REMARKS ON THE ATIYAH - HITCHIN METRIC*

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Abstract

We outline the construction of the Atiyah-Hitchin metric on the moduli space of SU(2) BPS monopoles with charge 2, first as an algebraic curve in $\mathbb{C}^3$ following Donaldson and then as a solution of the Toda field equations in the continual large $N$ limit. We adopt twistor methods to solve the underlying uniformization problem, which by the generalized Legendre transformation yield the Kahler coordinates and the Kahler potential of the metric. We also comment on the connection between twistors and the Seiberg-Witten construction of quantum moduli spaces, as they arise in three dimensional supersymmetric gauge theories, and briefly address the uniformization of algebraic curves in $\mathbb{C}^3$ in the context of large $N$ Toda theory.

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The Atiyah-Hitchin space is a four-dimensional hyper-Kahler manifold with $SO(3)$ isometry that was introduced long time ago to describe the moduli space of $SU(2)$ BPS monopoles of magnetic charge 2 [1, 2]. In more recent years, this space and various generalizations thereof were identified with the full quantum moduli space of $N = 4$ supersymmetric gauge theories in three dimensions [3]. The purpose of this contribution is to summarize briefly various results scattered in the literature on the algebro-geometric properties of the Atiyah-Hitchin space, with emphasis on the explicit description of its Kahler structure, and make connections between them by interpreting the classical twistor construction of hyper-Kahler manifolds (and the associated spectral curve) essentially as Seiberg-Witten construction of the geometry of non-perturbative quantum moduli spaces. The technical ingredient is the non-triholomorphic nature of the Killing vector fields that generate the full isometry group of the Atiyah-Hitchin space. In the two approaches that will be outlined in the sequel one chooses an abelian subgroup $SO(2) \subset SO(3)$ to construct the Kahler coordinates of the metric and derive all the relevant information about it in terms of elliptic integrals. One approach is Donaldson’s description of the moduli space as an algebraic curve in $C^{3}$, the other is based on the continual Toda field equation that arises in the large $N$ limit of 2-dim conformal field theories. We find as byproduct that $SU(\infty)$ Toda theory arises in the uniformization of algebraic curves in $C^{3}$, whereas $SU(2)$ Toda theory (ie Liouville’s equation) arises in the uniformization of algebraic curves in $C^{2}$ (ie Riemann surfaces), as it is well known for more than a century. This aspect certainly deserves further study and we hope to return to it elsewhere together with the possible use of twistor techniques in future developments of non-perurbative strings via M-theory [4], where the Seiberg-Witten curve has a pivotal role, or F-theory [5].

Recall Donaldson’s description of the moduli space of monopoles as the space of rational functions of a complex variable, say $W$, of degree 2 given in normal form as

$$F(W) = \frac{a_1 W + a_0}{W^2 - b_0}; \quad a_0^2 - b_0 a_1^2 \neq 0,$$  \hfill (1)
modulo the equivalence $F(W) \sim \lambda F(W)$ for $\lambda \in \mathbb{C}^*$ [6]. Here $a_0$, $a_1$, $b_0$ are complex numbers (moduli). Setting the resultant of these maps equal to 1, we obtain an algebraic curve in $\mathbb{C}^3$, 

$$a_0^2 - b_0 a_1^2 = 1,$$  

(2)

which describes the (universal) double covering of the Atiyah-Hitchin space; alternatively, the Atiyah-Hitchin space is the quotient of the algebraic curve (2) by the involution $Z_2 : (a_0, a_1, b_0) \to (-a_0, -a_1, b_0)$. Donaldson’s parametrization of the moduli space involves choosing a preferred direction in $\mathbb{R}^3$ (the space where BPS monopoles actually live) and the variable $W$ in $F(W)$ above parametrizes the 2-plane orthogonal to that direction. This description does not respect the full rotational symmetry of $\mathbb{R}^3$, $SO(3)$, which is the complete isometry group of Atiyah-Hitchin space, as it amounts to choosing an abelian subgroup $SO(2) \subset SO(3)$; it selects a preferred complex structure out of the three available in hyper-Kahler geometry, and as such it is more appropriate for comparison with the Toda field theory description of the metric, where, as we will see later, one chooses adapted coordinates to a non-triholomorphic $SO(2)$ isometry.

Next, we consider the uniformization of the curve $a_0^2 - b_0 a_1^2 = 1$, which is only briefly discussed in [2]. For this we start first with the elliptic curve

$$\eta^2(\zeta) = K^2(k)\zeta \left( k k' \zeta^2 - 3(k'^2 - k^2) \zeta - k k' \right),$$  

(3)

which can be brought into Weierstrass form $y^2 = 4x^3 - g_2x - g_3$ using the change of variables $\eta = k_1 y$, $\zeta = x + k_2$, where $4k_1^2 = k k' K^2(k)$, $k_2 = (k'^2 - k^2) / 3 k k'$. The quantity $\omega_1 = 4k_1$ is the real period of the curve, whereas its period matrix $\omega_1 / \omega_2$ turns out to be $\tau = i K(k') / K(k)$. For generic values of $k$ the roots in Weierstrass form are all distinct and they are ordered increasingly on the real axis as $e_3 < e_2 < e_1$ with branch cuts running from $e_3$ to $e_2$ and $e_1$ to $\infty$. As $k \to 1$ we have $e_2 \to e_1$ in which case the $b$-cycle of the underlying Riemann surface degenerates and this corresponds to the Taub-NUT limit of
the Atiyah-Hitchin space. On the other hand, as $k \to 0$ we have $e_2 \to e_3$ in which case the $a$-cycle of the Riemann surface degenerates and this corresponds to a bolt structure in the metric, where the 3-dim orbits of the full isometry group $SO(3)$ collapse to a two-sphere (more precisely $RP_2 \simeq S^2/Z_2$, since the first homotopy group of the moduli space is $Z_2$). These two limits are related by the modular transformation $k \leftrightarrow k'$ that exchanges the $a$- and $b$-cycles of the Riemann surface.

Next, it is convenient to “roll up” the curve (3) by applying an $SU(2)$ fractional transformation

$$
\zeta \to \tilde{\zeta} = \frac{c\zeta - d}{d\zeta + c}, \quad \eta \to \tilde{\eta} = \frac{\eta}{(d\zeta + c)^2}, \quad cc + dd = 1,
$$

where the elements $c, d$ of $SU(2)$ are parametrized using Euler angles $\theta, \phi, \psi$ as follows,

$$
c = e^{\frac{i}{2}\phi}\left(\sqrt{\frac{1 - k^2}{2}\sin \frac{\theta}{2} e^{-\frac{i}{2}\psi}} - i\sqrt{\frac{1 + k^2}{2}\cos \frac{\theta}{2} e^{\frac{i}{2}\psi}}\right),
$$

$$
d = e^{-\frac{i}{2}\phi}\left(-\sqrt{\frac{1 + k^2}{2}\sin \frac{\theta}{2} e^{\frac{i}{2}\psi}} + i\sqrt{\frac{1 - k^2}{2}\cos \frac{\theta}{2} e^{-\frac{i}{2}\psi}}\right).
$$

Then the curve (3) becomes

$$
\tilde{\eta}^2(\tilde{\zeta}) = z + v\tilde{\zeta}^2 + w\tilde{\zeta}^4,
$$

where the coefficients of the quartic polynomial turn out to be

$$
z = -\frac{1}{4} K^2(k) e^{-2i\phi}\sin^2 \theta \left(1 + k^2 \sinh^2 \nu\right), \quad v = -\frac{i}{2} K^2(k) \sin 2\theta e^{-i\phi}(1 + k^2 \cos \psi \tan \theta \sinh \nu)
$$

$$
w = \frac{1}{2} K^2(k) \left(2 - k^2 + 3 \sin^2 \theta \left(k^2 \cos^2 \psi - 1\right)\right); \quad \text{where } \nu = \log \left(\tan \frac{\theta}{2}\right) + i\psi.
$$

Note that the $\phi$ dependence of the coefficients in (6) can be removed by appropriate rotation of $\tilde{\zeta}$ and $\tilde{\eta}(\tilde{\zeta})$, which is in turn consistent with the isometry $SO(2) \subset SO(3)$ that was selected having $\partial/\partial \phi$ as Killing vector field in terms of Euler angles.

The uniformization proceeds by choosing a complex variable $u$ so that

$$
\mathcal{P}(u) = \frac{d}{c} - k_2 \equiv \frac{d}{c} + e_2,
$$
where \( \mathcal{P}(u) \) denotes the Weierstrass function of the cubic curve \( y^2 = 4x^3 - g_2x - g_3 \). Then, according to Atiyah and Hitchin [2], use of twistor methods lead to the following expressions for \( b_0, a_0, a_1, \)

\[
\sqrt{b_0} = \frac{1}{2} \tilde{c} \omega_1 \mathcal{P}'(u),
\]

\[
\log \left( a_0 + \sqrt{b_0} a_1 \right) = \omega_1 \left( \zeta(u) - \frac{\eta_1}{\omega_1} u \right) + \frac{1}{2} \tilde{c} \tilde{d} \omega_1 \mathcal{P}'(u),
\]

(9)

which together with the algebraic condition \( a_0^2 - b_0 a_1^2 = 1 \) complete the job. Here, \( \zeta(u) \) is the quasi-periodic \( \zeta \)-function in the theory of Riemann surfaces, \( \zeta'(u) = -\mathcal{P}(u) \), with real quasi-period \( \eta_1, \zeta(u + \omega_1) = \zeta(u) + \eta_1 \) (not to be confused with the twistor curve variables \( \zeta \) and \( \eta(\zeta) \)). The expression (9) can be written in more compact form using appropriate contour integrals of the rotated spectral curve (6). First, by the defining properties of the Weierstrass \( \mathcal{P} \)-function and the choice (8) of the uniformizing parameter \( u \), we find

\[
b_0 = 4\tilde{\eta}^2(\tilde{\zeta} = 0) \equiv 4z.
\]

(10)

Second, following the changes of variables that transform the curve (3) from its Weierstrass form into the fully rotated spectral curve (6), we find upon integration on an \( a \)-cycle,

\[
\log \left( a_0 + \sqrt{b_0} a_1 \right) = \frac{1}{2} \tilde{\eta}(\tilde{\zeta} = 0) \oint_a \frac{d\tilde{\zeta}}{\tilde{\zeta} \tilde{\eta}}.
\]

(11)

Note at this point that there are two discrete transformations whose effect is more easily described in terms of the uniformizing parameter \( u \),

\[
I_1 : u \to u + \frac{1}{2}(\omega_1 + \omega_2), \quad I_2 : u \to u + \frac{1}{2} \omega_2,
\]

(12)

which fold pairwise the four Weierstrass points onto each other. The latter are defined to be the zeros of \( \mathcal{P}'(u) \) and they occur at \( u = \omega_1/2 \) with \( \mathcal{P}(u) = e_1 \), \( u = (\omega_1 + \omega_2)/2 \) with \( \mathcal{P}(u) = e_2 \), \( u = \omega_2/2 \) with \( \mathcal{P}(u) = e_3 \) and \( u = 0 \) with \( \mathcal{P}(u) = \infty \). Under the transformations (12) the coefficients \( z, v \) and \( w \) of the spectral curve (6) remain invariant; we also find that under each one of them \( b_0 \) remains invariant. On the other hand, \( \log(a_0 + \sqrt{b_0} a_1) \) gets
shifted by \( \pi i \) by applying either \( I_1 \) or \( I_2 \), because \( \omega_1 \eta_2 - \omega_2 \eta_1 = 2\pi i \). Therefore both \( I_1 \) and \( I_2 \) correspond to the involution \((a_0, a_1, b_0) \rightarrow (-a_0, -a_1, b_0)\) and so the algebro-geometric description of the Atiyah-Hitchin space as a curve in \( \mathbb{C}^3 \), \( a_0^2 - b_0 a_1^2 = 1 \), modulo the involution \((a_0, a_1, b_0) \rightarrow (-a_0, -a_1, b_0)\), amounts to factoring out both of them. In the language of Euler angles, \( I_1 \) exchanges the position of the two monopoles, thus treating them as identical particles, and results to a bolt structure \( S^2 \) as \( k \rightarrow 0 \) by restricting the range of the angle \( \psi \) from 0 to \( 2\pi \) instead of \( 4\pi \), whereas \( I_2 \), which is the discrete remnant of an additional continuous triholomorphic isometry \( \psi \rightarrow \psi + \text{const.} \) that only appears asymptotically as \( k \rightarrow 1 \), is responsible for the bolt structure \( \mathbb{RP}_2 \simeq S^2/Z_2 \) that arises upon factorization.

The second approach to the problem is based on Toda field theory which results by selecting an abelian subgroup \( SO(2) \) from the full isometry group \( SO(3) \) of Atiyah-Hitchin space and using adapted coordinates to this isometry. Then, since all isometries are non-triholomorphic, the hyper-Kahler condition for the metric amounts to a non-linear differential equation in three dimensions for a single function \( \Psi \), namely Toda field equation, whose solution determines all components of the metric. More explicitly, consider Kahler coordinates \( q \) and \( p \) with respect to a selected complex structure and let \( \mathcal{K}(q, \bar{q}, p, \bar{p}) \) be the Kahler potential that depends on them and their complex conjugates. By definition of the Kahler potential we have \( ds^2 = 2\mathcal{K}_{Q \bar{Q} A B} dQ^A d\bar{Q}^B \), where \( Q^A \) are \( q \) or \( p \) and the hyper-Kahler condition for the metric reads as \( \mathcal{K}_{q \bar{q}} \mathcal{K}_{p \bar{p}} - \mathcal{K}_{q p} \mathcal{K}_{\bar{q} \bar{p}} = 1 \). For manifolds with a non-triholomorphic isometry the Kahler potential depends on the combination \( p\bar{p} \) and not on \( p \) and \( \bar{p} \) separately, or to put it differently there is a Killing vector field \( \xi = i(\rho \partial_{\rho} - \bar{\rho} \partial_{\bar{\rho}}) \) with the property \( \xi \mathcal{K} = 0 \). Then, introducing the notation \( p\bar{p} \equiv r \), whereas \( p/\bar{p} \equiv \exp(i\sigma) \) defines the corresponding phase variable, we cast the hyper-Kahler condition into the simpler form

\[
(r \mathcal{K}_r)_r \mathcal{K}_{q \bar{q}} - r \mathcal{K}_{qr} \mathcal{K}_{\bar{q} \bar{q}} = 1. \tag{13}
\]
Furthermore, by introducing a variable $\rho$ conjugate to $\log r$ with respect to $K$, $rK_r = \rho$, we may use $(q, \bar{q}, \rho)$ as a new set of independent variables to rewrite the metric and the condition (13) for the (hyper)-Kahler manifold. We obtain the following result for the Kahler metric

\[ ds^2 = \frac{1}{2} \left( \frac{r}{r_\rho} \right) \left( d\sigma + \frac{i}{r}(r_\rho dq - r_\bar{q} d\bar{q}) \right)^2 + \frac{1}{2} \left( \frac{r_\rho}{r} \right) (d\rho^2 + 4rdq d\bar{q}), \]  

(14)

whereas the hyper-Kahler condition (13) becomes [8]

\[ (\log r)_{qq} + r_{,\rho\rho} = 0, \]  

(15)

which is the Toda field equation for $SU(N)$ in the continual large $N$ limit [9, 10].

The local coordinate system (14) is called the Toda frame and naturally the function $r = \exp \Psi$ is called the Toda potential of the metric. The change of variables $(q, \bar{q}, r) \rightarrow (q, \bar{q}, \rho)$ can be viewed as a Legendre transform in the sense that the function

\[ F(q, \bar{q}, \rho) \equiv \rho \log r - K(q, \bar{q}, r) \]  

(16)

is the “Hamiltonian” for the “Lagrangian” $K$ with “momentum” $\rho$ and “velocity” variable $\log r$. According to this we have the relation $\partial K/\partial \log r = rK_r = \rho$, as required. Then, $F_{,\rho} = \log r$ and $r = \exp \Psi$ is considered as a function of $q$, $\bar{q}$ and $\rho$. This concludes our general description of the Kahler structure for 4-dim hyper-Kahler manifolds with a selected non-triholomorphic isometry, generated by $\partial/\partial \sigma$, using the Toda theory approach.

For the Atiyah-Hitchin metric we consider the non-triholomorphic isometry generated by $\partial/\partial \phi$ and then by application of twistor methods one shows that an appropriate choice for $F$ is [11]

\[ F = -\frac{1}{\pi i} \oint_0 \frac{d\zeta}{\zeta^3} \bar{\eta}(\zeta) + \oint_a \frac{d\zeta}{\zeta^2} \bar{\eta}(\zeta). \]  

(17)

Here, the first term is given by a contour integral around the origin and the second by a contour integral over the $a$-cycle of the Riemann surface for the twistor space curve (6),
which is the same as in Donaldson’s description of the moduli space. In this context, \( F \) is a function of the variables \( z, v \) and \( w \) given by eq. (7), but it follows that it can be used as a generating function for the Kahler coordinate \( q \). Namely, we have

\[
q = \frac{\partial F}{\partial v} = \frac{1}{2} \oint_a \frac{d\zeta}{\zeta \eta}, \tag{18}
\]

whereas \( F,w = 0 \) follows by inspection. As for the other Kahler coordinate \( p \) we have

\[
p = \eta^2(\zeta = 0) = -\frac{1}{4} K^2(k) e^{-2i\phi} \sin^2 \theta \left( 1 + k'^2 \sin^2 \nu \right), \tag{19}
\]

and so \(-2\phi + \text{arg}(1 + k'^2 \sin^2 \nu)\) provides the (shifted) Killing coordinate \( \sigma/2 \).

The construction of Kahler coordinates \( q, p \) by twistor methods provide a highly non-trivial transcendental solution of the underlying continual Toda field equation. The choice (17) for \( F \) is such that it remains inert under the \( SO(2) \) rotations \( U = \exp(i\phi) \) taking \( \zeta \rightarrow U\zeta \) and \( \eta \rightarrow U\eta \). In turn, the variables \( q \) and \( p \) given by eqs. (18) and (19) transform as follows, \( q \rightarrow U^{-1}q \) and \( p \rightarrow U^2p \). We note at this point that there is another pair of Kahler coordinates, since their choice is not unique,

\[
\tilde{p} = 2\sqrt{\tilde{p}} \equiv \sqrt{b_0}, \quad \tilde{q} = q\sqrt{\tilde{p}} \equiv \log \left( a_0 + \sqrt{b_0a_1} \right), \tag{20}
\]

which is preferable in the sense that the \( SO(2) \) rotations leave \( \tilde{q} \) inert, whereas \( \tilde{p} \rightarrow U\tilde{p} \), and connect directly with the variables of the curve \( a_0^2 - b_0a_1^2 = 1 \) used in Donaldson’s description (see eqs. (10), (11)). We may now drop the tildes and use Donaldson’s variables (20) to describe the Kahler coordinates of the metric (14) and the corresponding Toda field equation (15), viewing the pair (18) and (19) only as an intermediate choice for the Atiyah-Hitchin space. In any case, the standard holomorphic 2-form on the affine surface in \( \mathbb{C}^3 \),

\[
\Omega = \frac{da_1 \wedge db_0}{a_0} = 2dq \wedge dp, \tag{21}
\]

is not inert under the \( SO(2) \) rotations \( U \), since \( \Omega \rightarrow U\Omega \). As such it coincides with \( F_1 + iF_2 \), where \( F_1 \) and \( F_2 \) are the two real Kahler forms of the space that form a doublet under the selected \( SO(2) \) isometry; the third one, \( F_3 \), is a singlet and hence inert under \( U \).
We conclude with a number of comments which we plan to address in detail elsewhere [12]. First, the Atiyah-Hitchin space inherits the full isometry group $SO(3)$ by its very construction, since fractional $SU(2)$ transformations (5) with Euler angles $\theta, \phi$ and $\psi$ have been used to roll up the elliptic curve (3) to (6). The four Weierstrass points of the Riemann surface provide the zeros of the variable $z$ and it can be verified that they occur at $\cos^2 \theta = k^2$ and $\psi = \pi/2$ or $3\pi/2$, where (6) reduces back to (3). These points are the free field points of the underlying Toda field equation (15) in the sense that $\exp \Psi$ vanishes there; at these points the Atiyah-Hitchin metric develops coordinate singularity in the form (14), since one has $\det g = (\Psi, \exp \Psi)^2$ and $\Psi, \rho \neq 0$. Guided by the algebro-geometric construction of the full space, starting from the free field curve (3) and rolling it up to (6) by fractional $SU(2)$ transformations, one hopes to device a free field configuration in the 2-dim subspace with coordinates $(q, \bar{q})$ so that the transcendental solution for the Toda potential of the Atiyah-Hitchin space is reconstructed out of free fields in a group theoretical fashion, as it is always the case with 2-dim Toda field equations (though here the structure group is infinite dimensional, $SU(\infty)$). However, the ordinary free field realization of solutions is nothing else but a way to sum up the perturbative expansion around the free field configuration, which is provided by group theory in closed form, thanks to the integrability of the underlying non-linear differential equations. The correct way to address this question here is to consider solutions of the Toda field equations defined on a Riemann surface, as indicated by the uniformization formulae (9) for the Kahler coordinates, and not on a 2-dim flat Euclidean space which is only appropriate for considering the asymptotic form of the metric in the Taub-NUT limit [13]. Then, the non-perturbative corrections, due to instantons, which turn the perturbative quantum moduli space from Taub-NUT into Atiyah-Hitchin in 3-dim $N = 4$ supersymmetric gauge theories [3], could be reinterpreted as toron configurations in the Toda frame. This point of view is certainly useful for future generalizations to more complicated examples of algebraic surfaces in $C^3$ due to Dancer [7].

Second, the non-perturbative construction of Kahler coordinates (in particular $q$) and
their generating function $\mathcal{F}$ by twistor methods in hyper-Kahler geometry, where one introduces an auxiliary curve $\tilde{\eta}(\tilde{\zeta})$ and then integrates over non-contractible cycles, is identical in vein with the construction of Kahler coordinates in quantum moduli spaces of $N = 2$ supersymmetric gauge theories a la Seiberg-Witten. We think that this analogy should be explored further in view of future applications of twistors in the non-perturbative formulation of strings via M-theory or F-theory, as well as in 10-dim supersymmetric Yang-Mills theories.

Finally, we should explore further the geometric role of large $N$ Toda theory in the uniformization of algebraic curves in $C^3$ that admit a non-triholomorphic $SO(2)$ action, like the Atiyah-Hitchin space and its generalizations thereof. In this way we hope to gain better understanding of non-perturbative aspects of string theory, whereas the perturbative formulation of strings, as we know it in terms of world-sheets, is provided by algebraic curves in $C^2$ (ie Riemann surfaces) and their uniformization via Liouville theory. This is also an interesting problem in geometry in its own right.

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References


