = \frac{1}{\sqrt{1 + \lambda}} \left[ g^{\mu\gamma} \nabla^\gamma s_{\gamma\sigma} - \frac{1}{2} g^{\gamma\sigma} \nabla^\mu s_{\gamma\sigma} \right]. \quad (3)$$

For gauge fixing functions linear in the metric fluctuations, ghosts decouple from the fluctuations $s_{\mu\nu}$ and only couple to the background fields. The one loop effective action for the background metric is obtained from integrating out quantum fluctuations and implies the evaluation of functional determinants for gravitons and ghosts in the presence of the background fields. For the pure gravitational action $S_G$, the one loop divergencies in the DeWitt gauge $\lambda = 0$ have been calculated long ago using dimensional regularization and turn out to be local terms quadratic in the curvature tensors [13]. They read

$$\Delta S_G^{\text{div}} = \frac{2}{(4 - d)96\pi^2} \int d^4x \sqrt{-g} \left[ \frac{21}{10} R_{\mu\nu} R^{\mu\nu} + \frac{1}{20} R^2 \right], \quad (4)$$

where we have omitted the Gauss-Bonnet term, which is a topological invariant in $d = 4$ spacetime dimensions. Apart from the local parts, the one loop EA also has non-local components. These have been computed up to quadratic order in the curvature tensors through a resummation procedure of the Schwinger DeWitt expansion for the action [14]. In what follows we shall be working to order $O(R^3)$ at the level of the action and to order $O(R^2)$ in the equations of motion. The non-local, non-analytic terms proportional to $\ln(-\Box)$ are the relevant ones in order to compute the leading quantum corrections. They can be read off from the divergencies in Eq. (4) in a manner outlined in [8,14]

$$\Delta S_G^{\text{nl}} = -\frac{1}{96\pi^2} \int d^4x \sqrt{-g} \left[ \frac{21}{10} R_{\mu\nu} \ln(-\Box) R^{\mu\nu} + \frac{1}{20} R \ln(-\Box) R \right]. \quad (5)$$
The second term in Eq. (1) introduces an additional contribution to the EA. Following the method described in [15], one can compute the new divergence, from which it is possible to find the $\ln(-\Box)$ part of the EA arising from the presence of the mass $M$. After a long calculation we get

$$\Delta S^\text{nl}_M = -\frac{1}{64\pi^2} \int d^4 x \sqrt{-g} \left[ M_{\mu\nu,\rho\sigma} \ln(-\Box) M^{\rho\sigma\mu\nu} + 2 M_{\mu\nu,\rho\sigma} \ln(-\Box) \left( \pi_{\rho\sigma} - \frac{1}{6} R \delta^{(\rho}_{(\mu} \delta^{\sigma)}_{\nu)} \right) \right],$$

(6)

where

$$M_{\mu\nu,\rho\sigma}(y) = \frac{M \kappa^2}{8} \int d\tau \delta^4(y - x(\tau)) \times \left[ g_{\mu\nu} \dot{x}_\rho \dot{x}_\sigma + 2 \ddot{x}_\rho \ddot{x}_\sigma \dot{x}_\mu \dot{x}_\nu \right],$$

(7)

and

$$\pi_{\rho\sigma} = 2 R^{(\rho\sigma)} + 2 \delta^{(\rho}_{[\mu} R^{\sigma]}_{\nu]} - g^{\rho\sigma} R_{\mu\nu} - g^{\mu\nu} R^{\rho\sigma} - R \delta^{(\rho}_{(\mu} \delta^{\sigma)}_{\nu)} + \frac{1}{2} g^{\rho\sigma} g^{\mu\nu} R.$$  

(8)

Here indices in parenthesis or brackets imply symmetrization with a $1/2$ factor.

As we will calculate long distance corrections to the Newtonian potential, we can assume that the mass $M$ is a classical static “point mass”, although its size should be much larger than its Schwarzschild radius and the Planck length, in order to justify the weak field approximation to be done in what follows. Its contribution to the nonlocal part of the EA is

$$\Delta S^\text{nl}_M = \frac{7M \kappa^2}{1536\pi^2} \int d^4 x \sqrt{-g} R \ln(-\nabla^2) \delta^3(\vec{x}),$$

(9)

where the nonlocal operator $\ln(-\nabla^2)$ acts on the delta function as $\ln(-\nabla^2) \delta^3(\vec{x}) = -\frac{1}{\pi} \frac{1}{\vec{x}}$ [16]. Adding the classical and quantum contributions of the EA and taking functional derivations with respect to the metric, it is straightforward to compute the semiclassical Einstein equations including backreaction of gravitons. They can be derived from the in-in EA, or by taking twice the real and causal part of the in-out equations of motion. Up to linear order in curvatures they are

$$\frac{1}{8\pi G} (R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}) = M \delta^{(\mu}_{\nu} \delta^{(\rho}_{\sigma} \delta^{\delta)(x)} - \frac{1}{96\pi^2} \left[ \frac{21}{10} \ln(-\nabla^2) H_{\mu\nu}^{(2)} + \frac{1}{20} \ln(-\nabla^2) H_{\mu\nu}^{(1)} \right] + \frac{7M \kappa^2}{768\pi^2} (\nabla_{\mu} \nabla_{\nu} - g_{\mu\nu} \nabla^2) \ln(-\nabla^2) \delta^3(\vec{x}),$$

(10)

where we have introduced the tensors $H_{\mu\nu}^{(1)} = 4 \nabla_{\mu} \nabla_{\nu} R - 4 g_{\mu\nu} \nabla^2 R$ and $H_{\mu\nu}^{(2)} = 2 \nabla_{\mu} \nabla_{\nu} R - g_{\mu\nu} \nabla^2 R - 2 \nabla^2 R_{\mu\nu}$. Here we have used the fact that the mass $M$ is static to replace $\Box \rightarrow \nabla^2$.

In order to solve these equations for the background metric we shall make perturbations around flat spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $\eta_{\mu\nu} = \text{diag}(---++)$. We choose the harmonic gauge for the background perturbation metric. It is worth mentioning that this choice is completely independent of the gauge fixing problem for the quantum fluctuations. The 00 component for the perturbation $h_{\mu\nu}$ turns out to be

$$h_{00}(\lambda = 0) = \frac{2GM}{r} \left[ 1 + \frac{43G}{30\pi r^2} - \frac{7G}{12\pi r^2} \right].$$

(11)

The first term is due to the presence of the classical mass $M$ (for simplicity we consider only the Newtonian limit, that is, we do not include classical corrections from general relativity). The second and third terms are quantum corrections. The former stems pure gravitational contributions (vacuum polarization) while the latter arises from the coupling of the mass $M$ to gravitons.

The above result is valid only for the DeWitt $\lambda = 0$ gauge. For any other gauge (not only for the $\lambda$ family of gauge fixings) one has to add a new contribution to the nonlocal part, which should vanish on-shell. Keeping up to quadratic order in curvatures, the requirement that the effective action be gauge fixing independent on-shell fixes the most general form such a nonlocal term can have

$$\Delta S = \int d^4 x \sqrt{-g} \left[ a R_{\mu\nu} \ln(-\Box) \mathcal{E}^{\mu\nu} + b R g_{\mu\nu} \ln(-\Box) \mathcal{E}^{\mu\nu} + \mathcal{O}(\mathcal{E}^2) \right].$$

(12)

Here $\mathcal{E}^{\mu\nu}$ is the classical extremal $\mathcal{E}^{\mu\nu} = -\frac{2}{\pi} (R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu}) + \frac{1}{2} T^{\mu\nu}$, where
\[ T^{\mu\nu}(y) = M \int d\tau \dot{x}^{\mu} \dot{x}^{\nu} \delta^{4}(y - x(\tau)), \]  

is the energy-momentum tensor of the classical source and \( a \) and \( b \) are constants that depend on which particular gauge is used. The reason for omitting terms quadratic in the extremal in Eq. (12) is that, when the equations of motion are perturbatively solved, they vanish identically. This new contribution to the EA modifies the semiclassical equations (10), which will now depend on \( a \) and \( b \). Solving the modified equations we obtain the metric in a general gauge

\[ h_{00} = \frac{2GM}{r} \left[ 1 + \frac{43G}{30\pi r^2} - \frac{7G}{12\pi r^2} + \frac{a - 2b}{r^2} \right]. \]  

The last term is the extra contribution to the perturbation arising from a gauge fixing different from the DeWitt one. For example, for the \( \lambda \) family we have \( a(\lambda) = -\frac{3\lambda}{8\pi^2} \) and \( b(\lambda) = \frac{3\lambda}{96\pi^2} \). It is then clear that the metric that solves the backreaction equations for the one loop quantized gravity depends on which particular function one chooses to fix the gauge.

### III. Quantum Corrected Geodesic Equation: Newtonian Limit

The dependence on the gauge fixing of the gravitons is an obstacle to think of a solution to the SEE as the metric of spacetime. This obstacle is not ‘technical’ (as implicitly assumed in previous works) but physical: since any classical device couples to gravitons, the solution to the SEE will not, in general, have a clear physical interpretation.

To analyze this problem, we will consider the simplest classical device: a test particle of mass \( m \) in the presence of the quantized gravitational field \( g_{\mu\nu} \). A physical observable should be the motion of this particle. We consider that the mass of this particle is much smaller than \( M \), which allows us to neglect all contributions of the test particle to the solution of the one loop corrected equation (14). Now comes the key ingredient: in order to determine how this test particle moves, one also has to take into account the fact that it couples to the quantum metric \( \bar{g}_{\mu\nu} \) through a term \(-m \int \sqrt{-\bar{g}_{\mu\nu}(x)} dx^{\mu} dx^{\nu} \), where \( x^{\mu} \) denotes the path of the test particle. Therefore there will be an extra contribution to the one loop EA due to this coupling to gravitons, which in turn will introduce a correction to the geodesic equation. This contribution is, up to linear order in \( m \),

\[ \Delta S_{m} = \int d^{4}x \sqrt{-g} \left[ -\frac{1}{64\pi^2} m_{\mu\nu\rho\sigma} \ln(-\Box) M^{\rho\sigma\mu\nu} - \frac{1}{32\pi^2} m_{\mu\nu\rho\sigma} \ln(-\Box) \left( P^{\rho\sigma\mu\nu} - \frac{1}{6} R \delta^{(\rho)(\sigma)\mu\nu} \right) + \frac{a}{2} R_{\mu\nu} \ln(-\Box) T^{\mu\nu}_{m} + \frac{b}{2} R g_{\mu\nu} \ln(-\Box) T^{\mu\nu}_{m} \right], \]  

where the tensor \( m_{\mu\nu\rho\sigma} \) is the one given in Eq.(7) with \( M \) replaced by \( m \), and \( T^{\mu\nu}_{m} \) is the energy-momentum tensor for the test particle, given in Eq.(13), with the same replacement. The first two terms correspond to the \( \lambda = 0 \) gauge fixing, and the last two are extra terms appearing for any other gauge. In the weak, nonrelativistic Newtonian limit, the quantum corrected geodesic equation reads

\[ \frac{d^{2}\dot{x}}{dt^{2}} - \frac{1}{2} \nabla h_{00} = \frac{1}{m} \frac{\delta \Delta S_{m}}{\delta \dot{x}}. \]  

Note that \( h_{00} \), given in Eq. (14), depends on \( a \) and \( b \). The term on the rhs can be computed following the same methods we used to solve the backreaction problem. In that way

\[ \frac{\delta \Delta S_{m}}{\delta \dot{x}} = \left[ \frac{5G}{12\pi} - a + 2b \right] \nabla \left( \frac{GM}{r^{3}} \right). \]  

Plugging this expression into the corrected geodesic equation we see that those gauge fixing dependent terms arising from the backreaction metric cancel exactly those coming from the coupling of the test particle to gravitons. In this way we obtain a physical, gauge fixing independent Newtonian potential \( V(r) \) which we read from \( d^{2}\dot{x}/dt^{2} = -\nabla V \), namely

\[ V(r) = \frac{GM}{r} \left[ 1 + \frac{43Gh}{30\pi r^{2}c^{3}} - \frac{7Gh}{12\pi r^{2}c^{3}} + \frac{5Gh}{12\pi r^{2}c^{3}} \right], \]  

where we have restored units (\( \hbar \) and \( c \)). Note that the long distance quantum correction above is extremely small to be measured. However the specific number is less important than the conceptual fact that the potential and motion of the test particle are gauge fixing independent.
IV. COSMOLOGICAL BACKGROUND GEOMETRIES

Up to here we have considered the quantum corrections to the metric and to the test particle trajectory in the Newtonian approximation. We will now briefly comment about the quantum corrections to the geodesics in a cosmological background.

For simplicity, instead of working with a general gauge fixing term (as we did before), we will fix completely the gauge of the quantum fluctuations and describe gravitons by two massless, minimally coupled scalar fields. Moreover, we will assume that we are able to obtain a cosmological solution to the SEE, including graviton fluctuations, and will focus only in the computation of quantum corrections to the geodesic equation. The solution to the SEE will be described by the line element

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

for some function $a(t)$.

The corrections to the geodesic equation can be computed following a procedure similar to the one described in the previous section. The coupling between the gravitons and the test particle modifies the classical action for the particle. Keeping only the corrections that are linear in the mass $m$ the corrected geodesic equation reads

$$\frac{d^2t}{d\tau^2} + a\left(\frac{dx^2}{d\tau}\right)^2 = 4\pi G \frac{dx^j}{d\tau} \frac{dx^k}{d\tau} \frac{dx^l}{d\tau} \frac{dx^m}{d\tau} \frac{\partial}{\partial t} G_{ijkl}$$

Here a dot denotes derivative with respect to $t$. $G_{ijkl}$ is the (renormalized) coincidence limit of the graviton two point function $\langle h_{ij}(x)h_{kl}(x') \rangle$. Note that $G_{ijkl}$ only depends on $t$.

Taking into account the symmetries of the problem, the above equations can be easily solved. Let us assume that, when the graviton contribution is neglected, the particle moves in the $x$ direction ($x = x^1$). Hence, in this case,

$$\frac{dx}{d\tau} = \frac{\alpha}{a^2} \left[1 + \frac{32}{3}\pi G < \varphi^2 > \frac{a^2}{a^2}\right]$$

$$\frac{dt}{d\tau} = \sqrt{1 + \frac{\alpha^2}{a^2}} \left[1 + \frac{32}{3}\pi G < \varphi^2 > \frac{a^2}{1 + \frac{\alpha^2}{a^2}}\right]$$

where $< \varphi^2 >$ is the renormalized coincidence limit of the two point function for a massless, minimally coupled scalar field in the background described by $a(t)$.

From the above equation we can find the quantum correction to the physical velocity of the test particle, induced by its coupling to the gravitons,

$$\frac{dx}{dt} = \frac{\alpha a^{-2}}{\sqrt{1 + \frac{\alpha^2}{a^2}}} \left[1 + \frac{32}{3}\pi G < \varphi^2 > \frac{a^2}{1 + \frac{\alpha^2}{a^2}}\right]$$

In the null limit $\frac{dx}{dt} \simeq \frac{1}{\pi} \left[1 + \frac{32}{3}\pi G < \varphi^2 > \right]$ describes the graviton correction to the cosmological redshift. We expect this correction to be very small for $t \gg t_{Planck}$.

Had we considered a different gauge fixing, we would have obtained a different expression for the quantum corrections to the geodesic equation in a cosmological context. However, the solution to the SEE described by $a(t)$ would also depend on the gauge fixing. As in the case of the Newtonian potential, both dependences should cancel when computing the trajectory of the test particle.
V. CONCLUSIONS

We hope to have convinced you that if one is interested in solving the backreaction problem including the graviton contribution, it is not enough to solve the semiclassical Einstein equations. The solutions are gauge fixing dependent and not physical. Rather one has to look for physical observables. As an illustration of this point we have shown, in the Newtonian approximation, that a classical test particle does not follow a geodesic in the background metric. Moreover, the trajectory is a physical observable independent of the gauge fixing procedure. We have also computed the quantum corrected geodesic equation for a test particle in a Robertson Walker background.

If the calculations presented here are performed using the Vilkovisky-DeWitt effective action, we expect the dependence on the supermetric to cancel and the resulting quantum corrected geodesic equations to coincide with the ones obtained by means of the conventional effective action.

Similar ideas to the one proposed here can be applied to more general situations and even to the analysis of the mean value equations of any gauge theory, for example when computing gluon backreaction effects on classical solutions to Yang Mills theories.

The results presented in this talk are contained in Refs. [17,18].

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[12] Our metric has signature (- + + +) and the curvature tensor is defined as $\bar{R}^\mu_{\nu\alpha\beta} = \partial_\nu \Gamma^\mu_{\alpha\beta} - \ldots$, $\bar{R}_{\alpha\beta} = \bar{R}^\mu_{\alpha\mu\beta}$ and $\bar{R} = \bar{g}^{\alpha\beta} \bar{R}_{\alpha\beta}$. We use units $\hbar = c = 1$.