Curvature and Acoustic Instabilities in Rotating Fluid Disks

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ABSTRACT

The stability of a rotating fluid disk to the formation of spiral arms is studied in the tightwinding approximation in the linear regime. The dispersion relation for spirals that was derived by Bertin et al. is shown to contain a new, acoustic instability beyond the Lindblad resonances that depends only on pressure and rotation. In this regime, pressure and gravity exchange roles as drivers and inhibitors of spiral wave structures. Other instabilities that are enhanced by pressure are also found in the general dispersion relation by including higher order terms in the small parameter $1/kr$ for wavenumber $k$ and radius $r$. We identify two important dimensionless physical parameters: $\epsilon = 2\pi G\sigma_0/(r\kappa^2)$, which is essentially the ratio of disk mass to total mass (disk and halo), and $a/(\kappa r)$, which is the ratio of epicyclic radius to disk radius ($\sigma_0$ is the mass column density, $\kappa$ is the epicyclic frequency, and $a$ is the sound speed). The small term $\zeta = (k^2r^2 + m^2)^{-1/2}$ is an additional parameter that is purely geometrical for number of arms $m$. When these terms are included in the dispersion relation, the oscillation frequency becomes complex, leading to the growth of perturbations even for large values of Toomre’s parameter $Q$. The growth rate is proportional to a linear combination of terms that depend on $\epsilon$ and $a/(\kappa r)$. Instabilities that arise from $\epsilon$ are termed gravitational-curvature instabilities because $\epsilon$ depends on the disk mass and is largest when the radius
is small, i.e., when the orbital curvature is large. Instabilities that arise from $a/(kr)$ are termed \textit{acoustic-curvature} instabilities, because they arise from only the pressure terms at small $r$.

Unstable growth rates are determined for these instabilities in four cases: a self-gravitating disk with a flat rotation curve, a self-gravitating disk with solid body rotation, a non-self-gravitating disk with solid body rotation, and a non-self-gravitating disk with Keplerian rotation. The most important application appears to be as a source of spiral structure, possibly leading to accretion in non-self-gravitating disks, such as some galactic nuclear disks, disks around black holes, and proto-planetary disks. All of these examples have short orbital times so the unstable growth time can be small, even when only terms of order $\epsilon$ contribute.

1. Introduction

Spiral galaxies are characterized by bright "arms" spiraling out from a region near the center. Differential rotation will shear and wind these arms quickly if they are material features, so Lindblad (1958) and Lin & Shu (1964) developed a theory of density waves to overcome this winding dilemma. Lin & Shu (1964,1966) also obtained the dispersion relation for these waves, which is the relation between frequency and wavenumber. An important discriminant in this dispersion relation is the stability parameter $Q$ for axisymmetric disturbances (Toomre 1964); when $Q > 1$, the disk is stable against ring-like disturbances.

Lau & Bertin (1978) included additional terms that treated tangential forces for fluid spiral waves in a uniform disk, finding an additional destabilizing term they called $J$. They used a WKB approximation and ignored curvature terms, which scale inversely with galactocentric radius. Goldreich & Lynden-Bell (1965), Zang (1976), and Toomre (1981) also studied azimuthal forces, by considering the temporal response of shearing wavelets. Toomre (1981) termed the mechanism responsible for the spectacular growth of shearing waves a "swing amplifier". He found that spiral waves can grow for a short time even when $Q > 1$, as long as $Q$ is not too large.

In section 3.2 below, we discuss a new instability in the usual spiral wave equations derived by Bertin et al. (1989; hereafter BLLT) that is relevant beyond the Lindblad resonances, i.e., inside the ILR and outside the OLR, even when $Q > 1$. This is a regime that BLLT did not consider. The new instability depends on shear and self-gravity as in the BLLT derivation, but it also has a component in the absence of self-gravity that arises only from pressure and rotation. We therefore refer to it as an \textit{acoustic} instability.
We also derive dispersion relations for fluid disks considering the curvature terms and other terms that were ignored in these previous studies, such as radial variations of the basic properties of the disk. Our additional terms depend on two dimensionless parameters,

$$\epsilon \equiv \frac{2\pi G\sigma_0}{r\kappa^2}$$

(1)

and $a/(\kappa r)$, for mass column density $\sigma_0$, radius $r$, epicyclic frequency $\kappa$, and sound speed $a$. Typically $\epsilon \sim 0.1$, which is small, so our new results are not important modifications to previous studies that considered only small $Q$. However, in regions where $Q$ is large, the additional terms lead to residual instabilities that can be important in some situations.

Numerical and analytical solutions to the modified dispersion relation are found here for typical regions in galactic and other disks. These include the main disks of spiral galaxies, where the rotation curves are approximately flat (Rubin et al. 1985); the inner disks of galaxies, where the rotation curves are approximately solid body; inner solid-body gaseous disks that are not self-gravitating (e.g., NGC 2207; Elmegreen et al. 1998), and non-self-gravitating Keplerian disks, as might be appropriate for proto-planetary disks or galactic nuclear regions surrounding black holes (Nakai et al. 1993).

2. The General Dispersion Relation

The dynamical response of an infinitely thin fluid disk to perturbation density waves will be studied here, considering various degrees of approximations using algebraic expansions in terms of small parameters. The disk response to spiral waves is considered to be weak enough for the linearized equations of motion to be valid. The effects of self-gravity, pressure, and differential rotation are included. The pressure is assumed to depend only on the density; in the formulation, enthalpy is used. In the analysis, perturbation variables are assumed to be of the form $g_1(r, \theta, t) = G(r)e^{i \int k(r)dr}e^{i(\omega t - m\theta)}$, where $r$ is the radius, $\theta$ is the azimuthal angle, $\omega$ is the frequency of oscillation if it is real, and the growth or decay rate if it is imaginary, $m$ is the number of arms, $k(r)$ is the radial wavenumber, and $G(r)$ is the slowly varying amplitude. The spiral waves have an interarm spacing that is much shorter than the radius, that is $\zeta \equiv 1/|\hat{k}r| \ll 1$ for total wavenumber $\hat{k} = \sqrt{k^2 + m^2/r^2}$. This condition is satisfied either for very short waves or for open spirals with many arms, and it allows asymptotic solutions to the density response. The same condition is used to express the density as a linear function of the gravitational potential (Bertin & Mark 1979).

The linearized equations of motion are combined with the continuity equation to relate the perturbation enthalpy $h_1$ to the perturbation gravitational potential $\phi_1$ (Goldreich &
where \( \mathcal{L} = \frac{d^2}{dr^2} + A \frac{d}{dr} + B \) and the coefficients are \( A = -\frac{1}{r^2} d \ln \mathcal{A}/d \ln r \), \( B = -\frac{m}{r^2} + (2m\Omega/r^2\kappa\nu) d \ln (\kappa^2 (1 - \nu^2)/\sigma_0\Omega)/d \ln r \), and \( C = -\kappa^2 (1 - \nu^2)/a^2 \); also \( \mathcal{A} = \kappa^2 (1 - \nu^2)/(\sigma_0 r) \), where \( \nu \) is the dimensionless frequency, \( \nu = (\omega - m\Omega)/\kappa \), \( m \) is the number of arms in the spiral pattern, \( \kappa \) is the epicyclic frequency, \( \sigma_0 \) is the surface density of the disk, \( \Omega(r) \) is the angular frequency, and \( a \) is the sound speed in the disk. The perturbation gravitational potential can be expressed in the form

\[
\phi_1(r) = \Phi(r) e^{i \int k(r) dr};
\]

then Poisson’s equation is (Bertin & Mark 1979):

\[
\sigma_1 = -\frac{\sigma_0}{a^2} f(r) \phi_1,
\]

with the definition

\[
f(r) \equiv \frac{1}{2\pi Gr K(\alpha, m)} \left[ 1 + i\hat{\mathcal{A}}(\alpha) r \frac{d \alpha}{dr} + \hat{\mathcal{B}}(\alpha) r^2 \frac{d^2 \alpha}{dr^2} + \hat{\mathcal{C}}(\alpha) \left( r \frac{d \alpha}{dr} \right)^2 \right],
\]

and the approximation

\[
K(\alpha, m) = \tau \left( 1 + \frac{m + 1/2}{2} r^2 \right),
\]

\[
\tau = (\alpha^2 + (m + 1/2)^2)^{-1/2},
\]

\[
\alpha = kr - irr\Phi'/\Phi - i/2.
\]

This expansion for \( f(r) \) is correct to third order in \( \alpha \). The terms \( \hat{\mathcal{A}}, \hat{\mathcal{B}}, \) and \( \hat{\mathcal{C}} \) are defined in Bertin & Mark (1979); they are:

\[
\hat{\mathcal{A}}(\alpha) = K_2 - K_1^2
\]

\[
\hat{\mathcal{B}}(\alpha) = K_1^3 + K_3 - 2K_1K_2
\]

\[
\hat{\mathcal{C}}(\alpha) = 9K_1^2K_2 - 6K_1K_3 + 3K_4 - 3K_1^4 - 3K_2^2,
\]

where

\[
K_n = \frac{1}{n! K(\alpha, m)} \frac{\partial^n K(\alpha, m)}{\partial \alpha^n}.
\]

The enthalpy, \( h_1 = a^2 \sigma_1/\sigma_0 \), can be expressed in terms of the potential \( \phi_1 \) using equation (3) to obtain

\[
h_1 = -f(r) \phi_1.
\]
The expression for \( f \), equation (4), can be expanded in the small parameter \( \zeta \) to get
\[
 f(r) = \left( \frac{\hat{k}}{k_J} \right) \left[ 1 + i f_1 \zeta + f_2 \zeta^2 + (f_3 + i f_4) \zeta^3 + \ldots \right].
\]
Here, \( k_J \equiv 2\pi G\sigma_0/a^2 \) is the two-dimensional equivalent of the Jeans wavenumber. The terms \( f_i \) are real and depend on derivatives of \( k \) and \( \Phi \). For example,
\[
f_1 = \left( \frac{\hat{k}}{k} \right) \left[ -1/2 - \frac{1}{2} \frac{\Phi'}{\Phi} - \left( 1 + \frac{r k'}{k} \right) \left( \frac{m^2}{2\hat{k}^2 r^2} \right) \right].
\]

If only the first term is kept in the expansion of \( f(r) \) and all radial gradients and the \( m \)-dependence of \( \hat{k} \) is dropped, the Lin & Shu (1966) dispersion relation is obtained:
\[
(\omega - m\Omega)^2 = \kappa^2 - 2\pi G\sigma_0 |k| + k^2 a^2.
\]
In terms of the dimensionless frequency, \( \nu = (\omega - m\Omega)/\kappa \), dimensionless wavelength, \( \eta = k_{crit}/|k| \geq 0 \), where \( k_{crit} = \kappa^2/2\pi G\sigma_0 \), and Toomre’s stability parameter \( Q = \kappa a/\pi G\sigma_0 \), the Lin-Shu relation is
\[
\nu^2 = 1 - \frac{1}{\eta} + \frac{Q^2}{4\eta^2}.
\]

3. Tangential forces and the stability parameter \( J \)

3.1. The Bertin-Lin-Lowe-Thurstans dispersion relation

In the derivation of the Lin-Shu dispersion relation, which is equation (7) above, terms of magnitude \( m/kr \) are ignored. Thus the dispersion relation is accurate for radial oscillations only. When the azimuthal wavenumber \( m/r \) is included, the gravitational instability is stronger (Lau & Bertin 1978). In the derivation of the corresponding dispersion relation, Lau and Bertin made the assumptions that in Poisson’s equation the out of phase (i.e., imaginary) terms can be ignored and the wavenumber \(|k| \sim k_J/2\). Defining the total wavelength to be \( \lambda_m = 2\pi/\sqrt{k^2 + m^2/r^2} \), Poisson’s equation becomes
\[
-\phi_1 = G\sigma_1 \lambda_m,
\]
and equation (2) is
\[
\frac{(\sigma_1/\sigma_0)}{\text{in phase}} = \frac{h_1 + \phi_1}{\kappa^2 - (\omega - m\Omega)^2} \left[ -\frac{4\pi^2}{\lambda_m^2} - \frac{T_1}{(1 - \nu^2)} \right],
\]
where \( T_1 = -(2m\Omega/\kappa r)^2 (d\ln\Omega/d\ln r) \). Note that the last term in the equation above contains \((1 - \nu^2)\), which was not present in (C15) in Lau & Bertin (1978) because they were considering solutions near corotation (\( \nu \sim 1 \)). However, \( T_1/(1 - \nu^2) \) can be derived from their equations (B6) and (B9), it comes from their second term in equation (B9); in fact, Bertin et al. (1989) included it in their dispersion relation. Lau & Bertin (1978) also
dropped the fifth term in (C14) when they derived (C15) because it is higher order in $1/kr$. We do the same for equation (9) because this section is about the low order terms as well. We include all of these terms in the higher order analysis in the rest of the paper.

The dispersion relation for spiral waves, which is analogous to equation (7), is now

$$Q^2/4 = \hat{\eta} - \frac{(1 - \nu^2)}{\hat{\eta}^{-2} + J^2/(1 - \nu^2)},$$

(10)

where $\hat{\eta} = k_{\text{crit}}/\hat{k}$ and $J^2 = T_1/k_{\text{crit}}^2$, as defined in Bertin et al. (1989). We call equation (10) the Bertin-Lin-Lowe-Thurstan (BLLT) dispersion relation with dimensionless frequency $\nu_{BLLT}$. It describes the response of a differentially rotating disk to spiral perturbations. Evidently, the response is stronger than for axisymmetric perturbations by a factor that depends on the parameter $J$.

Equation (10) was studied extensively by Lau & Bertin (1978) and Bertin et al. (1989) in the limit when $\nu \approx 0$, which is near corotation. In this limit, equation (10) predicts an instability when the frequency is purely imaginary, and this occurs when

$$1 + \left(\frac{Q^2}{4\hat{\eta}^2} - \frac{1}{\hat{\eta}}\right)\left(1 + J^2\hat{\eta}^2\right) < 0.$$  

(11)

For ring-like perturbations ($m = 0$ and $J = 0$), equation (11) is satisfied when $Q < 1$; that is, equation (11) reduces to Toomre’s (1964) instability condition, $Q < 1$, for the axisymmetric case.

It is seen from equations (7) and (10) that when the imaginary terms in the equation of motion and Poisson’s equation are ignored (Hunter 1983), the dimensionless frequency is pure real or pure imaginary according to the values of $\hat{\eta}$ and $Q$ and for small values of $J^2$. The exclusion of these imaginary terms is justified in the limits $|kr| >> 1$ and $k_{\text{crit}}r >> 1$. This latter quantity is $\epsilon^{-1}$, defined by equation (1). If the complex terms are included in the equation of motion and Poisson’s equation, then the frequency solutions are complex functions of $\hat{\eta}$ and $Q$. In that case, the frequency $\nu$ contains a non-vanishing imaginary part in all of the parameter space $(\eta, Q)$. This means there is always some instability present, consisting of an oscillation plus growth, so $Q$ is not an absolute discriminant of stability for small $J$ when higher order terms in $\epsilon$ are included. These new instabilities will be discussed in detail in sections 4 and 5, but first we consider the low-order BLLT equation in the region beyond the Lindblad resonances.
3.2. A modification to the BLLT equation beyond the Lindblad Resonances

In addition to the instability condition given by equation (11), the BLLT dispersion relation (Eq. 10) predicts another instability when the frequency $\nu$ is complex and has a real component with an absolute value larger than 1.

This is a different regime of position relative to the resonances than considered by BLLT. They were concerned mostly with instabilities near corotation, where the waves are evanescent. For this reason, they took $\nu \sim 0$. In this section, we consider stability properties inside the inner Lindblad resonance ($\nu < -1$) and outside the outer Lindblad resonance ($\nu > 1$), using the same order of approximation as in BLLT. These are regions that were considered to be damped and radiative, respectively, in the BLLT model. We show that the same dispersion relations also allow solutions that grow as they oscillate, i.e., with complex frequencies.

The condition for this second instability may be obtained from the square root part of the solution for $\nu^2$ in equation (10), and is:

$$\left( \frac{Q^2}{4\hat{\eta}^2} - \frac{1}{\hat{\eta}} \right) \left( \frac{Q^2}{4\hat{\eta}^2} - \frac{1}{\hat{\eta}} - 4J^2\hat{\eta}^2 \right) < 0.$$  (12)

This condition can be written in the form

$$\frac{1}{\hat{\eta}} < \frac{Q^2}{4\hat{\eta}^2} < \frac{1}{\hat{\eta}} + 4J^2\hat{\eta}^2,$$  (13)

which is the same as

$$\epsilon \hat{k}r < \frac{a^2\hat{k}^2}{\kappa^2} < \epsilon \hat{k}r + \frac{4s^2m^2}{\hat{k}r^2},$$  (14)

if we substitute $Q\epsilon/2 = a/(\kappa r)$ and $\epsilon \hat{\eta} = 1/kr$, and define $J^2/\epsilon^2 \equiv s^2m^2$, where $s = 2(-\Omega r \Omega')^{1/2}/\kappa$ and is of order 1. Equation (14) is a new condition for instability. When this condition is satisfied, the self-gravitating disk is unstable to the growth of spiral waves. The right hand side of equation (14) contains two terms. The first term depends on the self-gravity of the disk and the second depends on shear. When gravity is negligible, there is still instability from the second term, coming entirely from pressure, shear, and Coriolis forces. We refer to this as an acoustic instability; it has apparently not been considered previously in the literature.

Figure 1 shows the unstable regions for a five-arm spiral ($m = 5$) in the ($k_{\text{crit}}/|k|, Q^2$) plane from the BLLT dispersion relation, equation (10), considering a self-gravitating disk with a flat rotation curve ($s^2 = 2$); this case is studied in more detail in the next section. The growth rate is represented as a gray scale, and the borders of the regions of instability
are represented as lines, obtained from the instability conditions. The most unstable region is in the bottom left corner of the figure, where the bottom line shows the stability limit for the Lin-Shu dispersion relation ($m = 0$), which is obtained from equation (7). For $m$ and $J^2 \neq 0$ the border of this region of instability shifts to the line given by the BLLT condition (Eq. 11). The acoustic instability is bracketed by the two upper lines described by equation (13). The lower line corresponds to $\nu^2 = 1$. This occurs at a Lindblad resonance when the Doppler-shifted frequency of oscillation, $(\omega - m\Omega)$, matches the epicyclic frequency, $\kappa$, which is where the self-gravity of the disk is balanced by the pressure force (the Jeans condition) according to equation (10). The upper line corresponds to $\nu^2 = 1 + 2J^2\eta^2$.

We can investigate the instability conditions (11) and (14) further by writing the BLLT dispersion relation without self-gravity. This can be done by multiplying equation (10) by $\epsilon^2$, and then substituting as above. We then let $\epsilon \to 0$ to turn off gravity. The BLLT dispersion relation becomes

$$\nu^4 - \left(2 + a^2\frac{k^2}{\kappa^2}\right)\nu^2 + 1 + \frac{a^2\left(\frac{k^2 + s^2m^2}{r^2}\right)}{\kappa^2} = 0. \quad (15)$$

We combine the contributions to the dispersion relation from the sound speed and the epicyclic frequency by defining an angle $\gamma = \tan^{-1}(a\hat{k}/\kappa)$. We also define an angle $p = \pi + \tan^{-1}(m/kr)$ for $k < 0$; this angle is between $\pi/2$ and $\pi$, giving $\sin p > 0$ and $\cos p < 0$. The standard definition of a spiral arm pitch angle is $\pi - p$. For $\epsilon = 0$, equation (11) is never satisfied, so the BLLT instability disappears, as recognized by these authors. However, the acoustic instability remains, with an instability criterion given by equation (14) with $\epsilon = 0$; this is

$$\frac{a}{kr} < \frac{2s \sin p}{kr} = \frac{2sm}{k^2r^2 + m^2}. \quad (16)$$

Another way to write equation (16) is to remove the explicit radial dependence; then the instability condition becomes

$$\tan \gamma = \frac{a\hat{k}}{\kappa} < 2s \sin p. \quad (17)$$

The left hand side of the inequality in equation (17) is the ratio of the length scale for the epicyclic oscillation to the interarm spacing. This ratio has to be less than order unity for the instability to develop, which means that there has to be room for epicyclic motions within the distance that separates the spiral arms. That is, spiral waves will grow at all wavelengths that have enough room for epicyclic motions at the local sound speed.

When equation (17) is satisfied, a non-self-gravitating fluid disk with differential rotation will be unstable to spiral perturbations inside the ILR and outside the OLR. For a disk with solid body rotation, $s = 0$, for a flat rotation curve, $s = \sqrt{2}$, and for a Keplerian
disk, \( s = \sqrt{6} \), so condition (17) is more easily satisfied, and the growth of instabilities is stronger, with greater shear. From equation (15), the phase velocity, \( c_{\text{ph}} \), and the group velocity, \( c_g \), of the acoustic waves in the radial direction can be obtained. Define \( z = (2s \sin p/ \tan \gamma)^2 \), and \( w_\pm = (1 \pm \sqrt{1 - z})/2 \); then

\[
c_{\text{ph}} = \frac{m \Omega}{k} \pm \frac{\kappa}{\kappa} \sqrt{1 + w_\pm \tan^2 \gamma},
\]

\[
c_g = \frac{\pm \sqrt{1 - z} \sqrt{1 + w_\pm \tan^2 \gamma}}{\sqrt{1 + w_\pm \tan^2 \gamma}}.
\]

In the unstable regime, \( z > 1 \), which implies that \( c_{\text{ph}} \), and \( c_g \) are complex; only the real parts should be taken for the physical phase and group velocities. Note that for trailing waves, which are the only waves considered here, \( \cos p < 0 \). This instability, along with additional instabilities resulting from higher order terms, will be studied further in section 5 for the cases with flat rotation curves and Kepler rotation.

### 3.3. Physical insights to the BL
t extension beyond the Lindblad resonances

The acoustic instability determined by the condition (17) has a different physical origin than the higher-order instability discussed in the next sections. For example, the higher-order instability works with or without shear, but the BL
t extension beyond the Lindblad resonances requires shear ( \( s \neq 0 \) in equation 17). We show in section 6 that the physical origin of the higher order instability is a geometric growth of incoming wavetrains near the nucleus of a galaxy. We do not actually think of this higher-order growth as an instability because it is limited in time to the propagation time over the radius. This is unlike the acoustic instability discussed in the previous section, which is a true instability. The acoustic instability is very similar to the gravity-driven instability of BL
t near corotation, i.e., between the Lindblad resonances, but it is pressure-driven instead, and beyond the Lindblad resonances. We explain here in physical terms how it works.

In normal galactic spirals between the Lindblad resonances, and in bars between corotation and the ILR, the spiral or bar perturbation grows with time because more and more stellar (or fluid particle) orbits lock into phase with the perturbation, and because each new aligned orbit reinforces the perturbation, causing greater and greater forcing. This works for two reasons: (1) In this radial range, an unperturbed epicycle precesses slower than the pattern speed, i.e., the precession speed, \( \Omega - \kappa/m \), is less than the pattern speed, \( \Omega_p \). (2) The inward forcing from the perturbation, gravity in this case, is greatest near the apocenter of the epicyclic orbit. For a spiral arm, this apocenter occurs just outside the
potential minimum of the arm, and is directed inward because of the arm gravity. For a bar, the apocenter is on the bar major axis, and is directed inward because of the gravity of the bar.

Reason (1) implies that in the absence of forcing, a fluid element with its apocenter at the crest of one arm will come in and go out again to the next apocenter before it reaches the next arm. This is because the precession rate is slow and the apocenter of the epicycle twists around in angle more slowly than the spiral pattern. (In other words, the Coriolis force (in $\kappa$) is too large, so the angular velocity perturbation causes too large a radial velocity perturbation and the radial oscillation period is short.) However, with gravity, the excess inward forcing at the apocenter in the first arm crest gives the orbit an extra kick in the radial direction, and this flings the fluid element all the way around to the next arm before it has its next apocenter. Moreover, this kick occurs during the part of the orbit when the fluid is most susceptible to gaining momentum, i.e., when it is moving most slowly and spending the most time (at apocenter). Thus the forced orbit aligns with the perturbation, always having its apocenter in the arm crest. The same occurs for a bar: the presence of an excess inward bar forcing on the major axis of the bar flings the fluid elements around so they have their next apocenter at the other major axis, rather than too early. Thus we see how the gravitational force from spirals and bars causes the epicyclic motions of individual fluid elements to align with the perturbation and strengthen it.

Inside the inner Lindblad resonance, the precession speed of an unforced stellar orbit is greater than the pattern speed, i.e., $\Omega - \kappa/m > \Omega_p$, so normal spiral or bar gravity kicks the stellar orbits the wrong way. That is, the gravity forcing makes an epicycle that already has its next apocenter come too late, meet the next arm even later. For the case of the bar, this leads to a perpendicular alignment of the orbits, so the point of maximum inward forcing, on the bar axis, is at the place in the epicycle, its pericenter, where the fluid element is least susceptible to acquire excess momentum, i.e., where it is moving most quickly. A previous description of this process was given in Elmegreen (1997).

Now consider the influence of pressure on these waves. The pressure forcing in a spiral is out of phase from the gravity forcing. When the gravity forcing is a maximum inward, just outside the spiral potential minimum, the pressure forcing is a maximum in the outward direction, because of the pressure gradient from the compression in the arm crest. Thus pressure is a stabilizing influence on normal spirals and bars between the Lindblad resonances, as is well known. Pressure forcing has to be less than gravity forcing for the spiral to grow. This is the usual condition for the dispersion relation, which equates the wave oscillation frequency to positive (and therefore stabilizing) contributions from acoustic and epicyclic oscillations, plus a negative (and therefore destabilizing) contribution from
self-gravity.

Inside the ILR and outside the OLR, the role of pressure and gravity change. Whereas self-gravity opposes the alignment of epicycles beyond the Lindblad resonances, as discussed above, pressure is in the right phase to support this alignment of epicycles. The maximum outward force from pressure is near the epicycle apocenter both inside and outside the ILR, and the existence of this outward force slows down the fluid at its apocenter in both cases too. But inside the ILR and outside the OLR, this slow down causes the next apocenter to occur in the next arm, rather than after the next arm, which would be the case without the pressure forcing.

The acoustic instability beyond the Lindblad resonances is therefore due to a reversal in the role of gravity and pressure as driving agents for spiral density waves on either side of the Lindblad resonances. Between the ILR and OLR, gravity changes the orbits in such a way that they reinforce an initial perturbation, while pressure opposes this change. Beyond the Lindblad resonances, pressure changes the orbits to reinforce the initial perturbation, while gravity opposes. When gravity is weak beyond the Lindblad resonances, pressure alone is left to drive spiral instabilities.

The sensitivity of the instability condition (17) to shear \((s)\) and pitch angle \((\pi - p)\), which is the same as the requirement that \(\nu^2 < 1 + 2J^2\dot{\eta}^2\), makes sense for such pressure driven spirals. When the pitch angle is large, the maximum inward pressure force occurs closest to the minor axis of the epicycle, and the maximum outward pressure force occurs closest to the major axis. This situation leads to the maximum possible forcing from the pressure gradients. The shear is important because this is what causes the epicycles to precess forward or backward relative to the pattern. Without shear, the precession speed is zero, and no amount of pressure forcing can enhance the spiral alignment of orbits.

4. Higher Order Terms in the General Dispersion Relation

For the general case with self-gravity, it is possible to solve for the complex frequency if we know the basic state of the disk. If the rotation curve, the density distribution, and the sound speed distribution in the disk are known, then the dispersion relation in the tightwinding approximation can be obtained to second order in \(\epsilon\).

The dispersion relation for \(\nu\) is obtained by turning equation (2) into an algebraic expression. This is done by using the definition of the enthalpy and equation (3) to express the enthalpy as a function of the potential and then using the asymptotic form of the potential, \(\phi_1 = \Phi e^{i\int k(r)dr}\). We will consider only trailing spirals \((k < 0)\). Note that
\[ \nu' = -(m\Omega' / \kappa + \nu \kappa' / \kappa) \]

for radial derivatives denoted by primes. Equation (2) can be written in the form:

\[
\left[ \frac{r^2 d^2}{dr^2} + (Ar) \frac{r d}{dr} + Br^2 \right] (h_1 + \phi_1) = \delta^{-1} (1 - \nu^2) h_1,
\]

(18)

where \( \delta = a^2 / (\kappa^2 r^2) \). Multiply equation (18) by \( \delta / h_1 \) and define \( D_0 = \delta (1 - \frac{1}{f}) \), \( D_1 = \delta \frac{d}{d_r} (\phi_1 + h_1) \), and \( D_2 = \delta \frac{1}{\eta} (r^2 d^2 - m^2) (\phi_1 + h_1) \). Then

\[
D_2 + (Ar) D_1 + (Br^2 + m^2) D_0 + \nu^2 - 1 = 0.
\]

(19)

The terms \( D_i \) are

\[
D_0 = \delta \left( 1 - \frac{1}{f} \right),
\]

\[
D_1 = \delta \left[ (ikr + \frac{r \Phi'}{\Phi}) \left( 1 - \frac{1}{f} \right) + \frac{rf'}{f} \right],
\]

\[
D_2 = \delta \left[ (-\hat{k}^2 r^2 + ikr \left( \frac{2r \Phi'}{\Phi} + \frac{r k'}{k} \right) + \frac{2r \Phi''}{\Phi} \right] \left( 1 - \frac{1}{f} \right) + \frac{rf'}{f} \left( ikr + \frac{r \Phi'}{\Phi} + \frac{r f''}{f} \right),
\]

These terms are used to find numerically the roots of the dispersion relation. They can be expanded in the small parameter \( 1/kr \) by using the Bertin & Mark expression of Poisson’s equation (Eqs. 4 and 5). Their expansion is correct to third order in \( 1/kr \), so our dispersion relation is limited to third order in this quantity as well. In terms of \( Q^2 \) and \( \hat{\eta} \), and to lowest order in \( \epsilon \), \( D_i \) become:

\[
D_0 = \epsilon^2 \left( \frac{Q^2}{4} - \hat{\eta} \right) + i \epsilon^3 \hat{\eta}^2 f_1 + ... = \epsilon^2 d_0 + i \epsilon^3 d_{03} + ...
\]

\[
D_1 = i \epsilon \cos p \left( \frac{Q^2}{4 \hat{\eta}} - 1 \right) + \epsilon^2 \left[ \frac{Q^2}{4} \left( \frac{r \Phi'}{\Phi} + \frac{r k'}{k} - \frac{r k_j}{k_j} \right) + \hat{\eta} \left( f_1 \cos p - \frac{r \Phi'}{\Phi} \right) \right] + ...
\]

\[
D_2 = \frac{1}{\hat{\eta}} - \frac{Q^2}{4 \hat{\eta}^2} + i \epsilon \left[ \cos p \left( \frac{Q^2}{4 \hat{\eta}} - 1 \right) \left( \frac{2r \Phi'}{\Phi} + \frac{r k'}{k} \right) - f_1 + \cos p \frac{Q^2}{2 \hat{\eta}} \left( \frac{r k'}{k} - \frac{r k_j}{k_j} \right) \right]
\]

\[
+ \epsilon^2 \left( \frac{Q^2 \Phi''}{4 \Phi} + \hat{\eta} \left( \frac{r \Phi''}{\Phi} + \cos p f_1 \left( \frac{2r \Phi'}{\Phi} + \frac{r k'}{k} \right) + f_1^2 + f_2 \right) \right)
\]

\[
= d_{20} + i \epsilon d_{21} + \epsilon^2 d_{22} + ...
\]

These equations define the terms \( d_{ij} \); note that alternate terms are imaginary as is typical for WKB approximation methods. Also note that \( r k'/k = \cos^2 pr k'/k - \sin^2 p \), and that
\(rk_{j}'/k_{j} = r\sigma_{0}'/\sigma_{0} - 2ra'/a\). We take \(k\) to be constant and real. The terms \((Ar)\) and \((Br^{2} + m^{2})\) contain contributions of order unity divided by \(\nu\) and \((\nu^{2} - 1)\). To get a polynomial expression for \(\nu\), we calculate the expressions

\[
\nu(\nu^{2} - 1)Ar = a_{1}\nu + a_{2}\nu^{2} + a_{3}\nu^{3}
\]

\[
\nu(\nu^{2} - 1)(Br^{2} + m^{2}) = b_{0} + b_{1}\nu + b_{2}\nu^{2}
\]

with

\[
a_{1} = \frac{2r\kappa'}{\kappa} - 1 - \frac{r\sigma_{0}'}{\sigma_{0}}
\]

\[
a_{2} = \frac{2m\Omega r\Omega'}{\kappa \Omega'}
\]

\[
a_{3} = 1 + \frac{r\sigma_{0}}{\sigma_{0}}
\]

\[
b_{0} = \frac{2m\Omega}{\kappa} \left( \frac{r\sigma_{0}'}{\sigma_{0}} + \frac{r\Omega'}{\Omega} - \frac{2r\kappa'}{\kappa} \right)
\]

\[
b_{1} = -\left( \frac{2m\Omega}{\kappa} \right)^{2} \frac{r\Omega'}{\Omega} \equiv J^{2}/\epsilon^{2} = s^{2}m^{2}
\]

\[
b_{2} = -\frac{2m\Omega}{\kappa} \left( \frac{r\sigma_{0}'}{\sigma_{0}} + \frac{r\Omega'}{\Omega} \right).
\]

Equation (19) is now multiplied by \(\nu(\nu^{2} - 1)\) to obtain a general dispersion relation for fluid disks:

\[
\nu^{5} + c_{3}\nu^{3} + c_{2}\nu^{2} + c_{1}\nu + c_{0} = 0,
\]

where

\[
c_{3} = -2 + d_{20} + i\epsilon(d_{21} + a_{3}d_{11}) + \epsilon^{2}(d_{22} + a_{3}d_{12}) + ...
\]

\[
= c_{30} + i\epsilon c_{31} + \epsilon^{2}c_{32} + ..., \]

\[
c_{2} = i\epsilon a_{2}d_{11} + \epsilon^{2}(a_{2}d_{12} + b_{2}d_{02}) + ...
\]

\[
= i\epsilon c_{21} + \epsilon^{2}c_{22} + ..., \]

\[
c_{1} = 1 - d_{20} + d_{02}J^{2} + i\epsilon(-d_{21} + a_{1}d_{11} + J^{2}d_{03}) + ...
\]

\[
= c_{10} + i\epsilon c_{11} + \epsilon^{2}c_{12} + ..., \]

\[
c_{0} = \epsilon^{2}b_{0}d_{02} + i\epsilon^{3}b_{0}d_{03} + ...
\]

\[
= \epsilon^{2}c_{02} + i\epsilon^{3}c_{03} + .... \]
This dispersion relation includes terms that have been neglected in previous studies. The effect of the higher order terms can be followed by the dependence of the coefficients $c_i$ on the small parameter $\epsilon$. In the limit of $\epsilon \to 0$, but with a finite $\hat{\eta}$ and finite $Q^2/\hat{\eta}^2$, the general dispersion relation (Eq. 20) becomes the BLLT dispersion relation (Eq. 10).

We investigate the effects of the higher order terms by expressing $\nu$ as an expansion in the parameter $\epsilon$, that is, $\nu = \nu_0 + \epsilon \nu_1 + \epsilon^2 \nu_2 + \ldots$, and by solving for the roots of equation (19). Substituting the expansion for $\nu$ into equation (19) and setting coefficients of equal powers of $\epsilon$ to zero, we obtain expressions for the expansion terms $\nu_i$. The zeroth-order root, $\nu_0$, satisfies the equation

$$\nu_0 \left[ \nu_0^4 - \nu_0^2 (2 - d_{20}) + 1 - d_{20} \left(1 + J^2 \hat{\eta}^2\right) \right] = 0. \tag{21}$$

The expression in the squared brackets of equation (21) is the BLLT dispersion relation as discussed above. The other solution ($\nu_0 = 0$) has no terms of order $\epsilon$; i.e., it is of the form $\nu = \nu_2 \epsilon^2 + \nu_3 \epsilon^3 + \ldots$.

The first-order term that corresponds to the nonzero solution $\nu_0$ is

$$\nu_1 = -\frac{i \nu_0 (c_{11} + c_{21} \nu_0 + c_{31} \nu_0^2)}{c_{10} + 3 c_{30} \nu_0^2 + 5 \nu_0^4}; \tag{22}$$

for real $\nu_0$ (i.e., stability in the BLLT equation), this $\nu_1$ is purely imaginary; for imaginary $\nu_0$, it is complex.

The coefficients, $c_i$, for the next term in the expansion, $\nu_2$, are pure real and, if $\nu_0$ is real, then this term is real also, and a factor of $\epsilon^2$ smaller. When $\nu_0$ is real, the growth rate to first order in $\epsilon$ is attributed to $\nu_1$. The next contribution to the growth rate will be from $\nu_3$, which is of order $\epsilon^2$ smaller.

In summary, we have found in this analysis a general dispersion relation that includes the effects of radial variations in the basic parameters of the disk and is accurate to higher order in the small parameter $\epsilon = (k_{\text{crit}} r)^{-1}$. Furthermore, the effects included in this analysis change significantly the criterion for stability of the disk as shown explicitly by the models in the next section.

5. Instability models including the high order terms

Several models will be studied to illustrate the effects of the higher order terms in the dispersion relation and to investigate how different assumptions affect the stability of the disk. Four models will be considered: a self-gravitating disk with a flat rotation
curve, a self-gravitating disk with solid body rotation, a non-self-gravitating disk with solid body rotation, and a non-self-gravitating disk with Keplerian rotation. The amplitude of the wave is assumed to be slowly varying so $r \Phi'/\Phi \ll 1$. This gives an arm/interarm contrast that increases with radius beyond one scale length, in agreement with observations (Schweitzer 1980; Elmegreen & Elmegreen 1984).

All disks considered here are assumed to have an exponential mass column density profile with a scale length $r_d$ and a constant sound speed, $a$. Then $r \sigma_0'/\sigma_0 = -r/r_d$ and $r a'/a = 0$.

We are considering solutions to the dispersion relation obtained from a local analysis where there are gradients in the physical quantities of the equilibrium disk. The local analysis is relevant when the growth time for the perturbations is shorter than the time needed for the disturbances to travel to the boundaries (e.g., see Lin & Shu 1964; Toomre 1981). That is usually $10^9$ years to the outer boundary and $10^7$ years to the center for a circumnuclear disk, but in this case the center boundary usually serves as a sink, as waves are shocked and energy is dissipated. Therefore we are justified in using a local analysis in nuclear disks. For main galaxy disks the growth time of spiral waves is also typically less than the propagation time. Bertin et al. (1989) considered a non-local analysis, including the effects of gradients and boundary conditions. This leads to the standard modal theory of spiral structure.

Gradients of disk properties, as well as curvature, can lead to spatial variations in the amplitude of spiral waves, including singularities. The curvature effects are considered in more detail in section 6.

General dispersion relations like these can be solved by assuming $k$ real and $\omega$ complex, or $k$ complex and $\omega$ real. In the remainder of this section, we consider $k$ real and constant and look for solutions with imaginary $\omega$. The result will be sinusoidal waves that grow exponentially with time, as in the usual stability analyses.

A third method of analysis is to consider the initial value problem of time-dependent growth with shearing sinusoidal perturbations, as in Goldreich & Lynden-Bell (1965) and Toomre (1981). When gravity is important, this leads to the swing amplifier theory.

In the following subsections we will investigate analytically and numerically the dispersion relation for disks with different rotational properties. The relevant dispersion relation is Eq. (19). The same dispersion relation with explicit expansion in terms of the small parameter $\epsilon$ is Eq. (20) for self-gravitating disks. Another dispersion relation is derived for non-self-gravitating disks in the indicated subsections.
5.1. Exponential self-gravitating disk with constant rotation velocity

We first find the roots of Eq. (19) at two scale lengths for an exponential disk with a constant rotation speed. In this case $r \Omega / \Omega = r \kappa' / \kappa = -1$, and $\Omega / \kappa = 1/\sqrt{2}$. The value of $\epsilon = 1/(k_{\text{crit}})$ depends on the ratio of the disk to total mass (disk and halo) in the spiral region. A value of $\epsilon \sim 0.11$ corresponds to the Solar radius in the Galaxy, using the rotation curve model in Schmidt (1983) and a disk mass surface density of 48 $M_\odot$ pc$^{-2}$ (Kuijken & Gilmore 1989, 1991). We use a value of $\epsilon = 0.1$.

There are five roots of the dispersion relation. The root that corresponds to the greatest growth is always plotted in the figures here; this is the root with most negative imaginary component.

Figure 2 shows the components of the normalized frequencies $\nu$ in the $(k_{\text{crit}}/|k| \equiv \eta, Q^2)$ plane for two values of the azimuthal wavenumber, $m = 2$, and 5, obtained numerically using the full dispersion relation, equation (19) with coefficients up to third order in the small parameter $\zeta \equiv \epsilon \hat{\eta}$. To be clear, we write $k_{\text{crit}}/|k|$ instead of $\eta$ in the figures. The top figures show the negative of the imaginary component of the frequency, i.e., the growth rate normalized to the epicyclic frequency $\kappa$, with contour values $2i/4$ for $i = -20$ to 10. The bottom figures show the absolute values of the corresponding real frequencies with the same contours. The left figures correspond to $m = 2$ and the right correspond to $m = 5$. The values of the real and imaginary components are tabulated for some values of $Q^2$ and $k_{\text{crit}}/|k|$ in table 1; this will facilitate the interpretation of the contours.

The thick lines in the top plots of figure 2 indicate the loci of points where the normalized frequency, $\nu_{\text{BLLT}}$, equals 0 (corotation, lower line), $\pm 1$ (inner and outer Lindblad resonances, middle line) and $\pm \sqrt{1+2J^2\eta^2}$ (upper line) in the BLLT dispersion relation, equation (10). The BLLT instability condition, equation (11), is satisfied below the lower thick line. The new acoustic instability condition, equation (14), is satisfied between the middle and the upper thick lines.

The thick lines in the top plots of figure 2 indicate the loci of points where the normalized frequency, $\nu_{\text{BLLT}}$, equals 0 (corotation, lower line), $\pm 1$ (inner and outer Lindblad resonances, middle line) and $\pm \sqrt{1+2J^2\eta^2}$ (upper line) in the BLLT dispersion relation, equation (10). The BLLT instability condition, equation (11), is satisfied below the lower thick line. The new acoustic instability condition, equation (14), is satisfied between the middle and the upper thick lines.

The figure and table show that the growth rate decreases but remains finite for $k_{\text{crit}}/|k| \to 0$, and that at $k_{\text{crit}}/|k| = 0$, it increases with increasing $Q$. At intermediate values of $k_{\text{crit}}/|k|$, say 0.5, the growth rate is largest for $Q < 1$ and decreases to a minimum at $Q^2 \approx 2$, but again increases for increasing $Q^2$. The growth rate decreases for increasing $k_{\text{crit}}/|k|$ beyond 0.5 for constant values of $Q^2$. This pattern is observed for both $m$ values. A significant difference between the figures for $m = 2$ and $m = 5$ is that for higher $m$, the growth rate is larger over the plotted $(k_{\text{crit}}/|k|, Q^2)$ plane than for low $m$, and for high $k_{\text{crit}}/|k|$, the growth rate remains relatively large for moderate values of $Q^2$ above the line $\nu_{\text{BLLT}} = 0$. This enhanced growth at high $m$ is because the $J$-parameter is proportional to
and is contributing to the higher order terms in the dispersion relation.

Note that there is a kink in the lower right corner \( (k_{\text{crit}}/|k| \approx 1.6) \) of the \( m=2 \) contour plot for the real component of the root. This occurs because in adjacent regions to the kink different real components have the most negative imaginary component.

Figure 2 and table 1 also indicate that the greatest growth occurs for small values of \( Q^2 \), just as predicted using the BLLT dispersion relation (cf. Sect. 3.2). Moreover, they indicate that the disk is unstable to form spirals for a wide range of \( Q \) and \( m \), although the growth rate is low, of order \( \epsilon \), when \( Q \) is large. This implies there is still a spiral instability at low gravity. For most bright galaxies, however, the region where the rotation curve is flat is also the region where \( Q \) is relatively small, so these high \( Q \) solutions are not important. They could be important in early type galaxies (Caldwell et al. 1992) or low surface brightness galaxies (van der Hulst et al. 1993) where \( Q \) is high in the main disk.

5.2. Exponential self-gravitating disk with solid body rotation

The inner parts of galaxies and small galaxies typically have rotation curves that are approximately solid body. This is the result of a strong bulge with a nearly uniform central density in some spiral galaxies, and a relatively dense dark matter halo in dwarf galaxies. Inner galaxy disks (Elmegreen et al. 1998) and dwarfs (Hunter et al. 1998) may also be weakly self-gravitating for some time (e.g., between accretion events and starbursts), and so the high-\( Q \) cases studied here may have applications there. Furthermore, inner disks and dwarfs have short rotation times, so the actual growth factor of a spiral instability can be large even if the normalized growth rate is small.

For solid body rotation, \( r \Omega'/\Omega = r \kappa'/\kappa = 0 \), and \( \Omega/\kappa = 1/2 \). We assume a value for \( \epsilon = 0.1 \) as in the previous section. In this case the term \( A(r) \) does not depend on \( \nu \) and \( B(r) \) has a \( 1/\nu \) dependence. The dispersion relation then becomes cubic in \( \nu \):

\[
\nu^3 + (-1 + D_2 + a_3 D_1) \nu + b_2 D_0 = 0,
\]

where the terms \( D_i, a_3, \) and \( b_2 \) were defined in the previous section. The roots can be expressed as an expansion in \( \epsilon \), writing \( \nu = \nu_0 + \nu_1 \epsilon + \nu_2 \epsilon^2 + \ldots \). The zero order term is the Lin-Shu dispersion relation, equation (7), with \( \eta \) replaced by \( \tilde{\eta} \). The first order term is

\[
\nu_1 \epsilon = -i \epsilon \cos \frac{p}{2 \nu_0} \left[ Q^2 \frac{4}{\tilde{\eta}} \left( 1 - \frac{r \sigma_0'}{\sigma_0} - 2 \sin^2 p \right) - \frac{1}{2} - \frac{r \sigma_0'}{\sigma_0} + \frac{\sin^2 p}{2} \right].
\]

In the region where \( \nu_0 \) is real, the growth rate is dominated by the first order term. In the region where \( |\nu_0| \) is of order 1, \( \nu_1 \sim -i \) and the growth rate is of order \( i \nu_1 \epsilon \approx \epsilon \). For an
Figure 3 and table 2 show real and imaginary components of the normalized roots of the full dispersion relation (23) for the rising rotation curve model at $r = 2r_d$. Again we display only the root that corresponds to the fastest growth. We can see from the left-hand regions in the $(k_{\text{crit}}/|k|, Q^2)$ plot, where the absolute values of the real components are large, that the growth rates become small for small $k_{\text{crit}}/|k|$. The opposite occurs for small values of the real component, which are in the lower region of the plot. Where the real component is of order 1, in the center of the plot, the growth rate is of order $\epsilon$.

The detailed behavior of the growth rate in this case can be followed from the approximate analytical solution written above as equation (24). For example, equation (24) gives the same growth rate as the full solution in table 2 for $\eta = k_{\text{crit}}/|k| = 0.2$ for both $m = 2$ and 5, because the approximate equation is relatively accurate for low $\eta$.

Equation (24) gives slightly different rates than table 2 for $\eta = 0.6$; at $m = 2$ and $Q^2 = 2, 5, \text{ and } 10$, equation (24) has growth rates of 0.232, 0.228 and 0.275 while table 2 has more precise growth rates of 0.227, 0.225 and 0.273. The rates given by equation (24) differ more significantly from those calculated by equation (23) when $\eta > 0.6$.

One can observe from table 2 that the real component corresponding to the greatest growth rate in the $(k_{\text{crit}}/|k|, Q^2)$ plane is always negative, i.e., it corresponds to the Lin-Shu and BLLT solutions inside of corotation in the disk.

Figures 2 and 3 show that there is a similarity between the growth rates for the flat and solid body rotation curve models. Both figures display a saddle shape for the growth rate contours; the greatest growth occurs as $Q^2 < 1$, and for $k_{\text{crit}}/|k| \approx 0.5$, the growth rate first decreases and then increases with increasing $Q^2$. The main difference between the two models occurs for large numbers of spiral arms, where the growth rate is smaller at $m = 5$ than $m = 2$ for the solid body case, and larger at $m = 5$ than $m = 2$ in the flat rotation curve case. This is because for solid body rotation, $J = 0$, so the absence of differential rotation reduces the growth rate of waves at any $m \neq 0$. The zero order BLLT instability condition (Eq. (11)) is reduced to the Lin-Shu instability condition (Eq. (7)), and the acoustic instability disappears as the upper unstable region collapses around $\nu^2 \approx 1$. In addition, the contributions of $J^2$ to the higher order terms $\nu_i$ are also absent so the growth rate is less than for $J^2 > 0$.

The solutions shown for all the self-gravitating models indicate that disks are weakly unstable to spiral waves when $\epsilon = 1/(k_{\text{crit}}r) > 0$, even in the limit of weak self-gravity.
This is the first time spiral disk instabilities have been found at large $Q$ in the absence of magnetic fields. We pursue this result further in the next section, which considers the growth of waves in the absence of self-gravity, that is, when $\epsilon = 0$.

### 5.3. Exponential disk with solid body rotation and no self-gravity

This section and the next consider fluid disks without self-gravity as an idealization of the high $Q$ cases found to be unstable in the previous sections. To be consistent with the radial dependence of the enthalpy amplitude, $H(r)$, used before, which was defined in terms of a slowly varying potential amplitude $\Phi$, we now assume $H(r) \propto -f(r)$, where $f$ was given in the discussion following equation (5).

From equation (24) we can see that the growth rate of the instability depends on both the self-gravity of the disk and the radial derivative of the background surface density. The normalized growth rate is, to first order, $\nu_1 \epsilon$, from the previous discussion. The first term in equation (24) is proportional to $Q^2 \cos p \epsilon/4\hat{\eta} = k a^2/\kappa^2 r$, which is independent of the self-gravity of the disk. It depends primarily on the disk curvature, i.e., on the ratio of the square of the semimajor axis of an epicycle caused by random motions $(a/\kappa)$, to the product of the wave scale $(k^{-1})$ and the disk radius $(r)$. The second term is proportional to $\epsilon \propto \text{mass}_d/\text{mass}_{\text{total}}$, which comes from the self-gravity of the disk. If the disk self-gravity is neglected, $\epsilon = 0$ and the second term is zero, but there is still growth from the first term, depending on orbital curvature. When $\epsilon = 0$, the expansion has to be made in terms of the small parameter $\zeta \equiv 1/\hat{k}r$. Then we get:

$$
\nu_0 = \pm \sqrt{1 + (a \hat{k})^2/\kappa^2},
$$
$$
\nu_1 \zeta = i - \frac{k a^2}{2 \kappa^2 r \nu_0} \left( 2 \sin^2 p - \frac{r}{r_d} - 1 \right).
$$

When the expression inside the parenthesis of equation (26) is zero for some particular pitch angle $\pi - p$, there is no growth at that radius, but there is growth at adjacent radii.

The numerical solutions to equation (23) when gravity is neglected are shown for $r = 2r_d$ in figure 4 and table 3, using normalized axes $(a/\kappa r)^2$ instead of $Q^2$ and $1/|kr|$ instead of $k_{\text{crit}}/|k|$. To compare the growth rates with the previous models, recall that the value of the vertical axis in figure 4 is obtained by multiplying the value of the vertical axis in our previous figures by $\epsilon^2 = 0.01$, and the value of the horizontal axis in figure 4 is obtained by multiplying the previous value of the horizontal axis by $\epsilon = 0.1$. This means that the growth rates in figure 4 are analogous to those in the upper right part of figure 3. Figure 4 and table 3 indicate that the growth rate remains finite, proportional to $(a/\kappa r)$, as
\(|kr| \to \infty\). We infer from this behavior that the instability is acoustic in nature, similar to that described in section 3.2, but in the absence of self-gravity and shear. It is driven by curvature and pressure gradients in the disk (cf. section 6).

5.4. Exponential disk with Keplerian rotation and no self-gravity

Accretion disks around black holes (Nakai et al. 1993) and protostars have negligible self-gravity and may have Keplerian rotation. In this case \(r\kappa'/\kappa = r\Omega'/\Omega = -3/2\), and equation (2) becomes a fifth order polynomial in \(\nu\), as for a flat rotation curve. As in the previous section, an acoustic instability is still present even in the absence of self-gravity. The dispersion relation for this acoustic instability is now obtained from equation (19) with the modifications

\[
D_0 = \delta \\
D_1 = \delta \left(ikr + \frac{rf'}{f}\right) \\
D_2 = \delta \left(-\hat{k}^2r^2 + 2ikr \frac{rf'}{f} + \frac{r^2f''}{f}\right).
\]

Because there is no self-gravity, this fifth order dispersion relation has to be expanded to successive orders in \(\zeta = 1/\hat{kr}\) instead of \(\epsilon\), giving \(\nu = \nu_0 + \nu_1\zeta + \nu_2\zeta^2\ldots\). The zero-order term in this expansion is the modified BLLT dispersion relation, equation (15). Recall that the geometric term for a Keplerian disk is \(s = \sqrt{6}\). The zero order term becomes complex when the instability condition, equation (17), is satisfied. When equation (17) is not satisfied, the growth rate is dominated by the first order term

\[
\nu_1\zeta = -i \frac{a^2k}{2\kappa^2r} \left(1 + r/r_d - 2\sin^2 p\right) \left(\nu_0^2 - 1\right) - 3 \left(1 + m\nu_0\right) \frac{\nu_0 \left(2\nu_0^2 - 2 - a^2k^2/\kappa^2\right)}{\nu_0^2}. \tag{27}
\]

Figure 5 and table 4 display numerical solutions to the fifth order polynomial, equation (19), for the dispersion relation in this Keplerian model using the modified expressions \(D_i\) when self-gravity is neglected. The real and imaginary components of the root with the largest growth rate are plotted using the same axes as in the previous section, \(1/|kr|, a^2/\kappa^2r^2\). The critical curve for stability, equation (16), is plotted in the top figures as a thick line. To the left of the critical curve, equation (16) is not satisfied and the disk is stable against acoustic instabilities to lowest order (higher order instabilities remain). To the right of the critical curve, equation (16) is satisfied and the acoustic instability to all
orders dominates the growth of perturbations. The growth rate is larger than that in the case of solid body rotation without self-gravity because shear stimulates growth. There are discontinuities in the contours for the real component, with kinks at the same locations in the contours of the imaginary growth rate near the critical curve. To the left of these discontinuities, the real part of $\nu$ is negative, corresponding to radii inside the ILR; to the right, the real part is positive, corresponding to radii outside the OLR.

Equation (27), which is the first order approximation to the growth rate, matches the full numerical solutions in figure 5 and table 4 to two significant digits for $\bar{kr} > 5$ and $(a^2/\kappa_2^2 r^2) < 0.1$.

### 6. Physical Insights to the Curvature Terms

We have just shown that differential rotation, curvature, and radial gradients in the basic properties of a fluid disk affect the propagation and growth of spiral disturbances. Here we simplify the problem by including only the effects of orbital curvature.

The curvature terms can be illustrated by considering an ideal disk with solid-body rotation, constant surface density, and negligible self-gravity. Such disks may be appropriate for the central regions of quiescent galaxies, such as NGC 2207 (Elmegreen et al. 1998). The governing equation (2) for such a disk, assuming constant surface density, becomes

$$\frac{d^2 h_1}{dr^2} + \frac{1}{r} \frac{dh_1}{dr} + \left( \frac{\kappa^2}{a^2} \left[ \nu^2 - 1 \right] - \frac{m^2}{r^2} \right) h_1 = 0. \quad (28)$$

This equation was derived with the center of the coordinate system at the center of rotation. It is the well-known Bessel equation, and can have mathematical singularities at $r = 0$. Often in wave equations, these mathematical singularities can be transformed away by a change in the coordinate system, adopting, for example, a rectilinear coordinate system instead of cylindrical. However, in the case of a galaxy, the singularities cannot be transformed away by a different coordinate system: rotation and galactic gravity define the coordinate system.

In the galactic Bessel equation, the time derivative in the equation of motion appears as the term $\nu^2$, as it did in the previous sections. The spatial variation in the azimuthal direction is also assumed to be the same as before, $e^{-im\theta}$, but in the radial direction it is written explicitly. We may look for the behavior of spirals by assuming radial solutions of the form $h_1 \propto e^{ikr}$. These solutions are trailing spirals when $k < 0$. They always contain pieces of waves that can come close to the origin, depending on their direction of propagation, so they can force out the singularities in the Bessel equation. The pure-ring
case with $m = 0$ may also approach the origin and increase in amplitude. In this case the increase is analogous to laboratory sonoluminescence, in which sound waves converge to the center of an air bubble in a liquid and increase in amplitude until they shock and emit light (Kondic, Gersten, & Yuan 1995).

In the case of spiral solutions, the radial derivatives in the Bessel equation are replaced by $ik$ and the frequency $\nu$ may be solved to give

$$\nu^2 = \left[ \left( \frac{a}{\kappa r} \right)^2 (\nu^2 k^2 + m^2) + 1 \right] - i r k. \tag{29}$$

This frequency is necessarily complex because of the first derivative term in equation (28). Because of this term, the general solutions are growing or decaying oscillations with spiral shapes having $m$ arms. It will become apparent shortly that the incoming waves are growing, and the outgoing waves are decaying, as expected from the nuclear singularity.

For $\nu_R >> \nu_I$, we recover the same result as equation (26) in the limit $r_d \to \infty$, assuming constant enthalpy amplitude:

$$\nu_R = \pm i \left[ \left( \frac{a}{\kappa r} \right)^2 (r^2 k^2 + m^2) + 1 \right]^{1/2}, \quad \nu_I = -\frac{1}{2} \left( \frac{a}{\kappa r} \right)^2 \frac{r k}{\nu_R}. \tag{30}$$

This gives the growth rate of a wave with azimuthal wavenumber $m$ and radial wavenumber $k$. Note that $|\nu_R| > 1$ in all cases here, which means the waves are only inside the inner Lindblad resonance or outside the outer Lindblad resonance. This was the case also for the new instability solutions discussed in section 3.2, which re-considered the BLLT equations in this new radial limit.

The nature of the growth implied by $\nu_I$ in equation (30) should be discussed more. Recall that the assumed time behavior of the wave in an inertial frame is $e^{i(kr + \omega t - m\theta)} = e^{i(kr + \nu t)}$ for $\nu = (\omega - m\Omega)/k$ and $\theta = \Omega t$ following the rotation. Also note that we have written $\nu = \nu_R + i\nu_I$. Thus we have a time behavior $e^{i\nu_R t}e^{-\nu_I t}$. When $\nu_I$ is negative, the wave grows in time. This occurs for trailing waves only when $\nu_R$ is the negative one of the two solutions given above, because $\nu_I \propto -k/\nu_R$, and $k < 0$ for trailing waves. Moreover, the negative $\nu_R$ solution is an incoming trailing wave, because the wave-like part of the solution, $e^{i(kr + \nu_R t)}$, has constant phase for decreasing $r$ with increasing time when $\nu_R < 0$, i.e., $r = - (\nu_R/k)t = -|\nu_R/k|\kappa t$ when both $\nu_R$ and $k$ are less than zero. As a result, the galactic Bessel equation has trailing spiral wave solutions that grow in time as they propagate toward the center of the galaxy.

These solutions are not instabilities in the usual sense, because $t$ cannot be allowed to go to infinity. The waves reach the center in finite time, i.e., in the time $t \sim r/a$. In
this sense, the growing solutions are like those in the galactic swing amplifier (Goldreich & Lynden Bell 1965; Julian & Toomre 1966), in which spiral waves grow in the shearing part of a disk for a finite time \( \Delta t \sim 2/A \) for Oort constant \( A \). The instabilities in a galactic nucleus are also not stationary waves that grow in amplitude without any change in shape. This is because the spiral solution is always undefined at the nucleus and can never be considered present at all radii. The waves are only pieces of spirals, moving inward or outward with a growth or decay in time following the wave crest, respectively. Thus the growth is also unlike the growth of infinite plane waves in a sheet, as might be the case for the Kelvin-Helmholtz instability, for example. Spiral wave growth in galactic nuclei involves inward propagation of finite wave trains.

The dispersion relation (29) may also be regarded as an equation for \( k \), in which \( \nu \) is held as a real variable. Then equation (28) has normal Bessel function solutions \( J_m(k_Br) \) and \( Y_m(k_Br) \) for

\[
\frac{k^2}{a^2}(\nu^2 - 1) \equiv k_B^2 > 0;
\]

\( k_B \) is the radial wavenumber. When \( k_B r \) is large, \( J_m \) and \( Y_m \) behave like sines and cosines, which may be combined as outgoing or incoming waves with \( \exp(i\omega t) \). When \( k_B r \) is small, the \( Y_m \) solutions grow algebraically with decreasing \( r \). The growth arises directly from the curvature terms, namely, the first derivative term and the \( m^2/r^2 \) term. These are the same terms that led to imaginary \( \nu \) in equations (29) and (30). The \( m^2/r^2 \) term actually defines a region within which the waves start to grow out of bounds, i.e., when \( r < m/k_B \) for \( m \neq 0 \), the \( Y_m(k_Br) \) solution begins to increase. In terms of the growth discussed for the time-dependent case, this is the radius at which an incoming wave has only one more epicycle in time before it reaches the nucleus propagating at the sound speed.

What happens to a trailing spiral wave in a real galaxy when it enters the \( rk_B/m < 1 \) regime? We expect that the amplitude will begin to increase geometrically until nonlinear and dissipative effects come into play. This means that the waves will break in the form of shocks shortly after they enter the inner region. The condition \( rk_B/m < 1 \) implies that the radius for this wave shocking increases with azimuthal wave number \( m \). This explains for the case of NGC 2207 (Elmegreen et al. 1998) why the multiple-arm features are only observed in the outer part of the nuclear disk, while the \( m = 1 \) and ring-like feature is close to the center. That is, the multiple arms (high \( m \)) become non-linear and damp out before they reach the inner radii, leaving only the low-\( m \) arms near the center. Other spirals that may travel outward in NGC 2207 are probably too weak to be seen because their amplitudes decrease as they propagate.

Galactic nuclear spiral waves also propagate in the azimuthal direction with angular speed \( \omega/m \) as long as \( \nu^2 - 1 > 0 \). This angular speed implies that waves with different \( m \)
will interact, forming complex structures. The waves are also dispersive, with dispersion relation

\[ \frac{\omega - m \Omega}{\kappa} = \pm \left( 1 + \frac{a^2 k_B^2}{k^2} \right)^{1/2}. \]

They form wave packets that propagate with group velocity

\[ c_g = \pm a \sin \gamma_B, \]

with \( \gamma_B = \tan^{-1} \left( \frac{ak_B}{\kappa} \right) \). Undoubtedly the waves will interact because of these various phase and group speeds. They will also get sheared by differential rotation in reality to form complex spiral structures. When \( \nu^2 - 1 < 0 \), the entire disk is evanescent; then we should not see any waves.

So far we have ignored the exponential density distribution of the disk. If it is taken into consideration, equation (28) will change to

\[ \frac{d^2 h_1}{dr^2} + \frac{1}{r} \left( 1 - \frac{m^2}{\nu rr_d} \right) \frac{dh_1}{dr} + \left[ \frac{\kappa^2}{a^2} (\nu^2 - 1) - \frac{m^2}{r^2} + \frac{m}{\nu rr_d} \right] h_1 = 0, \]  

(31)

where \( r_d \) is the scale length of the exponential disk. The additional factor in the first derivative term will modify the behavior of the Bessel functions when \( r \geq r_d \), and the additional term in the last parenthesis will complicate the wave behavior. But the qualitative nature of the Bessel function solutions does not change.

7. Summary

We have obtained dispersion relations for spiral waves with multiple arms, considering curvature and gradient terms that were ignored in previous derivations. These dispersion relations suggest the presence of several new instabilities. Four specific cases were studied, flat and rising rotation curves with self-gravity, rising rotation curves without self-gravity, and Kepler rotation curves without self-gravity. These cases seem to have applications in various regions of galaxies and accretion disks.

When self-gravity is present, instability at lowest order in the parameter \( \epsilon \) (cf. Eq. 1) is driven by both shear and self-gravity. Then there are two independent instability conditions, either of which can cause spiral waves. These are equations (11) and (14). The first of these comes from Bertin et al. (1989), and contains the Toomre (1964) instability condition, \( Q < 1 \), as a special case for ring-like perturbations \( (m = 0, \text{ which gives } J = 0) \). This first instability is the spiral instability that is commonly discussed in the literature as a source of multiple arm and grand design spiral structure in galaxy and protoplanetary disks.
The second of these conditions arises outside the Lindblad resonances from a combination of parameters different than the first when $Q^2 > 4\eta$ (cf. Eq. 13). When self-gravity is not present, this second case is still unstable as a result of pressure and differential rotation alone, as determined by the smallness of the parameter $a/(kr)$ (cf. Eq. 16). This pressure-rotation instability is apparently new, and we call it an acoustic instability. A physical explanation for it was given in section 3.3.

We also found additional instabilities coming from higher order terms in an expansion of the dispersion relation (19) around the small parameter $\epsilon$. These additional instabilities are present even when the BLLT and Toomre instability conditions are not satisfied, i.e., when the low order terms give stability. The source of these residual instabilities is a combination of orbital curvature [terms of order $1/(kr)$], self-gravity (terms of order $\epsilon$), and various disk gradients ($r\sigma'/\sigma$, $ra'/a$, etc.), including shear (the $J$ or $s$ terms). Growth rates for these residual instabilities were given to all orders in $\epsilon$ for flat and rising rotation curves by figures 2 and 3 and tables 1 and 2, and they were given to first order in $\epsilon$ by equation (24) for solid body rotation. The residual instability that arises from self-gravity and orbital curvature (through $\epsilon$), discussed in sections 5.1 and 5.2, will be called gravitational-curvature instability. The residual instability that arises from a combination of pressure and orbital curvature [through $a/(kr)$], discussed in sections 5.3 and 5.4, will be called an acoustic-curvature instability, because it operates even without self-gravity.

These three new instabilities should be important for fluid disks with negligible or weak self-gravity, including proto-planetary disks, gaseous disks around black holes, some galactic nuclear disks, low surface brightness galaxy disks, and some dwarf galaxies. In these cases, zero-order acoustic and higher-order acoustic-curvature and gravitational-curvature instabilities can lead to the growth of spiral or other structures in about an orbital time. They are most important in the region close to the center where the orbital time is small.

Non-linear effects arising from these waves may ultimately lead to visible dust lanes (Elmegreen et al. 1998) and associated gaseous shocks (Roberts 1969) in even the most weakly self-gravitating disks, with the possibility of heightened self-gravity and star formation in some of the compressed regions (e.g., Elmegreen 1994). Non-linear effects might also promote accretion flows (e.g., Larson 1990). Indeed, the ubiquity of acoustic waves in disks implies that galactic nuclear accretion should occur in a wide variety of environments with or without shear, self-gravity, or magnetic fields.

REFERENCES

Table 1. Flat rotation curve at \( r = 2 \, r_d \)

| \( Q^2 \) | \( \frac{k_{\text{crit}}}{|k|} \) | \( m = 2 \) growth rate | \( m = 2 \) frequency | \( m = 5 \) growth rate | \( m = 5 \) frequency |
|-----|----------------|------------------|------------------|------------------|------------------|
| 10  | 0.280          | 0.340            | -7.660           | 0.229            | -7.695           |
| 5   | 0.218          | 0.264            | -5.229           | 0.211            | -5.255           |
| 2   | 0.175          | 0.213            | -2.924           | 0.183            | -2.941           |
| 1   | 0.188          | 0.221            | -1.513           | 0.163            | -1.531           |
| 0.1 | 1.867          | 0.410            | -0.042           | 0.179            | -0.044           |

Table 2. Solid body rotation at \( r = 2 \, r_d \)

| \( Q^2 \) | \( \frac{k_{\text{crit}}}{|k|} \) | \( m = 2 \) growth rate | \( m = 2 \) frequency | \( m = 5 \) growth rate | \( m = 5 \) frequency |
|-----|----------------|------------------|------------------|------------------|------------------|
| 10  | 0.254          | 0.268            | -7.658           | 0.211            | -7.692           |
| 5   | 0.193          | 0.223            | -5.228           | 0.183            | -5.252           |
| 2   | 0.154          | 0.211            | -2.922           | 0.163            | -2.941           |
| 1   | 0.173          | 0.244            | -1.512           | 0.184            | -1.531           |
| 0.1 | 1.838          | 0.361            | -0.052           | 0.206            | -0.044           |

\( k_{\text{crit}} \) represents the critical growth rate for linear instability, \( |k| \) is the frequency in the rotating frame.
Table 3. Solid body rotation with no gravity at $r = 2 r_d$

| $(\frac{\omega}{\omega_0})^2$, $\frac{1}{|kr|}$ | 0.2 | 0.6 | 1.0 | 1.4 | 1.8 | 0.2 | 0.6 | 1.0 | 1.4 | 1.8 |
|---------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $m = 2$ growth rate             |     |     |     |     |     |     |     |     |     |     |
| 1.0                             | 1.205 | 0.482 | 0.376 | 0.433 | 0.462 | -5.584 | -2.816 | 0.777 | 0.917 | 0.995 |
| 0.5                             | 0.833 | 0.311 | 0.308 | 0.355 | 0.377 | -4.024 | -2.160 | 0.671 | 0.769 | 0.820 |
| 0.2                             | 0.499 | 0.168 | 0.139 | 0.137 | 0.131 | -2.673 | -1.608 | 0.493 | 0.559 | 0.598 |
| 0.1                             | 0.328 | 0.101 | 0.051 | 0.032 | 0.022 | -2.026 | -1.355 | -1.294 | -1.279 | -1.274 |
| 0.01                            | 0.058 | 0.014 | 0.007 | 0.004 | 0.003 | -1.148 | -1.045 | -1.037 | -1.035 | -1.034 |
| $m = 5$ growth rate             |     |     |     |     |     |     |     |     |     |     |
| 1.0                             | 0.684 | 0.179 | 0.097 | 0.067 | 0.051 | -7.173 | -5.420 | -5.271 | -5.230 | -5.214 |
| 0.5                             | 0.474 | 0.122 | 0.066 | 0.045 | 0.035 | -5.147 | -3.941 | -3.840 | -3.812 | -3.801 |
| 0.2                             | 0.287 | 0.071 | 0.039 | 0.026 | 0.020 | -3.375 | -2.657 | -2.597 | -2.581 | -2.575 |
| 0.1                             | 0.192 | 0.046 | 0.025 | 0.017 | 0.013 | -2.509 | -2.036 | -1.997 | -1.987 | -1.983 |
| 0.01                            | 0.039 | 0.008 | 0.004 | 0.003 | 0.002 | -1.252 | -1.162 | -1.156 | -1.154 | -1.153 |

Table 4. Keplerian rotation with no gravity at $r = 2 r_d$

| $(\frac{\omega}{\omega_0})^2$, $\frac{1}{|kr|}$ | 0.2 | 0.6 | 1.0 | 1.4 | 1.8 | 0.2 | 0.6 | 1.0 | 1.4 | 1.8 |
|---------------------------------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $m = 2$ growth rate             |     |     |     |     |     |     |     |     |     |     |
| 1.0                             | 1.606 | 1.088 | 1.266 | 1.365 | 1.414 | -5.736 | -2.960 | 2.170 | 2.081 | 2.040 |
| 0.5                             | 1.220 | 0.914 | 1.076 | 1.139 | 1.171 | -4.161 | 2.010 | 1.818 | 1.748 | 1.715 |
| 0.2                             | 0.861 | 0.739 | 0.821 | 0.856 | 0.875 | -2.787 | 1.621 | 1.489 | 1.439 | 1.415 |
| 0.1                             | 0.665 | 0.597 | 0.645 | 0.667 | 0.679 | -2.121 | 1.419 | 1.320 | 1.281 | 1.262 |
| 0.01                            | 0.249 | 0.236 | 0.242 | 0.246 | 0.248 | -1.197 | 1.102 | 1.068 | 1.055 | 1.048 |
| $m = 5$ growth rate             |     |     |     |     |     |     |     |     |     |     |
| 1.0                             | 1.413 | 0.961 | 1.097 | 1.140 | 1.160 | -7.262 | 4.327 | 4.007 | 3.885 | 3.821 |
| 0.5                             | 1.214 | 1.266 | 1.355 | 1.382 | 1.395 | -5.220 | 3.253 | 3.076 | 3.008 | 2.972 |
| 0.2                             | 1.041 | 1.231 | 1.276 | 1.291 | 1.298 | -3.450 | 2.394 | 2.291 | 2.250 | 2.228 |
| 0.1                             | 0.932 | 1.089 | 1.117 | 1.126 | 1.130 | -2.607 | 1.960 | 1.888 | 1.859 | 1.843 |
| 0.01                            | 0.506 | 0.539 | 0.544 | 0.546 | 0.547 | 1.370 | 1.229 | 1.205 | 1.195 | 1.190 |
Fig. 1.— Regions of instability in a self-gravitating disk with constant rotation for 5 arm spirals ($m = 5$). The lowest (thin) line borders the instability condition obtained from the Lin-Shu dispersion relation (Eq. (7)) for $m$ and $J^2 = 0$. The three upper lines bracket regions of zero-order instabilities obtained in the Lau-Bertin dispersion relation. The border of the Lin-Shu region of instability shifts to the lower thick line for nonzero $J^2$. This line is from Eq. (11) for the usual Lau-Bertin stability condition; the two upper thick lines are the boundaries enclosing the acoustic instability according to Eq. (13). The shading of the unstable regions gives an indication of the growth rate. The most unstable region is in the bottom left corner of the figure.

Fig. 2.— Contours showing the maximum growth rates (top) and corresponding frequencies (bottom) for instabilities to all orders of the small parameter $\epsilon$ in a self-gravitating disk with a constant rotation velocity and an exponential density profile, evaluated at two scale lengths. Solutions for two arm spirals ($m = 2$) are on the left, and for 5 arm spirals ($m = 5$) are on the right. The thick lines are obtained as in Fig. 1., they border the regions of zero-order instability for the Lau-Bertin dispersion relation. The bottom thick line is from Eq. (11) for the usual Lau-Bertin stability condition; the middle and upper lines are the boundaries enclosing the acoustic instability according to Eq. (13).

Fig. 3.— Same as Fig. 2 for a self-gravitating, exponential disk at two scale lengths, but now with solid body rotation. The thick lines on the top figures border the regions of instability for the zero-order Lau-Bertin dispersion relation in the case with no shear ($J = 0$ in Eq. (11)).

Fig. 4.— Same as Fig. 3, but to all orders of the small parameter $1/|kr|$ in the absence of self-gravity. All of the unstable growth exhibited in these solutions is from high order terms.

Fig. 5.— Growth rates and frequencies for a non-self-gravitating disk, as in Fig. 4, but with Keplerian rotation. The thick lines on the top figures border the regions of instability for the zero-order, Lau-Bertin dispersion relation in a Keplerian disk without self-gravity (Eq. 16).