Comments on Large $N$ Matrix Model

Hiroshige Kajiura$^a$, Akishi Kato$^{a*}$, Sachiko Ogushi$^b$

$^a$) Graduate School of Mathematical Sciences, University of Tokyo
Komaba 3-8-1, Meguro-ku, Tokyo 153-8914, Japan

$^b$) Department of Physics, Faculty of Science, Ochanomizu University
Otsuka 2-1-1, Bunkyo-ku, Tokyo 112-8610, Japan

Abstract

The large $N$ Matrix model is studied with attention to the quantum fluctuations around a given diagonal background. Feynman rules are explicitly derived and their relation to those in usual Yang-Mills theory is discussed. Background D-instanton configuration is naturally identified as a discretization of momentum space of a corresponding QFT. The structure of large $N$ divergence is also studied on the analogy of UV divergences in QFT.

PACS: 11.10.Kk; 11.15.Bt; 11.15.Pg; 11.25.-w;

Keywords: Matrix theory; Large $N$ limit; Perturbation theory; Gauge theory

† kuzzy@ms.u-tokyo.ac.jp
* akishi@ms.u-tokyo.ac.jp
* g9740505@edu.cc.ocha.ac.jp
1 Introduction

Matrix model provides a new paradigm for thinking about fundamental theories of physics. It originates in the observation [1] that the massless modes propagating along the world volume of $N$ coincident D-branes are those of the supersymmetric Yang-Mills theory, obtained by the dimensional reductions of the $D = 10 N = 1$ theory down to $p + 1$ spacetime dimensions.

According to so-called Matrix conjecture [2], 0 + 1-dimensional reduction can be regarded as the discrete light cone quantization of M-theory in which the spacetime is compactified on an almost light like circle. This proposes a concrete, a nonperturbative definition of quantum gravity, and quite remarkably, the conjecture has found quite nontrivial support [3, 4, 5, 6].

Meanwhile, type IIB matrix model proposed by [7] plays somewhat complementary role. Its action is 0+0-dimensional reduction of large $N$ super Yang-Mills theory in ten dimensions. The authors of [8] proposed a very interesting program to study dynamical formation of space-time using the type IIB matrix model.

Despite much remarkable success of Matrix model approach, the question “Why and how such a simple model could describe our real world?” is still elusive. The main difficulty consists in the absence of built-in rules concerning “How to take large $N$ limit.” For example, in the case of Matrix model approach to 2d gravity [9], there is a critical point $g_c$, and continuum limit is possible keeping certain relation between $N$ and $g - g_c$ (double scaling limit). For type IIB matrix model, however, the coupling constant $g$ can be absorbed into the rescaling of the fields (at least classically) and there is no nontrivial fixed point.

The result of matrix integration is just a number as it stands. To extract physical intuition, we need to separate field variables into two types: the classical background and the quantum fluctuation.

In the spirit of Born-Oppenheimer approximation, the effective dynamics of the slow variables (classical background) are of primary concern which is obtained only after fast variables (quantum fluctuations) are integrated out. This is the approach taken by many works.

In Matrix models, however, somewhat different approach might be of considerable interest. Recall that in the usual analysis of quantum field theory, gravitational effects are almost always ignored, although gravitons are massless and never decouple. Gravitational degree of freedom are not integrated over, but regarded as fixed, classical background. This treatment is justified
simply because the dimensionful coupling is so small in the energy scale accessible by the current
technology. Similarly for the observers living on the branes, natural time scale is set by that
of quantum fluctuations rather than the dynamical time scale of the background = spacetime.
Put it differently, “motion of the background is too slow to be treated quantum mechanically.”

In this paper, we will study the quantum dynamics of the Matrix model from the latter point
of view, hoping our work provide some insight about how to take large $N$ limit. The paper is
organized as follows.

In section two, starting from $0 + 0$-dimensional matrix action, we derive fatgraph Feynman
rules for the quantum fluctuations treating general multi D-instanton configuration as a fixed
background. The usage of the Feynman rules is shown with an example. Although we will work
in D-instanton backgrounds of IIB matrix model, we expect our analysis shed some light on
general D-$p$ branes in other Matrix theories as well, since type IIB matrix model compactified
on $S^1$ is equivalent to the $1 + 0$-dimensional Matrix model [10].

The matrix Feynman rules are very close to those in the usual $d$-dimensional SYM. In
section three, we will study a special backgrounds where D-instantons are concentrated along
d-dimensional sheet in the original $D$-dimensional spacetime. We will see finite $N$ theories can
be thought of as UV regulated versions of flat space Yang-Mills theory in which removing the
cutoff is equivalent to letting $N$ go to infinity. The crucial observation of this paper is that
from Yang-Mills perturbation point of view, going to Matrix model can be thought of as a
discretization of a momentum space rather than a coordinate space. This is shown explicitly by
comparing Feynman rules. This is yet another manifestation of spacetime uncertainty [11] or
UV/IR correspondence [12].

For the $d$-dimensional quantum field theory embedded in the Matrix model, the only source
of divergence is the large $N$ limit. In section four, we will study the structure of large $N$
divergences in Matrix theory and relate it to the renormalizability of QFT in the usual sense
of the term. We hope this line of argument give us a hint to deduce realistic physics from the
Matrix models. Finally, we conclude with a discussion of our results and some implications.
2 Matrix perturbation theory around D-instanton background

In this section, we elaborate the perturbation theory of the Matrix model around D-instanton background. Explicit forms of Feynman rules are derived and boson self energy diagrams are computed at one loop as an example.

2.1 The type IIB matrix model and its gauge fixing

Our starting point is the Euclidean type IIB matrix model, whose action is given by

\[ S = -\frac{1}{g^2} \text{Tr} \left( \frac{1}{4} [X_\mu, X_\nu]^2 + \frac{1}{2} \bar{\psi} \Gamma^\mu [X_\mu, \psi] \right) \] (2.1)

where \( X_\mu \) and \( \psi \) are \( D \)-dimensional vector and Majorana-Weyl spinor respectively, taking values in \( N \times N \) hermitian matrices. \(^1\) Throughout this paper, “Tr” denotes the trace taken over \( N \times N \) matrix indices.

The action enjoys the following symmetries

- rotation invariance
  \[ \delta X^\mu = \omega^{\mu\nu} X^\nu, \quad \delta \psi = \frac{i}{2} \omega^{\mu\nu} \Gamma^\mu \psi, \quad (\omega^{\mu\nu} = -\omega^{\nu\mu}) \]

- translation invariance
  \[ \delta X^\mu = \epsilon^\mu \] (2.2)

- \( \mathcal{N} = 2 \) SUSY
  \[ \delta^{(1)} \psi = \frac{i}{2} [X_\mu, X_\nu] \Gamma^\mu \epsilon_1, \quad \delta^{(1)} X_\mu = i \tilde{\epsilon}_1 \Gamma_\mu \psi, \quad \delta^{(2)} \psi = \epsilon_2, \quad \delta^{(2)} X_\mu = 0. \]

- \( U(N) \) Gauge invariance
  \[ X^\mu \mapsto U^{-1} X^\mu U, \quad \psi \mapsto U^{-1} \psi U, \quad (U \in U(N)) \] (2.3)

- scaling property:
  \[ X^\mu \mapsto \lambda X^\mu, \quad g \mapsto \lambda^2 g \quad (\lambda \in \mathbb{R}_{>0}) \] (2.4)

\(^1\)We choose \( D = 10 \) type IIB matrix model just for definiteness. We could start from any model reduced from \( D \)-dimensional SYM.
As stated in Introduction, we decompose \( X^\mu \) as a sum of classical background part \( \bar{X}^\mu \) and quantum fluctuation part \( \tilde{X}^\mu \). \( \bar{X}^\mu \) will be treated as fixed, classical number and we will be interested in the quantum field theory in this background. (\( \psi \) is assumed to have no classical vacuum expectation value.)

The background \( \bar{X}^\mu \) must be a solution to the equation of motion, \([X^\mu, [X^\nu, X^\mu]] = 0\). We will consider the cases where all the \( \bar{X}^\mu \)'s are simultaneously diagonalizable by the gauge action (2.3):

\[
X^\mu = \bar{X}^\mu + \tilde{X}^\mu, \quad \tilde{X}^\mu \equiv \begin{pmatrix}
  x_1^\mu \\
  x_2^\mu \\
  \vdots \\
  x_N^\mu
\end{pmatrix}.
\]  

(2.5)

The combination of \( D \) eigenvalues \( x_i \equiv (x_1^i, \cdots x_D^i) \in \mathbb{R}^D \) is interpreted as the location of the \( i \)-th D-instanton. For a generic background where all D-instantons are separated from each other, all the symmetries listed above are explicitly broken. In particular, \( U(N) \) gauge symmetry is broken down to \( U(1)^N \) and half of the \( \mathcal{N} = 2 \) SUSY (generated by \( \delta^{(1)} \)) survives indicating the BPS nature of the background (2.5).

Let us make a brief comment on the charges of the fields. All the quantum fields \( \tilde{X}^\mu_{ij}, \psi_{ij}, c_{ij}, b_{ij} \) have a charge

\[
( 0 \cdots 0 1 0 \cdots 0 -1 0 \cdots 0 )
\]

with respect to the unbroken \( U(1)^N \) gauge symmetry. These fluctuations correspond to the open string stretching between D-instantons \( i \) and \( j \). In particular, diagonal components \( \tilde{X}^\mu_{ii}, \psi_{ii}, c_{ii} \) and \( b_{ii} \) are neutral. In fact, as we will soon see, their kinetic terms vanish indicating they should be treated as collective coordinates rather than quantum variables, and thus need a separate treatment. Since these diagonal components could be absorbed into the shift of the background D-instanton configuration, incorporating these fluctuations would inevitably lead to the integral over the collective coordinates, which is beyond the scope of this paper.

We will study the quantum theory of fluctuations as parameterized by the classical background i.e. D-instanton positions \( \{x_i\}_{i=1}^N \). Plugging (2.5) into the action (2.1) and using the
relation $[\tilde{X}^\mu, \tilde{X}^\nu]_{ij} = (x_i^\mu - x_j^\mu)\tilde{X}^\nu_{ik}$, we have

$$S = \frac{1}{2g^2} \sum_{i,j} \{(x_{ij})^2 \tilde{X}^\nu_{ji}\tilde{X}^\mu_{ij} \} + \frac{1}{2g^2} \sum_{i,j} \{x_i^\mu x_j^\nu \tilde{X}^\mu_{ij} \tilde{X}^\nu_{ji} \}$$

$$+ \frac{1}{g^2} \sum_{i,j,k} \{(x_{ik} + x_{jk})^\mu \tilde{X}^\mu_{ij} \tilde{X}^\nu_{jk} \tilde{X}^\nu_{ki} \} + \frac{1}{g^2} \sum_{i,j,k,l} \{\tilde{X}^\mu_{ij} \tilde{X}^\mu_{jk} \tilde{X}^\mu_{kl} \tilde{X}^\mu_{li} \} - \frac{1}{2g^2} \sum_{i,j} \{\bar{\psi}_{ji} \Gamma^\mu x_{ij}^\mu \psi_{ij} \}$$

$$+ \frac{1}{2g^2} \sum_{i,j,k} \{\bar{\psi}_{ij} \Gamma^\mu x_{jk}^\mu \tilde{X}^\nu_{ki} - \bar{\psi}_{ij} \Gamma^\mu \tilde{X}^\nu_{jk} \psi_{ki} \}. \quad (2.7)$$

It should be noted that $x_i$’s always appear as difference $x_i - x_j$ due to translational invariance (2.2). Hereafter the notation $x_{ij}^\mu = x_i^\mu - x_j^\mu$ will be used to simplify the formulas.

To setup a perturbation theory, convenient to work with the background field gauge. This is achieved by adding the gauge fixing term $S_{g.f.} = -\frac{1}{2g^2} \text{Tr}[\tilde{X}^\mu, \tilde{X}^\nu]^2 = -\frac{1}{2g^2} \sum_{i,j} \{x_{ij}^\mu x_{ji}^\nu \tilde{X}^\mu_{ij} \tilde{X}^\nu_{ji} \}$

accompanied with the Faddeev-Popov ghost term

$$S_{F.P.} = -\frac{1}{g^2} \text{Tr}[\tilde{X}^\mu, b] [X^\mu, c]$$

$$= \frac{1}{2g^2} \sum_{i,j} \{(x_{ij}^\mu)^2 b_{ij} c_{ji} \} + \frac{1}{2g^2} \sum_{i,j,k} \{x_{ij}^\mu (c_{jk} \tilde{X}^\mu_{ki} - \tilde{X}^\mu_{jk} c_{ki}) \}. \quad (2.9)$$

The gauge fixing term (2.8) implies we have chosen a gauge such that the fluctuation $\tilde{X}_{ij}$ is transverse to the relative vector $x_{ij}$, i.e. $\sum_{\mu} x_{ij}^\mu \tilde{X}^\mu_{ij} = 0$.

The gauge fixed total action is given by

$$S_{\text{total}} = S + S_{g.f.} + S_{F.P.} \quad (2.10)$$

Note that $S_{g.f.}$ in (2.10) cancels with the second term of (2.7). The perturbation is valid when D-instanton separation $x_{ij}$ is much larger than $g^{1/2}$.

### 2.2 Feynman rules

Now we will derive Feynman rules from the gauge fixed action (2.10). Hereafter we will rescale the quantum fluctuations $\tilde{X}_{ij}^\mu \rightarrow g \tilde{X}_{ij}^\mu$, etc, in order that the coupling $g$ is removed from the propagators and moved to the interaction vertices.

The perturbative structure of large $N$ gauge theories are naturally described in terms of double line representation [13] of Feynman diagrams, so called fatgraphs. One considers, as in
Figs. 1 and 2, a graph with the lines thickened slightly into bands which meet smoothly at the vertices so as to form an oriented Riemann surface with boundary. In this representation, the fields in the adjoint representation of $U(N)$ are denoted by double lines.

Each edge carries a label $i,j,\ldots \in \{1,\ldots,N\}$ corresponding to the basis of fundamental representation of $U(N)$ (or its conjugate depending on its orientation). Here it is nothing but the label of a D-instanton. From the quadratic part of the total action (2.10), we can easily read off the fatgraph propagators as depicted in Fig. 1. Note that the denominator is the squared distance between the two D-instantons connected by the fields.

As for vertices, there is a crucial difference between ordinary (particle theory) vertex factor and fatgraph counterpart. In the former, interaction vertex of order $k$ is invariant under all possible $k!$ permutations of lines, while in the latter, it is invariant only under $k$ cyclic permutations. Thus, for example, two Yukawa coupling diagrams in Fig. 2 should be distinguished from each other.

In deriving the vertex factors, terms must be organized so that the index contraction should have manifest cyclic invariance. For instance, in order to deduce three-point and four-point vertex for $\tilde{X}$’s, we must rewrite corresponding terms in (2.7) as follows

$$\frac{1}{g^2} \sum_{i,j,k} \{ (x_{ik} + x_{jk})^\mu \tilde{X}^\mu_{ij} \tilde{X}^\nu_{jk} \tilde{X}^\nu_{ki} \}$$

$$= -\frac{1}{3g^2} \sum_{i,j,k} \sum_{\mu_1,\mu_2,\mu_3} \tilde{X}^\mu_{j} \tilde{X}^\mu_{k} \tilde{X}^\mu_{i}$$

$$\times \{ \delta^{\mu_1\mu_2}(x_{jk} - x_{ki})^{\mu_3} + \delta^{\mu_2\mu_3}(x_{ki} - x_{ij})^{\mu_1} + \delta^{\mu_3\mu_1}(x_{ij} - x_{jk})^{\mu_2} \}$$

$$\times \frac{1}{x_{ij}}$$
Figure 2: Vertices
\[
\frac{1}{2g^2} \sum_{i,j,k,\ell} \{ \tilde{X}^\mu_{ij} \tilde{X}^\nu_{jk} \tilde{X}^\nu_{\ell i} - \tilde{X}^\mu_{ij} \tilde{X}^\nu_{jk} \tilde{X}^\mu_{\ell i} \} = -\frac{1}{4g^2} \sum_{i,j,k,\ell} \sum_{\mu_1,\cdots,\mu_4} \tilde{X}^{\mu_1}_{ij} \tilde{X}^{\mu_2}_{jk} \tilde{X}^{\mu_3}_{\ell i} \tilde{X}^{\mu_4}_{ik} \\
\times \left\{ 2\delta^{\mu_1\mu_3}\delta^{\mu_2\mu_4} - \delta^{\mu_1\mu_2}\delta^{\mu_3\mu_4} - \delta^{\mu_1\mu_4}\delta^{\mu_2\mu_3} \right\}.
\]

Overall factors 1/3 and 1/4 are cancelled by the cyclic symmetry of the vertices.

Similar computation leads to the vertex factors listed in Fig. 2.

### 2.3 Example: one loop boson self energy

In order to illustrate how matrix perturbation theory works, let us calculate the one loop contribution to two point function

\[
\langle X^\mu_{ij} X^\nu_{ji} \rangle
\]

using the Feynman rules just derived. Relevant fatgraphs are shown in Fig. 3. In addition to the “external” D-instantons \(i\) and \(j\), we need to incorporate an “internal” D-instanton \(k\) as in Fig. 4.

Let us begin with the diagram (a). It can be seen as representing the history of an open string; the string \(ij\) splits into two pieces \(ik\) and \(kj\), and reconnect in the end. Applying the Feynman rules, this diagram contributes

\[
(a) = \frac{g^2}{x_{ij}} \sum_{k} \sum_{\kappa,\lambda} \frac{1}{x^2_{ik} x^2_{jk}} \times \left\{ \delta^{\kappa\lambda}(x_{ij} - x_{jk}) + \delta^{\kappa\lambda}(x_{ik} - x_{kj}) + \delta^{\kappa\lambda}(x_{ji} - x_{ik}) \right\}
\]

\[
\times \left\{ \delta^{\kappa\lambda}(x_{kj} - x_{ji}) + \delta^{\kappa\lambda}(x_{ki} - x_{kj}) + \delta^{\kappa\lambda}(x_{ki} - x_{kj}) \right\}. \tag{2.11}
\]

Here we denote by \(\lambda\) and \(\kappa\) the \(SO(D)\) vector indices associated to the upper and lower internal propagators respectively. Similarly, from the diagrams (b) and (c), we have

\[
(b) + (c) = -\frac{g^2}{x_{ij}} \sum_{k} \left( \frac{1}{x^2_{ik}} + \frac{1}{x^2_{jk}} \right) (1 - D) \delta_{\mu\nu}. \tag{2.12}
\]

Ghost contributions

\[
(d) + (e) = -\frac{2g^2}{x_{ij}} \sum_{k} \frac{x^\mu_{jk} x^\nu_{ik}}{x^2_{ik} x^2_{jk}}. \tag{2.13}
\]

\(^2\)No sum is taken for indices \(i\) or \(j\); it is invariant under \(U(1)^N\) gauge symmetry.
Figure 3: One loop boson self energy graphs

Figure 4: D-instanton configuration associated with the one-loop processes in Fig. 3
Finally fermion loop diagram gives
\[
(f) + (g) = -\frac{d \Gamma g^2}{2x_{ij}^2} \sum_k x_{jk}^\mu x_{ik}^\nu x_{jk} x_{ik}^\mu x_{ik}^\nu - 4 \delta^{\mu\nu} x_{jk} \cdot x_{ik},
\]
where \(d \Gamma\) is the size of gamma matrices in \(D\)-dimensions.

### 3 Correspondence to quantum field theories

From the sample calculation given in section 2.3, we notice a strong similarity between the matrix perturbation theory and a usual \(d\)-dimensional QFT. Roughly speaking, relative brane position \(x_{ij}\) corresponds to a momentum \(p\) whereas the sum over branes \(\sum_k\) looks like a loop integral \(\int d^d p\).

In this section, we will make this analogy more precise. In particular, we illustrate how \(d\)-dimensional gauge theories can be recovered from Matrix model, when D-instanton configuration has \(d\)-dimensional flat directions. In this context, the flat directions should be thought of as momentum coordinates rather than spatial coordinates, contrary to naive expectations. This can be considered as an example of IR/UV correspondence [12].

Infinite sums or integrals will pose a delicate problem of large \(N\) divergences. We will postpone discussing this issues to section 4.

#### 3.1 D-instanton distribution as discretized momentum space

We begin with recalling a general structure of \(d\)-dimensional gauge theory amplitudes. Let \(A^\mu(x)\) be \(U(n)\) gauge fields represented as \(n \times n\) matrices and \(A^\mu(p)\) their Fourier transform. Hereafter the symbol “\(\text{tr}\)” will denote the trace over \(n \times n\) matrix indices. Any gauge invariant correlation function can be decomposed as a sum of basic correlation functions of the form
\[
\langle \text{tr} A^{p_1}(p_1) A^{p_2}(p_2) \cdots A^{p_k}(p_k) \rangle, \quad (p_1 + \cdots + p_k = 0)
\]
which is invariant under the cyclic permutation of momenta and Lorenz indices:
\[
p_1 \rightarrow p_2, \quad \cdots, \quad p_k \rightarrow p_1, \\
\mu_1 \rightarrow \mu_2, \quad \cdots, \quad \mu_k \rightarrow \mu_1.
\]
In the amplitude (3.1), the cyclic order of \([(p_i, \mu_i)]_{i=1}^k\) has a definite meaning because \(U(n)\) indices are implicitly contracted. It is easy to check that there is one-to-one correspondence among the following three data:
Figure 5: Correspondence of a correlation function between gauge theory and Matrix model. 
(a): closed path formed by external momenta. (b): corresponding D-instanton configuration.

(A) gauge invariant amplitude (3.1),

(B) ordered set of momenta and Lorentz indices \( \{(p_1, \mu_1), (p_2, \mu_2), \ldots, (p_k, \mu_k)\} \) modulo cyclic permutation (3.2),

(C) closed oriented path in \( \mathbb{R}^d \) with edge labeling \( \mu_1, \ldots, \mu_k \) (Fig. 5 (a)).

We now come back to Matrix model correlation functions. In generic background (2.5) we still have unbroken \( U(1)^N \) gauge symmetry. Therefore, only gauge invariant “Wilson loops” such as

\[
\langle \tilde{X}^{\mu_1}_{i_1 i_2} \tilde{X}^{\mu_2}_{i_2 i_3} \cdots \tilde{X}^{\mu_k}_{i_k i_1} \rangle
\]

(3.3)
can be nonzero. Actually, we can draw a corresponding loop as in Fig. 5 (b): two D-instantons located at \( x_{i_r} \) and \( x_{i_{r+1}} \) are connected by a field (or an open string) \( \tilde{X}^{\mu_r}_{i_r i_{r+1}} \). The amplitude (3.3) is invariant under the cyclic permutation of D-instanton positions and Lorenz indices:

\[
i_1 \rightarrow i_2 \rightarrow \cdots \rightarrow i_k \rightarrow i_1,
\]

\[
\mu_1 \rightarrow \mu_2 \rightarrow \cdots \rightarrow \mu_k \rightarrow \mu_1.
\]

(3.4)

The correspondence between the two figures Fig. 5 (a) and (b) is now obvious; the momentum \( p_r \) in gauge theory is identified with the D-instanton separation \( x_{i_{r+1}} - x_{i_r} \). In this way, we can add two new entries to the previous list of one-to-one correspondence:

(D) D-instanton correlation functions (3.3),
loop passing through \( k \) D-instantons in \( \mathbb{R}^d \) with labeled edges \( \mu_1, \ldots, \mu_k \)
(Fig. 5 (b)).

Clearly, arbitrary sequence of \( d \)-dimensional momenta \((p_1, \ldots, p_k)\) can be realized as a loop in D-instanton configuration space, provided D-instantons are densely distributed over \( d \)-dimensional Euclidean space \( \mathbb{R}^d \). Conversely, if we start from finite \( N \) matrix theory, only discrete momentum points are available on gauge theory side. Of course, the error becomes smaller as the number of D-instantons is increased. Guided by these observation, we propose to identify a D-instanton distribution of Matrix theory as a discretization of momentum space seen by a Yang-Mills theory.

Precisely speaking, the “momentum path” (Fig. 5 (a)) can determine the “D-instanton path” (Fig. 5 (b)) only up to overall translation, \( x_i \rightarrow x_i + c \). This ambiguity can be resolved by fixing, say, their center of mass at the origin.

So far we neglected the problem how non-Abelian \( U(n) \) gauge symmetry can be recovered from D-instanton picture. This will be discussed in section 3.4.

In sum, we have argued that when the background D-instantons are continuously distributed along \( \mathbb{R}^d \), there is a one-to-one correspondence between the gauge theory amplitude (3.1) and (3.3).

### 3.2 Correspondence of Feynman diagrams

In perturbation theory, both amplitudes (3.1) and (3.3) are expressed as a sum over fatgraphs with fixed external lines. We now want to show that two computations, one as a Yang-Mills theory and the other as a Matrix theory, actually coincide for every fatgraph. To do this, we need to check the correspondence at the level of propagators and vertices.

Consider a fatgraph \( \Gamma \) made of several propagators and vertices. Recall that the graph form an oriented Riemann surface with boundaries. Thus for a given propagator with an orientation, it is meaningful to talk about its “left-” and “right-” edges.

Pick up a propagator and let \( i \) and \( j \) be its labels on left- and right-edges, repectively. In Matrix picture, the propgator represents a fluctuation \( X_{ij}^\mu \) connecting two D-instantons \( i \) and \( j \). As in section 3.1, we identify the relative separation of D-instantons

\[
x_{ij} = x_i - x_j
\]  

with the momentum carried by the propagator in a corresponding QFT.
This identification would be inconsistent if there was an interaction vertex which breaks their conservation law. In fact, any interaction vertex shown in Fig. 2 has following properties:
(i) edges of the propagators are glued together in a definite cyclic order around the vertex so that successive edges share a common D-instanton label, (ii) all incoming momenta are given as the differences of $x_i$’s associated to successive boundaries. These two facts guarantee the sum of the incoming momenta is automatically zero.\footnote{The momentum conservation is a direct consequence of $U(1)^N$ gauge invariance. It is, however, nontrivial whether one can represent the momenta as differences successively.} For example, consider an interaction vertex among three D-instantons depicted in Fig. 6 (a). Since the incoming momenta are defined as Fig. 6 (b), their sum $x_{ij} + x_{jk} + x_{ki}$ vanishes. It is now straightforward to check that Feynman rules given in Figs. 1 and 2 exactly coincide with those of a usual Yang-Mills theory.

### 3.3 From sums to integrals

In an ordinary perturbation method, we need to integrate over interaction positions in $d$-dimensional spacetime, which results in loop integrals. In the matrix perturbation theory, we need to sum over intermediate D-instanton positions. The correspondence between these two implies the equivalence of perturbation theory, because we have just seen that each fatgraph has the same factors both in Matrix and QFT pictures.

As discussed in section 3.1, D-instanton configuration along flat $\mathbb{R}^d$ can reproduce $d$-dimensional momentum space. More concretely, let us assume D-instantons fill uniformly a $d$-dimensional hyperplane

$$H := \left\{ (x^1, \ldots , x^d, 0, \ldots , 0) \in \mathbb{R}^D \right\} \simeq \mathbb{R}^d \quad (3.6)$$
with density \( \rho \) (i.e. there are \( \rho \) D-instantons per unit \( d \)-dimensional volume). Then, we can replace the sum over brane positions \( \sum_k \) by the loop integral \( \rho \int d^dp \).

For example, in the case of the two point function \( \langle X^\mu_{ij} X^\nu_{ji} \rangle \) of section 2.3, we can choose the point \( x_i \) as the origin of the momentum space. Then the translation dictionary reads

\[
\begin{align*}
\text{Matrix} & \quad \rightarrow \quad \text{QFT} \\
x_{ij} & \quad \rightarrow \quad q \quad \text{(external momentum)} \\
x_{jk} & \quad \rightarrow \quad p \quad \text{(loop momentum)} \\
x_{ik} & \quad \rightarrow \quad q + p \\
\sum_k \cdots & \quad \rightarrow \quad \rho \int d^dp \cdots 
\end{align*}
\]

Then eqs. (2.11), (2.13), (2.14) can be written as

\[
\begin{align*}
(a) &= -\frac{\rho \mu^2}{q^4} \int d^dp \frac{1}{(q+p)^2} \cdot \frac{p^2}{p^2 \\
&\times \{ \delta^{\mu\nu}(q-p)^\lambda + \delta^{\kappa\lambda}(2q+p)^\mu + \delta^{\lambda\mu}(-q-2p)^\kappa \} \\
&\times \{ \delta^{\kappa\nu}(p-q)^\lambda + \delta^{\nu\lambda}(p+2q)^\kappa + \delta^{\lambda\nu}(-2p-q)^\nu \} \\
(b)+(c) &= -2\rho \mu^2 \int d^dp \left( \frac{1}{(q+p)^2} + \frac{1}{p^2} \right) (1-d) \delta_{\mu\nu} \\
(d)+(e) &= -\frac{\rho \mu^2}{q^4} \int d^dp \frac{p^\mu(q+p)^\nu}{(q+p)^2 p^2} \\
(f)+(g) &= -\frac{d \mu^2}{2q^2} \int d^dp \frac{p^\mu(q+p)^\nu + p^\nu(q+p)^\mu - \delta^\mu\nu p \cdot (q+p)}{p^2(q+p)^2} 
\end{align*}
\]

which look more familiar as those in standard QFT textbooks.

### 3.4 Non-Abelian gauge symmetry in Matrix model

Now we come back to the problem how we can incorporate the non-Abelian gauge symmetry of \( d \)-dimensional SYM theory starting from Matrix models. Actually, without this non-Abelian structure, we cannot explain why the cyclic order of momenta is important in the amplitudes (3.1).

To achieve this, we need to consider the coincident D-branes as in [1]. Suppose we want to realize \( U(n) \) gauge symmetry. We need to put \( n \) D-instantons at the same point in \( \mathbb{R}^D \).
Hereafter the word “cluster” will be used to designate the \( n \) coincident D-instantons. The \( N \) D-instantons are thus grouped into \( M \equiv N/n \) clusters.

We choose a background in which cluster \( r \) is located at \( x_r = (x^1_r, \ldots, x^d_r, 0, \ldots, 0) \in \mathbb{R}^D \):  

\[
\bar{X}^\mu = \begin{pmatrix}
x^\mu_1 & 0 & \cdots & 0 \\
0 & x^\mu_1 & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & x^\mu_M \\
\end{pmatrix} \quad (\mu = 1, \ldots, d),
\]

\[
\bar{X}^m = 0 \quad (m = d + 1, \ldots, D).
\]

Here \( \mathbf{1} \) and \( \mathbf{0} \) denote unit and zero matrix of size \( n \), respectively.

In this background, \( U(N) \) gauge symmetry is broken down to \( U(n)^M \) generated by

\[
U = \begin{pmatrix}
U_{11} & 0 & \cdots & 0 \\
0 & U_{rr} & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & U_{MM} \\
\end{pmatrix}.
\]

As for fluctuations, it is useful to divide \( N \times N \) matrix into the blocks of size \( n \times n \) as

\[
\bar{X}^\mu = \begin{pmatrix}
\bar{X}^\mu_{11} & \cdots & \cdots & \bar{X}^\mu_{1M} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\bar{X}^\mu_{M1} & \cdots & \cdots & \bar{X}^\mu_{MM} \\
\end{pmatrix} \quad (\mu = 1, \ldots, d),
\]

\[
\bar{X}^m = \begin{pmatrix}
\bar{X}^m_{11} & \cdots & \cdots & \bar{X}^m_{1M} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\bar{X}^m_{M1} & \cdots & \cdots & \bar{X}^m_{MM} \\
\end{pmatrix} \quad (m = d + 1, \ldots, D),
\]

\[
\psi = \begin{pmatrix}
\psi_{11} & \cdots & \cdots & \psi_{1M} \\
\vdots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
\psi_{M1} & \cdots & \cdots & \psi_{MM} \\
\end{pmatrix}.
\]

\footnote{We use \( r,s,\ldots \in \{1, \ldots, M\} \) for cluster indices. \( i,j,\ldots \in \{1, \ldots, N\} \) are reserved for D-instanton labels.}

15
Figure 7: D-instanton configuration corresponding to $d$-dimensional SYM. Each cluster is denoted by a blob. The clusters are distributed along $\mathbb{R}^d$ with average spacing $b$.

Each $n \times n$ block $\tilde{X}^\mu_{rs}$, $\psi_{rs}$ transforms as a bi-fundamental representation of $U(n)_r \times U(n)_s$ subgroup of $U(n)^M$. As we will see shortly (cf. eq. (3.14) below), the fluctuations $\tilde{X}^\mu_{rs}$ ($\mu = 1, \ldots, d$) tangent to $H$ will be identified with the $U(n)$ gauge field whereas those in the transverse direction $\tilde{X}^m_{rs}$ ($m = d+1, \ldots, D$) will play the role of Higgs fields. Similarly, $D$-dimensional spinor $\psi_{rs}$ will be identified with their super partners.

In this notation, $U(n)^M$ invariant correlation functions are something like

$$\langle \text{tr} X^\mu_{r_1 r_2} X^\mu_{r_2 r_3} \cdots X^\mu_{r_k r_1} \rangle.$$ (3.12)

The correspondence given in sections 3.1, 3.2 and 3.3 is true to this non-Abelian case, provided “D-instantons” are now replaced by “clusters.”

3.5 $d$-dimensional super Yang-Mills action from Matrix model

So far, we have studied how a gauge theory is embedded into Matrix theory, through the correspondence of correlation functions.

To complete our analysis and to extract further intuition, it will be useful to rewrite the original Matrix model action into that of QFT: $d$-dimensional reduction of SYM in $D$ dimensions.
In the notation given in (3.9) and (3.11), the action (2.1) reads

\[ S = -\frac{1}{g^2} \sum_{r,s} \text{tr} \left\{ \frac{1}{4} (x^\mu_{rs} \tilde{X}^\nu_{rs} - x^\nu_{rs} \tilde{X}^\mu_{rs}) (x^\mu_{sr} \tilde{X}^\nu_{sr} - x^\nu_{sr} \tilde{X}^\mu_{sr}) + \frac{1}{2} (x^\mu_{rs} \tilde{X}^m_{rs} + [\tilde{X}^\mu, \tilde{X}^m]_{rs}) (x^\mu_{sr} \tilde{X}^m_{sr} + [\tilde{X}^\mu, \tilde{X}^m]_{sr}) \right\} \]

\[ + \frac{1}{4} [\tilde{X}^m_{rs}, \tilde{X}^m_{sr}] \]

(3.13)

where the trace “tr” is taken over \( n \times n \) matrix indices.

Let us approximate the sum by the \( d \)-dimensional integral, which looks more like a QFT. As in section 3.4, we assume the clusters \( \{x_r\} \) are uniformly distributed on \( d \)-dimensional hyperplane \( H \) with a constant cluster density \( \rho' \). Since each cluster consists of \( n \) D-instantons, \( \rho' \) is related to the D-instanton density \( \rho \) as \( n \rho' = \rho \).

Renaming the \( n \times n \) matrix valued fields as

\[ \tilde{X}^\mu_{rs} \Rightarrow \tilde{X}^\mu(p) \equiv A^\mu(p) \]

\[ \tilde{X}^m_{rs} \Rightarrow \tilde{X}^m(p) \equiv \Phi^m(p) \]

(3.14)

and the continuum approximation similar to (3.7),

\[ x^\mu_{rs} \Rightarrow p^\mu \]

\[ \sum_s \cdots \Rightarrow \rho' \int d^d p \text{tr} \cdots, \]

(3.15)

the action (3.13) reads

\[ S = -\frac{1}{g^2} \rho' \int d^d p \text{tr} \left\{ \frac{1}{4} (p^\mu \tilde{X}^\nu(p) - p^\nu \tilde{X}^\mu(p)) (\tilde{X}^\mu(p)(-p) + [\tilde{X}^\mu, \tilde{X}^\nu](-p)) \right\} \]

\[ + \frac{1}{2} (p^\mu \tilde{X}^m(p) + [\tilde{X}^\mu, \tilde{X}^m](p)) (\tilde{X}^m(p)(-p) + [\tilde{X}^\mu, \tilde{X}^m](-p)) \]

\[ + \frac{1}{4} [\tilde{X}^m(p), \tilde{X}^m](p)[\tilde{X}^m, \tilde{X}^m](-p) \]

\[ + \frac{1}{2} \tilde{\psi}(-p) \Gamma^\mu(p^\mu \tilde{\psi}(p) + [\tilde{X}^\mu, \tilde{\psi}](p)) \]

\[ + \frac{1}{2} \tilde{\psi}(-p) \Gamma^m[\tilde{X}^m, \tilde{\psi}](p). \]

(3.16)
This is a momentum space representation of the super Yang-Mills action in \(d\)-dimensions.

Using formula like

\[
x^\mu(p)\hat{X}^\nu(p) = -i \int d^d x (\partial^\mu A^\nu(x)) e^{-ip\cdot x}
\]

\[
[X^\mu, \hat{X}^\nu](p) = \rho'(2\pi)^d \int d^d x [A^\mu(x), A^\nu(x)] e^{-ip\cdot x},
\]  \( (3.17) \)

inverse Fourier transform of \((3.16)\) gives

\[
S = \frac{(\sum_r 1)}{g^2} \rho'(2\pi)^d \int d^d x \frac{1}{4} \left\{ (\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)) + i\rho'(2\pi)^d [A^\mu(x), A^\nu(x)] \right\}^2
\]

\[
+ \frac{1}{2} \left\{ (\partial^\mu \Phi^m(x) + i\rho'(2\pi)^d [A^\mu(x), \Phi^m(x)]) \right\}^2
\]

\[
- \frac{1}{4} \left\{ \rho'(2\pi)^d [\Phi^m(x), \Phi^m(x)] \right\}^2
\]

\[
+ \frac{i}{2} \left\{ \bar{\psi}(x) \Gamma^\mu (\partial^\mu \psi(x) + i\rho'(2\pi)^d [A^\mu(x), \psi(x)]) \right\}
\]

\[
- \frac{1}{2} \left\{ \bar{\psi}(x) \Gamma^m \rho'(2\pi)^d [\Phi^m(x), \psi(x)] \right\}.
\]  \( (3.18) \)

Rescaling the fields as

\[
\rho'(2\pi)^d A^\mu(x) \rightarrow A^\mu(x)
\]

\[
\rho'(2\pi)^d \Phi^m(x) \rightarrow \Phi^m(x)
\]

\[
\rho'(2\pi)^d \psi(x) \rightarrow \psi(x)
\]  \( (3.19) \)

and defining the \(d\)-dimensional coupling constant by

\[
g_d^2 \equiv (2\pi)^d \rho' g^2,
\]  \( (3.20) \)

we are finally lead to a familiar form of \(d\)-dimensional SYM coupled with adjoint matters:

\[
S_d = \int d^d x \frac{1}{g_d^2} \text{tr} \left\{ \frac{1}{4} (F^{\mu\nu})^2 + \frac{1}{2} (D^\mu \Phi^m)^2 - \frac{1}{4} [\Phi^m, \Phi^m]^2
\]

\[
+ \frac{i}{2} \bar{\psi} \Gamma^\mu (D^\mu \psi) - \frac{1}{2} \bar{\psi} \Gamma^m [\Phi^m, \psi] \right\}
\]  \( (3.21) \)

\[^5\text{We neglected the factor } (\sum_r 1). \text{ This reflects the fact that overall shift of D-instanton configuration results in the same momentum configuration in SYM picture. To make the mapping one-to-one, we need to specify the origin of momentum space as discussed toward the end of section 3.1. (Admittedly, the replacement } (3.7) \text{ is somewhat misleading.) Other possibility is to introduce an additional gauge symmetry to constrain the off-diagonal components as } X_{r,s} = X_{r+k,s+k} \text{ } (\forall k \in \mathbb{Z}). \text{ This gauge symmetry kills the ambiguity of overall shift, but the clusters need to be arranged periodically on a lattice. The latter method is equivalent to the } S^1 \text{ compactification proposed by W. Taylor } [14].\]
Here, the standard covariant derivative for the adjoint matter $D^\mu \Phi^m = \partial^\mu \Phi^m + i [A^\mu, \Phi^m]$ and the field strength $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + i [A^\mu, A^\nu]$ is used.

Our procedure can be schematically summarized as

$$\sum_i \cdots \Longleftrightarrow \sum_s \text{tr} \cdots \Longleftrightarrow \rho' \int d^d p \text{tr} \cdots . \quad (3.22)$$

Now we make a few comments on the hermiticity of the fields. If we naively interpreted the brane configuration as the discretization in the spatial coordinates (à la lattice gauge theory) rather than the momentum coordinates, then an $n \times n$ block $X^\mu_{rs}$ must be interpreted as a $U(n)$ connection (or parallel transport) matrix connecting two points $x_r$ and $x_s$. Then it must be anti-hermitian (or unitary) matrix. But, since $X^\mu_{rs}$ is just a part of a much larger $N \times N$ hermitian matrix, there is no a priori reason why it should take such special forms. It might be possible to devise a mechanism to put such a constraint, it would lead to additional complication. Actually, the hermiticity as a $N \times N$ matrix leads to $(\tilde{X}^\mu_{rs})^\dagger = \tilde{X}^\mu_{sr}$. This is consistent with our identification $X^\mu_{rs}$ with $A^\mu(p)$, because the hermiticity of the gauge field $A^\mu(x) = \int (\frac{dp}{2\pi})^d (A^\mu(p)e^{ip \cdot x} + \text{h.c.})$ leads to the condition $(A^\mu(p))^\dagger = A^\mu(-p)$.

Note also that the diagonal parts $X^\mu_{ii}$ of $N \times N$ matrix field correspond to the zero-momentum, Cartan component of $U(N)$ gauge field.

4 Divergences in large $N$ matrix theory

Although the Matrix model is defined for arbitrary value of $N$, interesting physics is believed to emerge from the large $N$ limit. But there seems to be no general argument or rule concerning what kind of large $N$ limit should be taken.

It is well known that in the so-called ’t-Hooft limit $N \to \infty$ with $g^2_{YM}N$ being fixed, $U(N)$ Yang-Mills theory simplifies drastically, and some exact analysis, say Borel summability became possible. Eguchi and Kawai [15] have argued that four dimensional large $N$ gauge theory can be replaced by Matrix models in zero dimensions. In a sense we are studying the reverse process of the Eguchi-Kawai reduction from a different viewpoint.

In type IIB matrix models, $g$ appears as the overall factor in the action and thus there is no nontrivial critical point for $g$. Yet, it is still controversial [16] how large $N$ limit should be taken; $g$ fixed? $g^2N$ fixed? or what else?
Figure 8: D-instanton configuration corresponding to $d$-dimensional SYM. $N$ D-instantons are distributed in a $d$-dimensional ball with radius $\Lambda$ within $\mathbb{R}^d$.

In the case of SYM theory embedded in the matrix perturbation theory, one is faced with an additional complication. Increasing $N$ allows two different interpretation: (i) larger gauge symmetry (ii) larger cutoff (i.e. more degrees of freedom). In the usual QFT case, they are clearly separated and never mixed. Here, however, there is a crosstalk between the two.

In the perturbative computation of $d$-dimensional quantum field theory, divergences arise from the integrals over loop momenta $\int d^dp$. We can regularize the integrals by introducing a UV cutoff $\Lambda$. Suppose the amplitude associated with a Feynman diagram $\Gamma$ diverges as $\Lambda^{D(\Gamma)}$ in the continuum limit $\Lambda \to \infty$. Basically, $D(\Gamma)$ can be determined from the superficial degree of divergence of $\Gamma$ or its subgraphs. Renormalizability is the property that all divergence can be removed if $\Lambda \to \infty$ limit is taken not keeping $g$ fixed but adjusting $g$ so as to maintain a certain functional relation $\mathcal{R}(g, \Lambda/\mu) = 0$. It is of course a very nontrivial problem to find the explicit form of renormalization trajectory $\mathcal{R}(g, \Lambda/\mu) = 0$, but at least perturbatively, it can be determined by carefully analyzing the structure of divergences in Feynman diagrams.

On the other hand, in the matrix perturbation theory we are working with, only source of divergence is sending $N$ to infinity.\textsuperscript{6} In the spirit of correspondence between Matrix model and

\textsuperscript{6}Of course, there are combinatorial divergences due to the infinitely many Feynman graphs. But this is common
Table 1: Various scales in D-instanton distribution

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Expression</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g^{1/2}$</td>
<td>$\sim N^{-\omega/4}$</td>
<td>core size, the minimum distance between D-instantons</td>
</tr>
<tr>
<td>$a$</td>
<td>$\sim N^{-\theta/d}$</td>
<td>D-instanton spacing, average distance to the nearest D-instantons</td>
</tr>
<tr>
<td>$\mu$</td>
<td>$\sim N^0$</td>
<td>renormalization scale, typical momentum scale of the external lines</td>
</tr>
<tr>
<td>$\Lambda$</td>
<td>$\sim N^{(1-\theta)/d}$</td>
<td>cutoff, distance to the farthest D-brane</td>
</tr>
</tbody>
</table>

QFT, continuum limit $\Lambda \to \infty$ should be related to the large $N$ limit. In other words, adding more and more D-instantons on the outskirts of the D-instanton cluster should be equivalent to increasing $\Lambda$. In this picture, the UV cutoff $\Lambda$ is nothing but the distance to the *farthest* D-instanton (see Fig. 7.), which is natural from UV/IR correspondence [12] or spacetime uncertainty [11].

The analysis of large $N$ behavior can be complicated because there is no unique way to add extra D-instantons; the relation between the two limits $N \to \infty$ and $\Lambda \to \infty$ is highly dependent on the strategy of putting new D-instantons.

In this section, we will take a phenomenological approach to clarify the relation between large $N$ limit and continuum limit $\Lambda \to \infty$. $N$ D-instantons are assumed to be concentrated along $d$-dimensional hyperplane with uniform density $\rho$ as in section 3.5. But here, since $N$ is finite, the radius $\Lambda$ of D-instanton cluster is also finite.

### 4.1 Scaling laws

So far we have encountered various length scales. The core size $g^{1/2}$, the D-instanton spacing $a$, the scale of external momenta $\mu$. For finite $N$, the distance $\Lambda$ to the farthest D-instanton will also play an important role. We will study the scaling laws for these length scales. For the readers’ convenience they are listed in Table 1. Although the word “length” will be frequently used, they represent momentum scales in the QFT picture as we argued in the previous section.

We are interested in how they should be varied as $N$ tends to infinity. In this paper, we assume a simple power law scaling and try to draw some bounds on the exponents from physically reasonable assumptions.

Since the quantities in Table 1 are all dimensionful while $N$ is dimensionless, one must decide to Matrix and to both theories and is not discussed in this paper.
\[
\begin{array}{|c|c|c|}
\hline
b & \sim N^{-\frac{\theta}{d}} & \sim \Lambda^{\frac{-\theta}{1-\theta}} \text{ cluster spacing} \\
\hline
\rho & \sim N^\theta & \sim \Lambda^{\frac{d\theta}{1-\theta}} \text{ D-instanton density} \\
\hline
\rho' & \sim N^\theta & \sim \Lambda^{\frac{d\theta}{1-\theta}} \text{ cluster density} \\
\hline
\mathcal{g}_{4d}^2 & \sim N^{\theta-\omega} & \sim \Lambda^{\frac{d(\theta-\omega)}{1-\theta}} \text{ YM coupling in } d\text{-dimensional QFT} \\
\hline
\end{array}
\]

Table 2: Other quantities with nontrivial scaling laws

which is kept fixed in the large \( N \) limit. For this purpose we choose \( \mu \), the momentum scale carried by the external lines in the QFT picture. In other words, all “lengths” discussed in this section are measured in the unit of \( \mu \). For example, we set \( \mu \sim x_{ij} \) when we compute \( \langle X_{ij}^\mu X_{ji}^\nu \rangle \).

The other three quantities \( g^{1/2}, a \) and \( \Lambda \) are assumed to scale with some power of \( N \) which is specified by three independent exponents \( d, \theta \) and \( \omega \) as in Table 1.

In addition to these basic “length” scales, there are some other quantities of interest, with nontrivial \( N \) dependence:

- D-instanton density \( \rho \)

  For finite \( N \), D-instantons are assumed to be distributed uniformly with density \( \rho \) within a \( d \)-dimensional ball of radius \( \Lambda \). Thus we have

  \[
  \rho \Lambda^d \sim N \tag{4.1}
  \]

  Since \( \Lambda \sim N^{(1-\theta)/d} \), (4.1) fixes the scaling of the \( \rho \) as

  \[
  \rho \sim N^\theta. \tag{4.2}
  \]

- cluster spacing \( b \)

  The D-instanton spacing \( a \) and cluster size \( b \) are related via \( na^d = b^d \). Since we fix the rank \( n \) of the gauge group, \( a \) and \( b \) will have the same scaling behavior, \( b \sim a \sim N^{-\theta/d} \).

- cluster density \( \rho' \)

  By the same token, the cluster density \( \rho' \), related to D-instanton density \( \rho \) via \( n\rho' = \rho \) will have the same scaling as \( \rho \).

  \[
  \rho' \sim N^\theta \tag{4.3}
  \]
YM coupling in $d$-dimensions $g_d$

YM coupling $g_d$ in $d$-dimensions is related to $g$ and $\rho'$ via (3.20). Thus it will have a nontrivial $N$ dependence:

$$g_d^2 \sim \rho' g^2 \sim N^{\theta - \omega} \quad (4.4)$$

We summarize the result in Table 2.

### 4.2 Physical bounds on scaling exponents

We have seen that a $d$-dimensional SYM theory emerges from the off-diagonal dynamics of the large $N$ Matrix model. In order to prove the claim, we need at least to show such a large $N$ limit is indeed possible — precisely specifying how to arrange the background D-instanton configuration as $N$ tends to infinity. It may be difficult to do this rigorously. We will content ourselves with obtaining some inequalities among the exponents $\theta, \omega, d$ introduced in section 4.1, so that there occurs no apparent inconsistency in QFT side. This would help us applying Matrix theory to more realistic situations in the future.

We will consider several physically reasonable assumptions, but we do not intend to claim that following conditions are all necessary or sufficient.

(i) Interpretation as $d$-dimensional QFT required replacing the discrete sum by $d$-dimensional integral. This coarse graining can be justified only when $a \sim b \ll \mu$. This is true in the large $N$ limit if

$$\theta \geq 0. \quad (4.5)$$

(ii) From the $d$-dimensional point of view, cutoff scale $\Lambda$ must tend to infinity. Thus we have

$$\theta \leq 1. \quad (4.6)$$

(iii) As we saw in section 2, the perturbation theory is essentially the expansion in $g/a^2$. Thus it is valid if $a \gg g^{1/2}$ is satisfied. This remains to be true in large $N$ limit if

$$d \omega \geq 4 \theta. \quad (4.7)$$
(iv) Actually, the same bound can be obtained from a different viewpoint. From exact results for matrix integrals [17, 18, 19], it is reasonable to assume there is a pairwise repulsive potential among D-instantons due to entropy factor. This could be effectively treated [8] as each D-instanton has a core size of order $g^{1/2}$. This implies $a \gtrsim g^{1/2}$. Sending $N$ to infinity, we obtain the inequality (4.7).

(v) To construct $U(n)$ gauge theory, $n$ D-instantons are put on the same point (see section 3.5). But this assumption might be too strong; it is possible that the $n$ D-instantons can disperse in the cluster of size $b$ but are still grouped via (slightly broken) $U(n)$ gauge action from QFT point of view.$^7$ Of course, the dispersion size $b$ must be sufficiently smaller than the renormalization scale,

$$b \ll \mu.$$  

This corresponds to the minimum momentum resolution seen by $U(n)$ Yang-Mills theory. The condition (4.8) leads to the bound, (4.5).

(vi) As far as a tree level amplitude or the form of Yang-Mills action is concerned, the argument given in (v) is sufficient. But if quantum effects are taken into account, it is another story. Just like the anomaly from one loop, large $N$ divergence from loop integrals may overwhelm the $b/\mu \sim N^{-\theta}$ suppression discussed in (v) and may yield non-negligible effects.

Let us estimate the effect of dispersing D-instantons using the one-loop two point function as an example. As a function of D-instanton configuration $\{x_k\}$, the most divergent contribution is roughly given by

$$\mathcal{A}^{\text{1-loop}}[\{x_k\}] \sim \frac{g^2}{\mu^2} \sum_k \frac{1}{x_{ik}^2} \sim \frac{g^2 \rho}{\mu^2} \int^{\Lambda} \frac{d^dp}{p^2 + \mu^2} \sim \frac{g^2 \rho}{\mu^2} \Lambda^{d-2}$$

$^7$This claim is not so strange as it sounds. In Nature, non-Abelian symmetry is exact in UV regime but hidden in IR regime through confinement or Higgs mechanism. In our context, the D-instantons within a cluster look almost coincident in much larger scale $\mu$.  

24
If the D-instanton positions \( \{x_k\} \) have a dispersion of order \( b \), \( A^{1\text{-loop}}[\{x_k\}] \) will change as
\[
\delta A^{1\text{-loop}} = A^{1\text{-loop}}[\{x_k + \delta x_k\}] - A^{1\text{-loop}}[\{x_k\}] \\
\sim \frac{g^2}{\mu^2} \sum_k \left\{ \frac{1}{(x_{ik} - \delta x_k)^2} - \frac{1}{x_{ik}^2} \right\} \\
\sim \frac{g^2 \rho}{\mu^2} \int_\Lambda d^d p \frac{\partial}{\partial p} \left( \frac{1}{p^2 + \mu^2} \right) \sim \frac{g^2 \rho}{\mu^2} b \Lambda^{d-3}.
\]

The last expression scales as \( N \) to the \((\theta - \omega - \frac{\theta}{d} + (d - 3) \frac{1-\omega}{d})\)-th power. Thus, it is negligible only if
\[
d(1 - \omega) < 3 - 2\theta
\]

This is the condition when the operators associated with the slight shift of D-instantons are irrelevant in the sense of large \( N \) renormalization group.

(vii) Let us study the problem of renormalizability of \( d \)-dimensional theory i.e. whether or not as \( N \to \infty \) only a finite number of amplitudes superficially diverge.

From the Wilsonian point of view, renormalizability is not a necessary condition for QFT, but a consequence of the renormalization procedure. But it is of some interest in presenting the analysis since in the context of Matrix model, situation is rather complicated.\(^8\)

For a given Feynman graph, the superficial degree of divergence is usually determined from the number of loops and propagators. The coupling constants just count the number of vertices and stay fixed when the cutoff is sent to infinity. As we all know, \( d = 4 \) is the critical dimension for gauge theories.

But here, we are talking about the divergence when \( N \) tends to infinity. Recall not only \( \Lambda \) but also \( g_d \) changes as a function of \( N \), i.e. \( g_d^2 \sim N^{\theta - \omega} \). Thus usual definition of the superficial degree of divergence does not work.

In a sense, we are studying a generalized large \( N \) limit in which \( g_d^2 N^{\omega - \theta} \) is kept fixed. Thus the standard QFT results should follow if we restrict to \( \theta = \omega \), whereas ’t Hooft limit would correspond to another special case, \( \theta + 1 = \omega \).

What is the new rule for the superficial degree of divergence? Note that \( g^2 \) always come in pair with a sum over D-instantons. From the substitution
\[
g^2 \sum_k \cdots \Rightarrow g^2 \rho \int d^d p \tr(\cdots) \Rightarrow g_3^2 \int d^d p \tr(\cdots),
\]
\(^8\)For example, it is not clear “What is IR limit?”
extra $d$-dimensional loop integral is always associated with the factor $g^2_d$. Each loop contributes $\Lambda^d$ while the coupling gives $g^2_d \sim N^\theta-\omega \sim \Lambda^{d(\theta-\omega)/(1-\theta)}$. It is easy to convince oneself that the net effect is to replace the spacetime dimension $d$ by an effective dimension

$$d_{\text{eff}} \equiv d + \frac{d(\theta - \omega)}{1 - \theta} = d \frac{1 - \omega}{1 - \theta}. \quad (4.13)$$

Thus we have a new criteria about large $N$ renormalizability as follows

$$d_{\text{eff}} \leq 4 \iff d(1 - \omega) \leq 4(1 - \theta). \quad (4.14)$$

As promised, $\theta = \omega$ recovers the standard result. Note that if $\omega = 1$, $d_{\text{eff}} = 0$ for any $d, \theta$. This corresponds to the well known fact that planar limit of the $0 + 0$-dimensional Matrix model absolutely converge.

(viii) As for the error in replacing sums by integrals, analysis in (vii) can be generalized to an arbitrary Feynman graph $\Gamma$. Suppose we know the amplitude diverges as

$$A(\Gamma) \sim N^{\gamma(\Gamma)},$$

including $N$ dependence of $g^2_d$. Then, the approximation can be justified if

$$\delta A \sim b \Lambda^{D_{\text{eff}}(\Gamma)} \sim N^{\gamma(\Gamma) - \frac{1}{d}} \ll 1.$$

Thus the error is negligible for graphs with sufficiently low degree of divergence:

$$\gamma(\Gamma) < \frac{1}{d}. \quad (4.15)$$

It may be useful to introduce effective superficial degree of divergence, $D_{\text{eff}}(\Gamma)$ defined through$^9$

$$\Lambda^{D_{\text{eff}}(\Gamma)} \sim N^{\gamma(\Gamma)},$$

or equivalently

$$D_{\text{eff}}(\Gamma) = \frac{d}{1 - \theta} \gamma(\Gamma).$$

Then, (4.15) can be expressed as

$$D_{\text{eff}}(\Gamma) < \frac{1}{1 - \theta} \iff \theta > 1 - \frac{1}{D_{\text{eff}}(\Gamma)} \quad (4.16)$$

$^9$ $D_{\text{eff}}(\Gamma)$ coincides with usual superficial degree of divergence $D(\Gamma)$ if $g_d$ is independent of $N$. 
Suppose $d$-dimensional theory is renormalizable in the sense that $\mathcal{D}_{\text{eff}}(\Gamma)$ has a $\Gamma$-independent upper bound $\mathcal{D}_{\text{max}}$. Then by choosing

$$\theta > 1 - \frac{1}{\mathcal{D}_{\text{max}}}$$

the error can be neglected for all Feynman integrals. In particular, $d = 4$ SYM case ($\mathcal{D}_{\text{max}} = 2$) leaves as a finite window $1 > \theta > \frac{1}{2}$.

If $d$-dimensional theory is non-renormalizable, $\mathcal{D}_{\text{eff}}$ grows with the number $L$ of loops. Thus from (4.16), for a given $\theta$, there is a maximal number of loops $L_{\text{max}}$ beyond which the approximation fails. But $L_{\text{max}}$ tends to infinity if we approach ’t Hooft limit, $\theta \to 1$.

### 4.3 Possible interpretation of exponents

So far we have chosen a particular D-instanton configuration depicted in Fig 8 the dimension $d$ in which QFT lives is determined by the number of flat directions. But this is clearly a very special configuration from $D$-dimensional viewpoint. Can we relax the assumption?

In Wilson’s approach to renormalization group, one can study the origin of ultraviolet divergences by isolating the dependence of the functional integral on the short distance degrees of freedom of the field. In Matrix approach, short distance degrees of freedom correspond to the long distance D-instantons. The number $\delta N$ of D-instantons contained in the momentum shell $\Lambda < |p| < \Lambda + \delta \Lambda$ is given by

$$\delta N \propto \Lambda^{d-1}\delta \Lambda \quad \text{as} \quad \Lambda \to \infty.$$ 

In fact, as far as the loop divergence is concerned, dimension $d$ will appear only through this relation.

Consider for example a more generic D-instanton configuration, Fig 9. Let $N(\Lambda)$ be the number of D-instantons within a $D$-dimensional ball of radius $\Lambda$. Then the space-time dimension can be “defined” as the rate of growth of $N$:

$$d = \frac{\partial \log N(\Lambda)}{\partial \log \Lambda}.$$  \hspace{1cm} (4.17)

For a uniform configuration like Fig 8, definition (4.17) gives the number of flat directions. Note that the new definition (4.17) make sense for non-integer dimension $d$, which may be useful to visualize the meaning of dimensional regularization.
In order to enumerate the physical degrees of freedom in a field theory, one needs to put the system into a finite box of volume $V$. The number of states is given by the available phase space volume.

$$ N = \frac{V}{(2\pi\hbar)^d} \int_\Lambda d^d p. \quad (4.18) $$

In order to realize a continuum field theory in infinite spacetime, one needs to take two limits:

\begin{align*}
V &\to \infty \quad \text{(Large volume limit)} , \\
\Lambda &\to \infty \quad \text{(Continuum limit)} .
\end{align*}

Singularities associated with the former and latter are usually called IR and UV divergences, respectively. In standard textbooks on QFT, $V \to \infty$ limit is taken first so that Feynman rules simplify in the momentum space. Subsequently, $\Lambda \to \infty$ limit is carefully investigated. This asymmetry between the two limits is due to the well known fact that the translational invariance in momentum space is actually broken by the hierarchical structure.

In the Matrix model, however, all limiting processes are “unified” into a single large $N$ limit. Comparing (3.22) and (4.18), we can say that we have investigated in section 4.2 all possible limits

$$ V \sim \rho' \sim N^\theta, \quad \Lambda \sim N^{(1-\theta)/d} \quad (4.19) $$

to get a continuum field theory.
5 Discussions

In this paper, we have studied the dynamics of large \( N \) Matrix models through the quantum fluctuations in a fixed D-instanton background. In particular, we have explicitly shown the correspondence of perturbation theories between the usual QFT and Matrix perturbation theory. The correspondence is exact if relative D-instanton positions are interpreted as momenta in QFT picture.

One might think that this is a kind of triviality. Indeed, Matrix model action is originated from the Yang-Mills action by dimensional reduction. It is no wonder Yang-Mills theory can be recovered from the Matrix model. However, since dimensional reduction is simply throwing away spacetime coordinate dependence, the reverse procedure would be just re-introducing \( x \) dependence to the matrix fields. But contrary to this naive expectation, the momentum space picture emerges first and coordinate picture is recovered only after Fourier transformation.

The correspondence exploited in this paper can be regarded as a “dual” version of Eguchi-Kawai reduction [15]. The original suggestion by Eguchi and Kawai is valid only at strong coupling [20], whereas we have shown the equivalence in a weak coupling regime.

Just like lattice gauge theories, Matrix model provides us with a natural gauge invariant regularization. But “Matrix regularization” has two important features. First, the quantum fields are discretized in the momentum space rather than ordinary space, and the hierarchical structure inherent to QFT can be understood in a geometrical fashion. Second, matrix regularization can, in principle, be “generally covariant” if the sum over all background configurations is taken into account. A permutation of D-instantons is a discrete analogue of coordinate reparametrization.

Matrix models pack too much degrees of freedom into a few matrices. As is often the case, this obscures the meaning of large \( N \) limit. Furthermore, in a theory with \( T \)-duality it is difficult to make distinction between IR and UV limits. The limits explicitly depend on the effective dynamics we are talking about. At any rate, it is obvious in Matrix theory that universal behavior is expected only in large \( N \) limit.

We initiated a preliminary study of what class of large \( N \) limit is possible in order to reproduce a QFT. Key idea is to classify the degree of divergence in terms of \( N \), the only source of divergence in Matrix theory. It is now possible to interpret renormalization group à la Brézin and Zinn-Justin [21, 22, 23] in terms of the usual renormalization of Yang-Mills theory. We hope
to report on this elsewhere.

One might well be puzzled by the interpretation of the Matrix dynamics as $d$-dimensional Yang-Mills theory. If the background D-instantons are distributed really uniformly, the rank of the gauge group would be just a matter of choice because it depends how we cut the D-instanton gas into pieces. Let $\rho$ be the D-instanton density. Suppose we decide to call D-instantons inside a $d$-dimensional hypercube of size $b$ as a cluster. Then we have

$$n = \rho b^d,$$

(rank of the gauge group)

$$\rho' = \rho/n = b^{-d},$$

(cluster density)

$$g_4^2 = (2\pi)^{d} \rho' g^2 = (2\pi)^d g^2 b^{-d}.$$  

(Yang-Mills coupling)

Note that $b$ dependence cancels in the 't Hooft coupling $\lambda \equiv g_4^2 n = (2\pi)^d g^2 \rho$. Therefore, matrix perturbation theory suggests that any universal property of $U(n)$ gauge theory with adjoint matters should depend, not separately on $g_d$ or $n$, but on 't Hooft coupling $\lambda \equiv g_4^2 n$, at least for sufficiently small $\lambda$.

At present, we do not know whether this is generally true or not, but the following evidence should be taken seriously. Consider a renormalization group beta function for $d = 4$ Yang-Mills coupling,

$$\beta(g_4) = -\beta_0 g_4^3 - \beta_1 g_4^5 - \cdots.$$  

In a $U(n)$ gauge theory with adjoint matters ($C_2(G) = n$), the coefficients are given by $\beta_0 = c_0 n$, $\beta_1 = c_1 n^2$, \ldots with $c_0, c_1, \ldots$ depending only on the matter contents. Thus we have

$$\beta(g_4) = -c_0 n g_4^3 - c_1 n^2 g_4^5 - \cdots.$$  

This can be rewritten as

$$\beta(\lambda) = -2(c_0 \lambda^3 + c_1 \lambda^5 + \cdots).$$

The right-hand side is a function of $\lambda$ only, in accordance with our expectation.

This work is supported in part by the Grant-in-Aid for Scientific Research on Priority Area 707 “Supersymmetry and Unified Theory of Elementary Particles”, Japan Ministry of Education.
References


32