Probing Large Extra Dimensions with Neutrinos

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Abstract

We study implications of theories with sub-millimeter extra dimensions and \(M_f \sim (1 - 10)\) TeV scale quantum gravity for neutrino physics. In these theories, the left-handed neutrinos as well as other standard model (SM) particles, are localized on a brane embedded in the bulk of large extra space. Mixing of neutrinos with (SM) singlet fermions propagating in the bulk is naturally suppressed by the volume factor \(M_f/M_P \sim 3 \cdot 10^{-16} - 3 \cdot 10^{-15}\), where \(M_P\) is the Planck mass. Properties of the neutrino oscillations and the resonance conversion to the bulk fermions are considered. We show that the resonance conversion of the electron neutrinos to the light bulk fermions can solve the solar neutrino problem. The signature of the solution is the peculiar distortion of the solar neutrino spectrum. The solution implies that the radius of at least one extra dimension should be in the range \(0.06 - 0.1\) mm \textit{irrespective} of total number of extra dimensions. The corresponding modification of the Newtonian law is within the range of sensitivity of proposed sub-millimeter experiments, thus providing a verifiable link between neutrino physics and the gravity measurements.
1 Introduction

It has been suggested recently [1, 2, 3] that the fundamental scale of quantum gravity, $M_f$, can be as low as few TeV. The observed weakness of gravity is the result of $N (\geq 2)$ new space dimensions in which gravity can propagate \(^1\). The observed (reduced) Planck scale, $M_P = (4\pi G_N)^{-1/2} = 3.4 \cdot 10^{18}$ GeV, where $G_N$ is the Newton constant, is then related to the reduced Planck scale in $4 + N$ dimensions, $M_f$ (fundamental scale), by

$$M_P = M_f \sqrt{M_f^N V_N},$$

where $V_N = L_1 L_2...L_N$ is the volume of the extra space, and $L_i$ is the size of the $i$-compact dimension. For definiteness we will assume that the volume has a configuration of torus in which case $L_i = 2\pi R_i$, where $R_i (i = 1, 2,...N)$ are the radii of extra dimensions, so that

$$V_N = (2\pi)^N R_1 R_2...R_N .$$

Using (1) and (2) we get the constraint on the extra dimension radii:

$$(2\pi)^N R_1 R_2...R_N = \frac{M_P^2}{M_f^{N+2}} .$$

(3)

(Notice that in some publications the factor $(2\pi)^N$ is removed from this relation by redefinition of the fundamental scale: $M_* = (2\pi)^{N/(N+2)} M_f$.)

The phenomenological acceptance requires that $N \geq 2$, since for $N = 1$ the radius would be of the solar system size. According to present measurement the distance above which the Newtonian law should not be changed is about 1 mm, and therefore

$$L_i = 2\pi R_i \leq 1 \text{ mm} .$$

(4)

For $N = 2$ and $M_f \sim \text{TeV}$ one gets from (1,2) $R_1 \sim R_2 \sim 0.1 \text{ mm}$ which satisfies (4)

Thus, in theories under consideration it is expected that the Newtonian $1/r$ law breaks down at the scales smaller than the largest extra dimension: $L_{max}$. The experimentally most exiting possibility would be if $L_{max} \sim 1 - 10^{-2} \text{ mm}$, that is, in the range of sensitivity of

\(^1\)In a different context an attempt of lowering the string scale to TeV, without lowering the fundamental Planck scale was considered in [4], based on an earlier observation in [5], see also [6] for lowering the GUT scale. Dynamical localization of the fields on a (solitonic) brane embedded in a higher dimensional space-time has been suggested earlier in the field theoretic context [7], [8],[9]. For some realizations of this scenario in the D-brane context see, [2], [10].
proposed experiments [11]. As we will argue in this paper the same range is suggested by neutrino physics, namely, by a solution of the solar neutrino problem based on existence of new dimensions.

Usually it is assumed that all large radii $R_i$ are of the same order of magnitude. In such a case $N > 2$ would be well out of sensitivity of any planned sub-millimeter gravitational measurements. On the other hand, for $N = 2$ the supernova analysis pushes the lower bound on $M_f$ to 30 TeV [3] or even to 50 TeV [12] implying that $R < 0.01$ mm, which is again beyond the planned experimental sensitivity. However, in the absence of any commonly accepted mechanism for stabilization of large radii\(^2\), the requirement of their equality is unjustified.

In the present paper we will assume that radii may take arbitrary values which satisfy the fixed over-all volume (3) and phenomenological (4) constraints. In such a case the theory with several extra dimensions still can be subject of sub-millimeter test, while avoiding astrophysical and other laboratory bounds. As we will see, these bounds are sensitive to the shape of extra dimensions.

In this paper we will discuss possible consequences of the high-dimensional theories for neutrino physics. In particular, we will suggest new high-dimensional solution of the solar neutrino puzzle. This solution implies that the radius of at least one extra dimension must be within $0.06 - 0.1$ mm range. This observation relies on new high dimension mechanism of neutrino mass generation suggested in [14] which we will briefly recall.

According to the framework elaborated in [1, 2, 3], all the standard model particles must be localized on a 3-dimensional hyper-surface ('brane') [7, 8, 9] embedded in the bulk of $N$ large extra dimensions. The same is true for any other state charged under the standard model group. The argument is due to the conservation of the gauge flux, which indicates that no state carrying a charge under a gauge field localized on the brane, can exist away from it\([9][1]\). Thus, all the particles split in two categories: those that live on the brane, and those which exist everywhere, 'bulk modes'. Graviton belongs to the second category. Besides the graviton there can be additional neutral states propagating in the bulk. In general, the couplings between the brane, $\psi_{brane}$, and the bulk $\psi_{bulk}$ modes are suppressed by a volume factor:

$$\frac{1}{\sqrt{M_f^N V_N}}\psi_{brane}\psi_{brane}\psi_{bulk}.$$  \hspace{1cm} (5)

\(^2\)For some ideas in this direction see [13].
According to (1) the coupling constant in (5) equals

\[ \frac{M_f}{M_P} = 3 \cdot 10^{-16} \frac{M_f}{1\text{TeV}} \]  

(6)

and it does not depend on number of extra dimensions. It was suggested [14] to use this small model-independent coupling to explain the smallness of the neutrino mass. The left handed neutrino, \( \nu_L \), having weak isospin and hypercharge must reside on the brane. Thus, it can get a naturally small Dirac mass through the mixing with some bulk fermion and the latter can be interpreted as the right-handed neutrino \( \nu_R \):

\[ \frac{hM_f}{M_P} H \bar{\nu}_L \nu_R. \]  

(7)

Here \( H \) is the Higgs doublet and \( h \) is the model-dependent Yukawa coupling. After electro weak symmetry breaking the interaction (7) will generate the mixing mass

\[ m_D = \frac{hvM_f}{M_P}, \]  

(8)

where \( v \) is the VEV of \( H \). For \( M_f \sim 1 \text{ TeV} \) and \( h = 1 \) this mass is about \( 5.6 \cdot 10^{-5} \text{ eV} \).

Being the bulk state, \( \nu_R \) has a whole tower of the Kaluza-Klein (KK) relatives. For \( N \) extra dimensions they can be labeled by a set of \( N \) integers \( n_1, n_2, \ldots n_N \) (which determine momenta in extra spaces): \( \nu_{n_1,n_2,\ldots n_N} R \). Masses of these states are given by:

\[ m_{n_1,n_2,\ldots n_N} = \sqrt{\sum_i n_i^2 / R_i^2}. \]  

(9)

Notice that the masses of the KK states are determined by the \textit{radii} whereas the scale of the Newton law modification is given by the \textit{size} of compact dimensions.

The left handed neutrino couples with all \( \nu_{nR} \) with the same mixing mass (8). This mixing is possible due to the spontaneous breaking of the translational invariance in the bulk by the brane.

In ref. [16] a general case has been considered with possible universal Majorana mass terms for the bulk fermions. Neutrino masses, mixings and vacuum oscillations have been studied for various sizes of mass parameters.

In this paper we continue to study the consequences of mixing of the usual neutrino with bulk fermions in the context elaborated in Ref. [14]. We consider both the neutrino oscillations (in vacuum and medium) and the resonance conversion. We show that the resonance
conversion of the electron neutrinos to the bulk states can solve the solar neutrino problem. This solution implies $R \sim 0.1$ mm, thus giving connection between neutrino physics and sub-millimeter gravity measurements. We also discuss production of the KK- neutrino states in the Early Universe and in supernovae.

2 Neutrino mixing with the bulk modes. Universality

Let us first assume that extra dimensions have the hierarchy of radii, so that only one extra dimension has radius $R$ in sub-millimeter range and therefore only one tower of corresponding Kaluza-Klein modes is relevant for the low energy neutrino physics. The number and the size of other dimensions will be chosen to satisfy the constraint (3). (We will comment on effects of two sub-millimeter dimensions in sect. 4 and 5.)

The right handed bulk states, $\nu_{iR}$, form with the left handed bulk components, $\nu_{iL}$, the Dirac mass terms which originate from the quantized internal momenta in extra dimension:

$$\sum_{n=-\infty}^{+\infty} m_n \bar{\nu}_n R \nu_{iL} + h.c., \quad m_n \equiv \frac{n}{R}. \quad (10)$$

The mass-split is determined by $1/R$. According to (7, 8) the bulk states mix with usual left handed neutrino (for definiteness we will consider the electron neutrino $\nu_{eL}$) by the Dirac type mass terms with universal mass parameter:

$$m_D \sum_{n=-\infty}^{+\infty} \bar{\nu}_n R \nu_{eL}, \quad m_D \equiv \frac{\hbar v M_f}{M_P} \approx 6 \cdot 10^{-5} eV h \frac{M_f}{1TeV}, \quad (11)$$

where $h$ is the renormalized Yukawa coupling.

The mass terms (10,11) can be rewritten as

$$m_D \bar{\nu}_0 R \nu_{eL} + m_D \sum_{n=1}^{\infty} (\bar{\nu}_n R + \bar{\nu}_{-n R}) \nu_{eL} + \sum_{n=1}^{\infty} \frac{n}{R} (\bar{\nu}_n R \nu_{nL} - \bar{\nu}_{-n R} \nu_{-n L}) + h.c.. \quad (12)$$

Notice that $\nu_{0L}$ decouples from the system. Introducing new states:

$$\bar{\nu}_{n L} = \frac{1}{\sqrt{2}} (\nu_{n L} - \nu_{-n L}), \quad \bar{\nu}_{n R} = \frac{1}{\sqrt{2}} (\nu_{n R} + \nu_{-n R}) \quad (13)$$

and denoting by $\nu_{n L}', \nu_{n R}'$ the orthogonal combinations we can write the mass terms in (10) as

$$m_D \bar{\nu}_0 R \nu_{eL} + \sqrt{2 m_D} \sum_{n=1}^{\infty} \bar{\nu}_n R \nu_{eL} + \sum_{n=1}^{\infty} \frac{n}{R} (\bar{\nu}_n R \nu_{nL} + \bar{\nu}_{n R} \nu_{n L}') + h.c.. \quad (14)$$
Notice that the zero mode has $\sqrt{2}$ smaller mixing mass with $\nu_e$ than non-zero modes; the states $\nu'_{nL}, \nu'_{nR}$ decouple from the rest of the system.

Diagonalization of the mass matrix formed by the mass terms (14) in the limit of $m_D R \ll 1$ gives (see Appendix) the mixing of neutrino with $n^{th}$ - bulk mode, $\tilde{\nu}_{nL}$:

$$\tan \theta_n \approx \frac{\sqrt{2} m_D}{m_n} = \frac{\xi}{n},$$

where

$$\xi \equiv \frac{\sqrt{2} h v M_f R}{M_P}$$

determines mixing with the first bulk mode. The lightest state, $\nu_0$, has the mass $m_0 \approx m_D$ and others, $\tilde{\nu}_n$:

$$m_n \approx \frac{n}{R}.$$  (17)

According to (15) the electron neutrino state can be written in terms of the mass eigenstates as

$$\nu_e \approx \frac{1}{N} \left( \nu_0 + \xi \sum_{n=1}^{1} \frac{1}{n} \tilde{\nu}_n \right),$$

where the normalization factor $N$ equals

$$N^2 = 1 + \xi^2 \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{\pi^2}{6} \xi^2.$$  (19)

From the phenomenological point of view the bulk modes (being the singlets of the SM symmetry group) can be considered as sterile neutrinos. Thus, we deal with the coupled system of the electron neutrino and infinite number of sterile neutrinos mixed. According to (18), the electron neutrino turns out to be the coherent mixture of states with increasing mass and decreasing mixing.

The following comment concerning normalization is in order. The contribution to the normalization $N^2$ (20) from mass states with $n = k, k + 1, \ldots$ (starting from number $k$) can be estimated substituting the sum by the integral:

$$\Delta_k \sim \xi^2 \int_k \frac{dn}{n^2} = \frac{\xi^2}{k},$$

and for $k \to \infty$ we get $\Delta_k \to 0$. Thus, due to decrease of mixing the effect of heavy states is suppressed.
In real physical processes with energy release $Q$ only low mass part of the state (18) can be produced. The states with $n > QR$ do not appear. This leads to breaking of universality, that is, to difference of normalization of the neutrino states produced in processes with different $Q$. As follows from (20) the contribution of states with $n > QR$ to the normalization equals

$$\Delta(Q) = \frac{\xi^2}{QR} \approx 10^{-10} \frac{h^2}{Q/1\text{MeV}}.$$  \hspace{1cm} (21)

(For $M_f = 10$ TeV and $R^{-1} = 3 \cdot 10^{-3}$ eV.) Since $Q \gg 1/R$ for sub-millimeter scale the deviation from universality is negligible. This is not true for two extra dimensions of common size [15] (see later). For the same reason a change of kinematics of processes due to emission of states with $m_n \sim Q$ is unobservable. The probability of emission of the heavy states is negligible.

3 Oscillations and Resonance Conversion

Let us consider the vacuum oscillations of neutrino state produced as $\nu_e$ (18) to the bulk modes. The state will evolve with time $t$ as

$$\nu_e(t) = \frac{1}{N} \left( \nu_0 + \xi \sum_{n=1}^{\infty} \frac{1}{n} e^{i\phi_n} \tilde{\nu}_n \right),$$  \hspace{1cm} (22)

where the phases $\phi_n$ equal

$$\phi_n \approx \frac{(m_n^2 - m_D^2)t}{2E},$$  \hspace{1cm} (23)

and $E$ is the energy of neutrinos. The survival probability of the $\nu_e \leftrightarrow \nu_e$ oscillations then equals:

$$P \equiv |\langle \nu_e | \nu_e(t) \rangle|^2 = \frac{1}{N^4} \left| 1 + \xi^2 \sum_{n=1}^{\infty} \frac{e^{i\phi_n}}{n^2} \right|^2.$$  \hspace{1cm} (24)

Since $\phi_n \propto n^2$, the oscillation picture consists of interference of the infinite number of modes with increasing frequencies $\propto n^2$ and decreasing amplitudes $\propto 1/n^2$. For practical purpose all high frequency modes can be averaged, so that only a few low frequency oscillations can be observed depending on the energy resolution of detector. Using (24) we find the probability averaged over all the modes:

$$\bar{P} = \frac{1}{[1 + (\pi^2/6)\xi^2]^2} \left[ 1 + \frac{\pi^4}{90} \xi^4 \right].$$  \hspace{1cm} (25)
It is smaller than the two neutrino probability with the same mixing parameter $\xi$ due to presence of infinite number of mixed states. In particular, for $\xi \ll 1$ we get $P \approx 1 - (\pi^2/3)\xi^2$, whereas in the 2$\nu$ case: $\tilde{P} \approx 1 - 2\xi^2$.

In the case when only lowest frequency mode (associated to $\nu_1$) is non-averaged, we get from (24) the survival probability

$$P = \frac{1}{(1 + (\pi^2/6)\xi^2)^2} \left[ (1 + \xi^2)^2 + \left( \frac{\pi^4}{90} - 1 \right) \xi^4 - 4\xi^2 \sin^2 \phi_1 \right].$$  \hspace{1cm} (26)

According to this equation the depth of oscillations equals

$$A_P = \frac{4\xi^2}{(1 + (\pi^2/6)\xi^2)^2}.$$

Notice that due to presence of (practically) infinite number of the bulk modes which give just averaged oscillation result, the depth of oscillations of the lowest mode can not be maximal. Moreover, the relation between the depth and the average probability differs from the standard 2$\nu$ - oscillation case (see also [16]).

Similarly one can find the probability with two non-averaged modes, etc..

In medium with constant density the mixing with different bulk states is modified depending on $m^2_k$ and the potential, $V$, which describes the interaction with medium:

$$V = G_F \frac{\rho}{m_N} \left( Y_e - \frac{1}{2} Y_n \right).$$  \hspace{1cm} (28)

Here $G_F$ is the Fermi coupling constant, $m_N$ is the nucleon mass, $Y_e$ and $Y_n$ are the numbers of electrons and neutrons per nucleon correspondingly. The mode for which the resonance condition $\approx V$ is fulfilled (resonance modes) will be enhanced: the effective mixing will be enhanced and the oscillations will proceed with large depth. The modes with lower frequencies (masses) will be suppressed; the high frequency modes, $m^2_k/2E \gg V$, will not be modified.

Let us consider propagation of neutrinos in medium with varying density $\rho(r)$ keeping in mind applications to solar and supernova neutrinos. The energies of bulk states do not depend on density:

$$H_i = \frac{m^2_i}{2E},$$  \hspace{1cm} (30)
whereas for the electron neutrino we have

$$H_e \approx V(\rho) .$$  (31)

Therefore the level crossing scheme (the \((H - \rho)\) - plot which shows the dependence of the energies of levels \(H_e, H_i\) on density) consists of infinite number of horizontal parallel lines (30) crossed by the electron neutrino line (31). In what follows we will concentrate on the case \(\xi \ll 1\), so that \(m_D \ll m_n\) for all \(n\), and the crossings (resonances) occur in the neutrino channels. The resonance density, \(\rho_n\), of \(H_e\) crossing with energy of \(n^{th}\) bulk state: \(H_n = H_e(\rho_n)\) equals according to (30, 31)

$$\rho_n = \frac{m_n^2 m_N}{2 E G_F (Y_e - \frac{1}{2} Y_n)} \propto n^2 .$$  (32)

For small mixing \((\xi \ll 1)\) the resonance layers for different bulk states (where the transitions, mainly, take place) are well separated:

$$\rho_{n+1} - \rho_n \gg \Delta \rho_{nR} = \rho_n \frac{2 \xi}{n} ,$$  (33)

here \(\Delta \rho_{nR}\) is the width of the \(n^{th}\)-resonance layer. Therefore transformation in each resonance occurs independently and the interference of effects from different resonances can be neglected. In this case the survival probability \(\nu_e \rightarrow \nu_e\) after crossing of \(k\) resonances is just the product of the survival probabilities in each resonance:

$$P = P_1 \times P_2 \times \ldots \times P_k .$$  (34)

Moreover, for \(P_i\) we can take the asymptotic formula which describes transition with initial density being much larger than the resonance density and the final density – much smaller than the resonance density. As the first approximation we can use the Landau-Zenner formula [18]:

$$P_n \approx \begin{cases} 1 & E < E_{nR} \\ e^{-\frac{1}{2} \xi^2 n} & E > E_{nR} \end{cases} ,$$  (35)

where \(E_{nR}\) is the resonance energy which corresponds to maximal (initial) density \(\rho_{max}\) in the region where neutrinos are produced:

$$E_{nR} \approx \frac{m_n^2 m_N}{2 E G_F \rho_{max} (Y_e - \frac{1}{2} Y_n)} ;$$  (36)
\[ \kappa_n = \frac{m_n^2 \sin^2 \theta_n}{2E \cos \theta_n} \frac{\rho}{d\rho/dr} \]  

is the adiabaticity parameter \[17\] and \( \sin^2 \theta_n \approx 4\xi^2/n^2 \). Since \( m_n^2 \propto n^2 \) whereas \( \sin^2 \theta_n \propto 1/n^2 \), the adiabaticity parameter, \( \kappa_n = \kappa_1 \propto m_n^2 \sin^2 \theta_n \), does not depend on \( n \) for a given energy. Using this property we can write final expression for the survival probability as

\[ P \approx e^{-\frac{\pi}{2} \kappa_1 f(E)} , \]  

where

\[ \kappa_1 \approx \frac{2\xi^2}{ER^2} \frac{\rho}{d\rho/dr} , \]  

and \( f(E) \) is the step-like function

\[ f(E) = \begin{cases} 0, & E < E_{1R} \\ n, & E_{nR} < E < E_{n+1R} \end{cases} . \]  

Since high level resonances turn on at higher energies and \( \kappa \propto 1/E \), the effect of conversion decreases with increase of the order of the resonance, \( n \). Moreover, since in real situation the density is restricted from above and the energies of neutrinos are in certain range, only finite number of levels is relevant and the largest effect is due to the lowest mass resonance.

### 4 Solution of the Solar Neutrino Problem

Let us apply the results of previous section to solution of the solar neutrino problem.

We choose the lowest (non-zero) bulk mass, \( m_1 = 1/R \), in such a way that \( \Delta m^2 = 1/R^2 \) is in the range of small mixing MSW solution due to conversion to sterile neutrino \( \nu_e \rightarrow \nu_s \) \[19\]:

\[ \frac{1}{R^2} = (4 - 10) \cdot 10^{-6} \text{ eV}^2 . \]  

This corresponds to \( 1/R = (2 - 3) \cdot 10^{-3} \text{ eV} \) or

\[ R = 0.06 - 0.10 \text{ mm} . \]  

Using mass squared difference (41) as well as maximal density and chemical composition of the sun we find from (36)

\[ E_{1R} \approx 0.4 \div 0.8 \text{ MeV} . \]
Thus the pp-neutrinos \((E_{pp} < 0.42 \text{ MeV})\) do not undergo resonance conversion: \(E_{pp} < E_{1R}\) (see (38, 40)), whereas the beryllium neutrinos \((E_{Be} = 0.86 \text{ MeV})\) cross the first resonance. (For smaller \(1/R^2\) the pp-neutrinos from the high energy part of their spectrum can cross the resonance and the flux can be partly suppressed.)

The energies of the next resonances equal: \(E_{nR} = n^2 E_{1R}\), or explicitly: \(E_{2R} = 1.6 - 3.2\) MeV, \(E_{3R} = 3.6 - 7.2\) MeV, \(E_{4R} = 6.4 - 12.8\) MeV, \(E_{5R} = 10 - 20\) MeV, \(E_{6R} = 14.4 - 28.8\) MeV. Higher resonances \((n > 6)\) turn on at energies higher than maximal energy of the solar neutrino spectrum and therefore are irrelevant. The dependence of the survival probability on energy is shown in the fig. 1. The dips of the survival probability at the energies \(\sim E_{iR}\) are due to turning the corresponding resonances.

The effects of higher resonances lead to additional suppression of the survival probability in comparison with the two neutrino case. Therefore the parameter \(4\xi^2\) which is equivalent to \(\sin^2 2\theta\) should be smaller. We find that

\[
4\xi^2 = (0.7 - 1.5) \cdot 10^{-3}
\]  

(44)
gives average suppression of the boron neutrino flux required by the SuperKamiokande results. According to (8), the value of \(\xi\) (44) determines the fundamental scale:

\[
M_f = \frac{\xi M_P}{\sqrt{2} h v R} = \frac{1}{h} (0.35 - 0.7) \text{ TeV}.
\]  

(45)

For small \(h\) the scale \(M_f\) can be large enough to satisfy various phenomenological bounds.

Let us consider features of the suggested solution of the solar neutrino problem. The solution gives the fit of total rates in all experiments as good as usual 2\(\nu\) flavor conversion does: the pp-neutrino flux is unchanged or weakly suppressed, the beryllium neutrino flux can be strongly suppressed, whereas the boron neutrino flux is moderately suppressed and this latter suppression can be tuned by small variations of \(\xi\).

Novel feature appears in distortion of the boron neutrino spectrum. As follows from fig. 1 three resonances turn on in the energy interval accessible by SuperKamiokande \((E > 5 \text{ MeV})\). The resonances lead to the wave-like modulation of the neutrino spectrum. (Sharp form (38, 40) is smeared due to integration over the production region.) The observation of such a regular wave structure with \(E \propto n^2\) would be an evidence of the extra dimensions. However, in practice this will be very difficult to realize.

The SuperKamiokande experiment measures the energy spectrum of the recoil electrons from the reaction \(\nu e - \nu e\) [20]. Integrations over the neutrino energy as well as the electron
Figure 1: The survival probability as the function of neutrino energy for the electron neutrino conversion to the bulk states in the Sun (solid line), $4\xi^2 = 10^{-3}$. Dot-dashed line shows the survival probability of the two neutrino conversion for the equivalent mixing $\sin^2 2\theta = 10^{-3}$. Dashed line corresponds to the survival probability of the two neutrino conversion for $\sin^2 2\theta = 4 \cdot 10^{-3}$ which gives good fit of the total rates in all experiments.

energy of the survival probability folded with the neutrino cross-section and the electron energy resolution function lead to strong smearing of the distortion in the recoil spectrum. As the result, the electron energy spectrum will have just small positive slope (the larger the energy the weaker suppression) with very weak ($< 2 - 3\%$) modulations. It is impossible to observe such a modulations with present statistics.

Notice that relative modulations become stronger if mixing, $\xi$, is larger than $10^{-3}$ and therefore suppression is stronger. This, however, requires larger original boron neutrino flux. The SNO experiment [21] will have better sensitivity to distortion of the spectrum.

The slope of distortion of the neutrino spectrum is substantially smaller than in the case of conversion to one sterile neutrino (see fig. 1). In view of smearing effects due to integration over neutrino and electron energies (due to finite energy resolution) we can approximate the step-like function $f(E)$ in (40) by smooth function $f_{\text{app}}(E) \approx \sqrt{E/E_{1R}}$. Then the smeared survival probability in the high energy range can be written as

$$P \approx e^{-\sqrt{E/E_0}},$$

where $\sqrt{E_0} = \pi \xi^2 \rho/(R^2 d\rho/dr \sqrt{E_{1R}})$.

In the case of two large dimensions with $R_1 \sim R_2 \sim 0.02 - 0.03$ mm (see sect. 5) the
number of bulk states, and therefore the number of relevant resonances increases quadratically: \( n^2 \). (Here \( n \sim 5-6 \) is the number of resonances in the energy range of solar neutrinos in the one dimension case.) Correspondingly, the approximating function \( f_{\text{app}}(E) \) will be proportional to \( E \). As the result, \( \kappa_1 \cdot f(E) = \text{const} \) and the smeared survival probability will not depend on energy. In this case \( P(E) \approx \text{const} \) for \( E > E_{1R} \) and there is no distortion of the recoil electron spectrum. For two different radii: \( R_2 < R_1 \) one can get any intermediate behaviour of the probability between that in (46) and \( P = \text{const} \).

Common signature of both standard \( \nu_e - \nu_s \) conversion and conversion to the bulk modes is the suppression of the neutral current (NC) interactions. The two can be, however, distinguished using the following fact. In the case of the \( \nu_e - \nu_s \) conversion there is certain correlation between suppression of the NC interactions and distortion of the spectrum. The weaker distortion the weaker suppression of the NC interactions and vice versa. In the case of \( \nu_e - \nu_{\text{bulk}} \) conversion a weak distortion can be accompanied by significant suppression of the NC events. This can be tested in the SNO experiment.

No significant Day-Night asymmetry is expected due to smallness of mixing angle.

Thus, the smeared distortion of the energy spectrum (for \( E > 5 \text{ MeV} \)) is weak or absent in the case of \( \nu_e - \nu_{\text{bulk}} \) conversion. However, in contrast to other energy independent solutions here pp-neutrino flux may not be suppressed, or the energy spectrum of pp-neutrinos can be significantly distorted.

Notice that it is impossible to reproduce the large mixing angle MSW solution of the solar neutrino problem [19] in this context. Indeed, for \( \xi \sim O(1) \) and \( 1/R^2 \) as in (41) transitions in all low mass resonances are adiabatic, and therefore the survival probability has the form:

\[
P(E) \sim P_n(E) = \sin^2 \theta_n \approx \frac{\xi^2}{n^2}, \quad E_{nR} < E < E_{n+1R}.
\]

(There are smooth transitions in the regions \( E \sim E_{nR} \).) For a given energy \( E \) the probability is determined by conversion in the nearest \( n^{\text{th}} \) resonance with \( E_{nR} < E \). According to (47) the probability decreases monotonously with energy, in contrast with observations. For instance, if \( P \sim 0.5 \) in the interval \( E = 0.5 \div 2 \text{ MeV} \), then it will be \( P \sim 0.2 \) for \( E = 2 \div 4.5 \text{ MeV} \), \( P \sim 0.06 \) for \( E = 4.5 \div 8 \text{ MeV} \) etc..

If \( \xi > 1 \), for all the modes \( k \) with \( k^2/(k+1) < \xi^2 \) the resonances will be in the antineutrino channels and for \( k^2/(k + 1) > \xi^2 \) in the neutrino channels [24].
Note that solution of the solar neutrino problem due to the long length vacuum oscillations into bulk modes implies $\Delta m^2 \approx (0.5 - 5) \cdot 10^{-10} \text{ eV}^2$ and large (maximal) mixing. This leads to the following estimations

$$\frac{1}{R} \sim \sqrt{\Delta m^2} \sim (0.7 - 2) \cdot 10^{-5} \text{eV}, \quad m_D = \frac{\xi}{R} = (0.5 - 1.5) \cdot 10^{-5} \text{eV} \quad (48)$$

for values $\xi = 0.5 - 0.7$. Now the size of the extra dimension equals $L = 6 - 20 \text{ cm}$ which is excluded by existing tests of the Newton law.

The approach opens however another possibility. Suppose that the radius $R$ is small enough so that KK-excitations have negligible mixing with usual neutrinos. In this case the effect of extra dimensions is reduced to interaction with zero mode, $\nu_0$, only. Suppose that the same bulk field couples with two usual neutrinos: $\nu_e$ and $\nu_\mu$ (or $\nu_\tau$) generating the Dirac mass terms

$$m_{eD}\bar{\nu}_0 R \nu_e + m_{\mu D}\bar{\nu}_0 R \nu_\mu. \quad (49)$$

These terms lead to Dirac neutrino with mass $m_D = \sqrt{m_e^2 + m_\mu^2}$ formed by $\nu_0$ and the combination $(m_{eD}\nu_e + m_{\mu D}\nu_\mu)/m_D$. The orthogonal component is massless. In this way the $\nu_e$ and $\nu_\mu$ turn out to be mixed with the angle determined by $\tan \theta = m_{eD}/m_{\mu D}$. (Similar mechanism of mixing has been considered in Ref. [16].) For $M_f \sim 1 \text{ TeV}$ and the original Yukawa couplings with bulk field $h_e \sim h_\mu \sim 1$ we get $m_{eD} \sim m_{\mu D} \sim 10^{-5} \text{ eV}$ which leads to $\Delta m^2 \sim 10^{-10} \text{ eV}^2$ and maximal (large) $\nu_e - \nu_\mu$ mixing. This reproduces values of parameters required for the $\nu_e \leftrightarrow \nu_\mu$ Just-so oscillation solution of the solar neutrino problem. The solution has however no generic signatures of the high dimensional theory and employs the latter as the source of neutrino mass only.

5 Solar Neutrinos and Parameters of Extra Dimensions

As follows from previous section, a solution of the solar neutrino problem via resonance conversion to the bulk modes implies the radius of extra dimension $R = 0.06 - 0.1 \text{ mm}$ and the fundamental scale $M_f > 0.5 - 1 \text{ TeV}$. To satisfy the relation (3) we need to introduce more extra dimensions. Let us assume that second large dimension exists with radius $R'$. From (3) we get

$$\frac{1}{R'} = \frac{(2\pi)^2 M_f^4 R}{M_P^2} \approx 1.3 \cdot 10^{-3} \left( \frac{M_f}{1 \text{ TeV}} \right) \text{eV}. \quad (50)$$
For $M_f = 1$ TeV both extra dimensions will have radii of the same size. In this case more bulk states are involved in conversion of solar neutrinos which will lead to absence of the distortion of the spectrum, as we have discussed in sect. 4.

For $M_f \geq 10$ TeV the bulk states associated to $1/R'$ dimension do not participate in solar neutrino conversion, however they are relevant for other neutrino processes. Let us consider this in more details. Now the electron neutrino state can be written as

$$\nu_e = \frac{1}{N} \left( \nu_0 + \xi \sum_n \frac{1}{n} \nu_{n,0} + \xi \sum_{n,k \geq 1} \frac{1}{\sqrt{n^2 + (R/R')^2 k^2}} \nu_{n,k} \right),$$  

(51)

where index $n$ refers to dimension of larger radius $R$, while $k$ enumerate the bulk states from dimension $R'$. The sum over the states is divided into two parts: the first sum contains the states with $k = 0$, that is, with small mass split only. This part corresponds to the one dimensional case discussed in sect. 2 - 4 and as we have shown in sect. 3, it does not lead to observable violation of universality. The second sum contains the towers of states with both large and small mass splits. Its contribution to the normalization of the state equals

$$\Delta \equiv \xi^2 \sum_{n,k} \frac{1}{n^2 + (R/R')^2 k^2}.$$  

(52)

We can estimate the sum substituting it by the integral over $n$ and $k$. Performing first integration over $n$ from 0 to $\infty$ and then over $k$ from 1 to $(QR')$ – the number of states which can be produced in the process with energy release $Q$, we find:

$$\Delta(Q) = \frac{\pi R'}{2 R \xi} \xi^2 \ln(QR') = \pi \left( \frac{h v}{M_P} \right)^2 \frac{V_2 M_f^2}{(2\pi)^2} \ln(QR').$$  

(53)

The difference of normalizations of the two states produced in processes with energy releases $Q_1$ and $Q_2$ gives measure of the universality breaking:

$$\Delta_{21} \equiv \Delta(Q_2) - \Delta(Q_1) = \frac{\pi R'}{2 R \xi} \xi^2 \ln(Q_2/Q_1).$$  

(54)

Taking $\xi^2 = (2 - 4) \cdot 10^{-4}$ as is implied by the solar neutrino data, $Q_1 \sim 1$ MeV and $Q_2 \sim 100$ GeV as well as $R'/R < 0.1$ we find $\Delta_{12} \sim (2 - 4) \cdot 10^{-4}$ which is below present sensitivity. Notice, however, that for $R' \sim R$ the violation can be at the level of existing bounds [15].

Let us consider the case of three extra dimensions with radii $R \gg R_2, R_3$. Assuming for simplicity the equality $R_2 = R_3 \equiv R'$, we find from (3):

$$\frac{1}{R'} = \frac{(2\pi)^{3/2} M_f^2}{M_P} \sqrt{M_f R}.$$  

(55)
For $1/R \sim 2.5 \cdot 10^{-3}$ eV and $M_f = 10$ TeV this equation gives $1/R' \sim 3 \cdot 10^{-2}$ GeV. Performing calculations similar to those for one additional dimension we find parameter of universality violation

$$
\Delta_{21} \approx 2 \left( \frac{hv}{M_p} \right)^2 \frac{V_3 M_f^3 (Q_2 - Q_1)}{(2\pi)^3 M_f} = \frac{(hv)^2(Q_2 - Q_1)}{M_f^3} \quad (56)
$$

($V_3 = (2\pi)^3 RR'^2$) which can be as low as $10^{-6}$ even for $Q_2 - Q_1 \sim 100$ GeV (in this estimation we used $M_f = 10$ TeV and $h = 1$).

6 Astrophysical Bounds. Atmospheric neutrinos

The important bound on mixing of usual neutrinos with sterile neutrinos as well as with bulk states follows from primordial nucleosynthesis: production of new relativistic degrees of freedom leads to faster expansion of the Universe and to larger abundance of $^4$He. There are two ways of production of the bulk states: (i) via oscillations and (ii) incoherently via chirality flip.

In the first case the electron neutrino produced as the coherent combination of mass eigenstates oscillates in medium to bulk states. Inelastic collision splits the state into active ($\nu_e$) and to sterile ($\nu_{bulk}$) parts and after each collision two parts will oscillate independently. Oscillations between two collisions average. Production rate is then the sum of the averaged oscillation effects over collisions. The condition that sterile states do not reach equilibrium puts stringent bound on oscillation parameters [23].

The masses squared and mixing angles of bulk states (41, 44) implied by solution of the solar neutrino problem satisfy the following relation:

$$
\Delta m_n^2 \cdot \sin^2 2\theta_n \approx \frac{4 \xi^4}{R^2} = (4 - 8) \cdot 10^{-9} \text{eV}^2, \quad (57)
$$

where $\sin^2 2\theta_n \equiv 4\xi^4/n^2$. This “trajectory” in the $\Delta m^2 - \sin^2 2\theta$ - plot lies far outside the region excluded by primordial nucleosynthesis [23]. That is, production of the bulk neutrinos via oscillations is strongly suppressed.

Let us consider incoherent production of the bulk states due to chirality flip. The production rate of an individual bulk neutrino ($\Gamma_1$) is suppressed relatively to the production rate of the left-handed neutrino ($\Gamma_{\nu_e}$) by the chirality - flip factor

$$
\frac{\Gamma_1}{\Gamma_{\nu_e}} \sim \left( \frac{m_D}{T} \right)^2, \quad (58)
$$

16
where the temperature $T$ in the denominator comes from the propagator of a primarily-produced left handed neutrino. The multiplicity of the final bulk states is $TR$, so that the total bulk neutrino emission rate is suppressed as

$$\frac{\Gamma_{\text{bulk}}}{\Gamma_{\nu_e}} \sim \left(\frac{m_D}{T}\right)^2 TR = \xi \left(\frac{m_D}{T}\right).$$

(59)

For the parameters implied by the solar neutrinos and $T \sim 1$ MeV this ratio is about $3 \cdot 10^{-11}$. Using (59) we find the temperature, $T_{\text{bulk}}$, at which production rate of the bulk states is comparable with expansion rate of the Universe: $\Gamma_{\text{bulk}} = \Gamma_{\text{exp}} \sim T^2/M_P$:

$$T_{\text{bulk}} = T_{\nu} \sqrt{\frac{T_{\nu}}{\xi m_D}},$$

(60)

where $T_{\nu} \sim 1$ MeV is the temperature of the neutrino decoupling. From (60) we find $T_{\text{bulk}} \sim 200$ GeV which is much above the “normalcy” temperature [3]. The modes from additional dimensions having smaller radii give even smaller contribution.

The KK-neutrinos as well as the KK-gravitons produced in stars, in particular, in supernovae, increase the rate of star cooling [3]. No additional sources of cooling have been found from observations of the SN1987A which put stringent bound on production of the bulk states. The rate of the incoherent production of the bulk neutrinos in supernovae is suppressed by the same factor (59). For temperature of the core of supernova $T \sim 30$ MeV the eq. (59) gives $\Gamma_{\text{bulk}}/\Gamma_{\nu_e} \sim 3 \cdot 10^{-12}$. Then taking into account that bulk neutrinos are emitted from the whole volume of the core, whereas usual neutrinos are emitted from the surface (neutrinosphere) we find that the luminosity in the bulk states is 5 - 6 orders of magnitude smaller than luminosity in neutrinos. Production of the bulk states via oscillations is strongly suppressed by matter effect. Matter suppression is weak or absent for bulk states with high mass: $m \sim 10^3$ eV. However, their production is suppressed by very small vacuum mixing.

So, we conclude that parameters required by solution of the solar neutrino problem satisfy astrophysical constraints.

Let us comment in this connection on solution of the atmospheric neutrino problem via oscillations of muon neutrinos to the bulk states $\nu_\mu \leftrightarrow \nu_{\text{bulk}}$. This solution requires smaller radius of the extra dimension: $1/R \approx \sqrt{\Delta m^2_{\text{atm}}} \sim (5-9) \cdot 10^{-2}$ eV or $R \sim (2-4) \cdot 10^{-3}$ mm and near to maximal mixing: $\xi \sim 1$. The latter means that the fundamental scale should be about
\( M_f \approx (10^3 \text{ TeV})/h \). that is, about 3 orders of magnitude larger than that for solar neutrinos. (Notice that approximation \( \xi \ll 1 \) can not be used now to diagonalize the mass matrix and results of sect. 3 and Appendix should be corrected \[24\]. Still mixing of large mass bulk states is suppressed by factor \( 1/n \) and these states lead to finite averaged oscillation result. Only a few low mass states are relevant for non-averaged oscillation picture.)

The oscillation parameters required by the solution of the atmospheric neutrino problem violate nucleosynthesis bound. Indeed, masses and mixing angles of the bulk modes satisfy now the relation:

\[
m_n^2 \cdot \sin^2 2\theta_n \sim \frac{2}{R^2} \sim 6 \cdot 10^{-3} \text{eV}^2.
\]

Nucleosynthesis bound reads \[23\]

\[
\Delta m^2 < \frac{3 \cdot 10^{-5} \text{eV}^2}{\sin^4 2\theta}.
\]

From these two equation we find that about 20 - 25 lightest bulk states are in the forbidden region. They turn out to be in equilibrium, whereas 1 or at most 2 are allowed. Production of the bulk states via oscillations can be suppressed if there is substantial (\( \sim 10^{-5} \)) leptonic asymmetry in the Universe \[25\].

Let us finally consider production of gravitons in the stars. The generic reason that saves the bulk gravitons from being ruled out by star cooling is the infrared-softness of the high dimensional gravity. On the language of four-dimensional KK modes this can be visualized as follows. The rate of each individual KK graviton emission in the star is suppressed by the universal volume factor

\[
\left( \frac{T}{M_P} \right)^2 \sim \left( \frac{T}{M_f} \right)^2 \frac{1}{M_f^N V_N}.
\]

The number of available final states is \( \sim T^N V_N \), so that the over-all rate is suppressed as

\[
\Gamma_{\text{grav}} \sim \left( \frac{T}{M_P} \right)^2 T^N V_N \sim \left( \frac{T}{M_f} \right)^{2+N}.
\]

According to this expression, the analysis of supernova core cooling gives for \( N = 2 \) the lower bound on the fundamental gravitational scale \( M_f \) about 30 TeV \[3\] - 50 TeV \[12\]. As it is clear from (64), the rate is determined by the value of \( M_f \) and it is insensitive to sizes of individual radii \( R_i \) as far as all \( 1/R_i < T \). On the other hand, if some radii, \( R_k \), do not satisfy this bound, then the corresponding KK modes can not be produced in the star and
the corresponding factor $TR_k$ in $T^N V_N$ of Eq. (64) has to be replaced by 1. The bottom-line of this discussion is that standard constraint can be avoided if the extra dimensions have different radii. For instance, with one radius $R \sim 0.03$ mm and two others $R_k > T$ (which can be realized even for $M_f$ as low as several TeV’s) the rate becomes

$$\Gamma_{\text{grav}} \sim \left( \frac{T}{M_P} \right)^2 (TR) \approx 10^{-31}. \quad (65)$$

Therefore, some of the radii can be of sub-millimeter size and, thus, can be a subject of direct experimental search in future gravitational measurements [11].

7 Conclusions

We have studied consequences of the neutrino mixing with fermions propagating in the bulk in the context of theories with large extra dimensions. The bulk fermions could be components of bulk gravitino or other singlets of the SM gauge group.

Phenomenology of this mixing is determined by the following features: (i) The bulk fermions can be considered as sterile neutrinos. (ii) Large number of these sterile neutrinos is involved in physical processes. (iii) For $m_D < 1/R$ usual neutrinos are combinations of mass eigenstates with increasing masses and decreasing admixtures.

The effect of bulk states with large masses is reduced to averaged oscillations. Low modes can lead to non-averaged oscillations in vacuum (uniform medium) or to multi-resonance conversion in medium with varying density.

The resonance conversion of the electron neutrino to the bulk states can solve the solar neutrino problem. Properties of this solution are similar to those due to conversion to sterile neutrino. The important difference is that significant suppression of the boron neutrino flux can be accompanied by weak distortion of the energy spectrum. Moreover, weak modulation of the boron neutrino spectrum is expected due to conversion to several KK-states.

Simultaneous explanation of the atmospheric, solar and LSND results in terms of neutrino oscillation/conversion implies existence of sterile neutrino. Moreover, the data favour $\nu_\mu \leftrightarrow \nu_\tau$ oscillations as the solution of the atmospheric neutrino problem, so that the solar neutrinos should be converted to sterile states. In this connection one can consider the following possibility. There is some usual (4 dimensional) mechanism of generation of the active neutrino masses. This mechanism produces neutrino mass pattern with heavy and strongly mixed $\nu_\mu$
and $\nu_\tau$ and very light $\nu_e$. Such a pattern can explain the atmospheric neutrino and LSND results. Neutrinos (in general of all flavors) couple also with the bulk fermion. These couplings (being of the same order for all neutrino species) generate negligible mixing of the KK-states with heavy $\nu_\mu$ and $\nu_\tau$ and large enough mixing with light $\nu_e$, so that the solar neutrino problem can be solved as is described in sect. 4.

The suggested solution of the solar neutrino problem implies that the radius of at least one extra dimension is in the range 0.06 - 0.10 mm, that is, in the range of sensitivity of proposed gravitational measurement. The fundamental scale should be about $hM_f \sim 1$ TeV. This mass satisfies the fixed overall volume condition provided additional large extra dimensions exist. In the case of one additional extra dimension its radius should be $1/R' = 1.3 \cdot 10^{-3} (M_f/1\text{TeV})^4$ eV. For $h \sim 1$ one has $M_f \sim 1$ TeV, and $1/R' \sim 2 \cdot 10^{-3}$ eV, so that the second extra dimension will influence the solar neutrino data too. If $h \ll 1$, the fundamental scale can be much larger than 1 TeV, and $R'$ can be much smaller than 1 mm. For $h = 0.1$ and $M_f = 10$ TeV we get $1/R' \sim 10$ eV. For two additional dimensions the common radius equals $1/R' \sim 3 \cdot 10^{-2}$ GeV, if $h = 1$ and $M_f = 10$ TeV.

For large fundamental scales ($M_f > 10 - 20$ TeV), direct laboratory searches at high energies will be practically impossible and neutrinos can give unique (complementary to gravitational measurements) opportunity to probe the effects of large extra dimensions.

Appendix

The mass terms (14) can be written as

$$\bar{\nu}_L M \nu_R,$$

where $\nu_L^T = (\nu_L, \tilde{\nu}_1L, \tilde{\nu}_2L, ...)$ and $\nu_R^T = (\nu_0R, \tilde{\nu}_1R, \tilde{\nu}_2R, ...)$; the modes $\nu_0L, \nu_{nL}, \nu_{nR}$ decouple from the system, and the mass matrix $M$ for $k + 1$ states equals

$$M = \begin{pmatrix}
  m_D & \sqrt{2}m_D & \sqrt{2}m_D & \cdots & \sqrt{2}m_D \\
  0 & \frac{1}{R} & 0 & \cdots & 0 \\
  0 & 0 & \frac{2}{R} & \cdots & 0 \\
  \cdots & \cdots & \cdots & \cdots & \cdots \\
  0 & 0 & 0 & \cdots & \frac{k}{R}
\end{pmatrix}.$$
Let us consider the matrix $MM^\dagger$ which determines mixing of the left handed neutrinos:

$$MM^\dagger = \frac{1}{R^2} \begin{pmatrix}
  (k+1/2)\xi^2 & \xi & 2\xi & \cdots & k\xi \\
  \xi & 1 & 0 & \cdots & 0 \\
  2\xi & 0 & 4 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  k\xi & 0 & 0 & \cdots & k^2
\end{pmatrix}, \quad (A1)$$

where $\xi \equiv \sqrt{2m_D} R$. Notice that the mass matrix (A1) corresponds to compactification on the circle; it can be shown that the same matrix follows from the $Z_2$ - orbifold compactification [16]. The diagonalization of matrix can be performed starting from the heaviest state $k \gg 1$. Then one can check that the limit $k \to \infty$ does not change results. The rotation by the angle $\theta_k$ in the plane $\nu_0L - \tilde{\nu}_kL$

$$\tan 2\theta_k = \frac{2\xi}{k} \cdot \frac{1}{1 - \frac{\xi^2}{k^2}} \quad (A2)$$

diagonalizes corresponding submatrix. It is easy to perform the diagonalization using $\xi$ as an expansion parameter: $\xi \ll 1$ as is implied by the solution of the solar neutrino problem. The rotation (A1) leads to modification of the first diagonal element $(k+1/2)\xi^2 \to (k-1/2)\xi^2 + O(\xi^4)$ and modification of the mixing terms as $n\xi \to \cos \theta_k n\xi$. For small $\xi$: $\tan \theta_k \approx \xi/k$ and the eigenvalues equal $m_k^2 \approx k^2/R^2$. After $k-1$ subsequent rotations we get for the first diagonal element: $3/2\xi^2 + O(\xi^4)$ and for off-diagonal term:

$$\xi \cos \theta_2 \cos \theta_3 \cdots \cos \theta_k \approx \xi \left(1 - \frac{1}{2}\xi^2 \sum_{n=2}^{k} 1/n^2 + \ldots \right) \approx \xi.$$

These results can be obtained from the exact characteristic equation for the mass eigenstates: $\text{Det}[MM^\dagger - m^2]$ which can be written explicitly as

$$\pi \xi^2 \cot(\pi mR) = 2mR.$$

which coincides with the characteristic equation in [16] for the same neutrino system.

**Acknowledgments**

One of us (A.S.) is grateful to E. Dudas for useful discussions.
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