Bosonic linear unitary Bogoliubov transformation reduction theorem

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We show that bosonic linear unitary Bogoliubov transformations may be reduced to a standard form. This leads to a better understanding of the action of general quadratic Hamiltonians. We show that this reduction theorem has a simple physical interpretation in quantum optics. Using this reduction we derive a no-go theorem for superpositions of macroscopically distinct states from single-photon detection. Finally, we construct a few minimal quantum optical circuits.

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The linear mixing of annihilation and creation operators (Bogoliubov transformations) provides a surprisingly rich class of theoretical models for quantum systems. This is especially true when this mixing is supplemented by the ‘cheap’ non-linearity afforded by particle detection and subsequent feedback or feedforward. If we exclude detection, then the system evolution may be described by a, possibly time-dependent, quadratic Hamiltonian. It has been known for some time how to analytically calculate the evolution of such systems [1], thus providing a powerful tool for the analysis of bosonic systems. Here we show that all such Bogoliubov transformations may be reduced to a standard form. This reduction yields a new kind of tool; one equally suited to the synthesis of a rich class of theoretical models for quantum systems.

Theorem (BLUBBeR reduction): For a general linear unitary Bogoliubov transformation of the form

\[ \hat{b}_j = \sum_k (A_{jk} \hat{a}_k + B_{jk} \hat{a}_k^\dagger) + \beta_j , \]  

where \( \hat{a}_j, \hat{b}_j \) are bosonic annihilation operators, the matrices \( A \) and \( B \) may be decomposed into a pair of unitary matrices \( U \) and \( V \) and a pair of non-negative diagonal matrices \( A_D \) and \( B_D \) satisfying

\[ A_D^T = B_D^2 + 1 , \]  

with \( 1 \) the identity matrix, by

\[ A = U A_D V^\dagger \]  

\[ B = U B_D V^T . \]  

Proof: Without loss of generality, we may set the displacements in Eq. (1) to zero, i.e., \( \beta_j = 0 \). The canonical commutation relations for \( \hat{b}_j \) in Eq. (1) impose the conditions

\[ AB^T = (AB^T)^T \]  

\[ AA^T = BB^T + 1 , \]  

since \( AA^T \) and \( BB^T \) are hermitian and according to Eq. (5) must commute, they also may be diagonalized in the same basis by some unitary matrix \( U \). However, using the singular value decomposition theorem \[4\] we can always diagonalize \( A = U A_D V^\dagger \) and \( B = U B_D V^T \) into non-negative matrices \( A_D \) and \( B_D \) satisfying Eq. (2) where \( V \) and \( W \) are a pair of unitary matrices. Unitarity of (1) guarantees a unique inverse which with the aid of Eqs. (4) and (5) may be easily computed to be

\[ \hat{a}_j = \sum_k (A_{kj}^* \hat{b}_k - B_{kj} \hat{b}_k^\dagger) . \]  

Imposing the canonical commutation relations again here yields the conditions

\[ A^\dagger B = (A^\dagger B)^T \]  

\[ A^\dagger A = (B^\dagger B)^T + 1 . \]  

Thus we see that \( A^\dagger A \) and \( (B^\dagger B)^T \) may be diagonalized in the same basis by a unitary matrix \( V = W^* \) which yields Eq. (3) as required. Finally, we note that this form for \( A \) and \( B \) automatically satisfies the subsidiary conditions of Eqs. (4) and (7).

Remark: This result is essentially a consequence of Louville’s theorem applied to linear phase-space transformations of classical systems via

\[ \left( \begin{array}{c} \vec{q}^{(\text{out})} \\ \vec{p}^{(\text{out})} \end{array} \right) = \left( \begin{array}{cc} \text{Re}(A + B) & \text{Im}(B - A) \\ \text{Im}(A + B) & \text{Re}(A - B) \end{array} \right) \left( \begin{array}{c} \vec{q}^{(\text{in})} \\ \vec{p}^{(\text{in})} \end{array} \right) . \]  

To discuss applications of BLUBBeR reduction we shall henceforth concentrate on photonic modes in quantum optics. The principle reason for this is that there
exists a well developed correspondence between lumped primitive laboratory components and theoretical mode couplings. BLUBBeR reduction will give us a powerful tool for understanding how these components may be combined.

Much of traditional optics involves purely linear elements, such as lenses, beam-splitters, mirrors, half-wave plates, etc. Mathematically, linear optical components are those whose Bogoliubov transformations take the particularly simple form

$$\hat{b}_j = \sum_k U_{jk} \hat{a}_k,$$

where $U$ is an arbitrary unitary matrix and there is no mixing of the mode annihilation and creation operators. Any such unitary $U$ may be explicitly constructed from linear optical primitive components [5].

A number of non-linear optical components may produce a linear mixing between annihilation and creation operators when some pumping field or fields are strong enough that their quantum fluctuations and pump depletion may be neglected (the so-called parametric approximation). Without attempting to be exhaustive in our labeling, we shall consider three basic variations of so-called down-converters.

**Squeezers (S):** single-mode down-converters (also known as parametric amplifiers) may be described by an interaction Hamiltonian of the form

$$\hat{H}_{\text{int}} = i r (\hat{\alpha}_1^\dagger \hat{\alpha}_2^\dagger - \hat{\alpha}_1 \hat{\alpha}_2),$$

where $r$ is the squeezing parameter and we drop extraneous phases from our descriptions without loss of generality.

**Two-mode down-converters (D$_2$):** may be described by

$$\hat{H}_{\text{int}} \propto i (\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2).$$

**Four-mode down-converters (E$_4$):** are given by

$$\hat{H}_{\text{int}} \propto i (\hat{a}_1^\dagger \hat{a}_2^\dagger + \hat{a}_3^\dagger \hat{a}_4^\dagger - \hat{a}_1 \hat{a}_2 - \hat{a}_3 \hat{a}_4).$$

These latter devices may be thought of as two-mode entangling down-converters if, for example, the even (odd) numbered modes represent differing polarization states for a mode heading left (right). We are now in a position to describe the first non-trivial consequence of BLUBBeR reduction.

**Corollary:** For optical modes, BLUBBeR reduction says that the general form of multimode evolution with linear Bogoliubov transformations may be decomposed into a multi-port linear interferometer, followed by the parallel application of a set of single-mode squeezers followed by yet another multi-port linear interferometer. This reduction is shown schematically in Fig. 1.

**Remark:** After BLUBBeR reduction, the modes acted upon by the single-mode squeezers will in general involve linear combinations of different frequency fields. Thus, the optical circuits so obtained do not necessarily correspond to an immediate physical decomposition. Such a physical decomposition may be readily obtained, though it will no longer have the minimal BLUBBeR reduced form.

One common application for down-converters is as sources of interesting quantum states. We shall use BLUBBeR reduction to tell us something about how versatile such devices may be. We shall assume that initial coherent states can always be cheaply made available after which we may use linear optics and down-converters. The simplest way of operating such a source is unconditionally for which we state the following result:

**Corollary:** Given a set of initially coherent or vacuum states, an arbitrarily complicated combination of linear multi-port interferometers, down-converters, squeezers, etc, will deterministically generate only Gaussian states.

There are at least two other modes of state generation which might be considered of interest:

**Conditional state generation:** where the required state leaves some part of the apparatus whenever a suitable sequence of photodetection events is found in another part. For example, a weakly coupled two-mode down-converter (12) can make a single-photon state to a good approximation in either of the two modes conditioned on a single-photon count in the other.

**Random state generation:** where the required state is ‘polluted’ by contributions from the vacuum state. In this case, the state may be inferred by destructive photodetection, but then it cannot leave the apparatus. For example, a weakly coupled four-mode down-converter (13) can make polarization entangled states
randomly (in the sense given above); interestingly, it is currently unknown whether such states can be produced conditionally from the coherent states, linear optics, down-converters and photodetectors.

We see from these examples that the ‘cheap’ non-linearity introduced by particle detection can increase the versatility of BLUBBeR. However, there still appear to be limitations:

**Theorem (no-go for macro-superpositions):** Detection of a single photon in one mode and no photons in any number of other modes cannot conditionally create superpositions of macroscopically distinct states given an initial vacuum state and using an arbitrarily complicated combination of linear multi-port interferometers, down-converters, squeezers, etc (all described by quadratic interactions).

**Proof:** Consider such a combination of components acting on the vacuum. By BLUBBeR reduction (see Fig. 1) the initial linear multi-port interferometer described by $V^\dagger$ preserves the vacuum state, so only the later components have an affect. Since the individual single-mode squeezers have evolution operators which may be trivially normally ordered we may immediately write out the general form for the outgoing Gaussian state as

$$|\psi_{\text{out}}\rangle \propto \exp\left(\frac{i}{2} \sum_{jk} B_{jk} b_j^{\dagger} b_k^{\dagger}\right) |0\rangle,$$  

where without loss of generality $B_{jk}$ may be chosen to be complex symmetric and $b_j^{\dagger}$ are the outgoing mode creation operators. Suppose a single photon is detected in some mode $b_t$ and vacuum in several others, the conditioned state is

$$|\psi_{\text{cond}}\rangle \propto \det(0) b_t |\psi_{\text{out}}\rangle \propto \sum'_m B_{tm} b_m^{\dagger} \det(0) |\psi_{\text{out}}\rangle,$$  

where $|0\rangle_{\det}$ is the vacuum state for the subset of detected modes and the sum runs only over non-detected modes. It is easy to see that $\det(0) |\psi_{\text{out}}\rangle$ is a Gaussian state on the remaining modes, so the conditionally created state from single photon detection is seen to be a sum of branches which differ by the placement of only a single photon in one mode or another.

**Remark:** Large amplitude coherent states are ‘macroscopic superpositions’ only in the sense that they are superpositions of macroscopic states (although these states are not macroscopically distinguishable). Thus, we have given a no-go theorem against creating so-called Schrödinger cat states for any such scheme without regard to the specific details of any particular implementation. A consequence of this result is that entanglement may not be ‘amplified’ by say injecting microscopic superpositions into strongly pumped down-converters as has been recently suggested [6,7] (though obviously superposition states may be sent through an amplifier [8]).

The BLUBBeR reduction theorem teaches us some important lessons about the interconvertibility of different kinds of sources. For example, we find that a single squeezed state is an irreducible resource which cannot be made from any number of lesser squeezed states and linear optics. Similarly, if some device requires some given number of squeezers in BLUBBeR reduced form then fewer squeezers plus linear optics will never suffice for the device’s construction. Let us use these observations to relate the three types of down-converters $S$, $D_2$ and $E_4$.

A non-entangling two-mode down-converter ($D_2$) with coupling (12) requires two squeezers in reduced form as is illustrated in Fig. 2. For weak coupling this device is a source of random photon pairs generated into distinct modes. The BLUBBeR reduction into two squeezers and a 50:50 beam-splitter gives us a more sophisticated understanding of the Hong-Ou-Mandel interferometer [9]. Away from the weak coupling limit we retrieve the twin-beam scheme for making two-mode squeezed states from a pair of independently squeezed states [10,11]. BLUBBeR reduction neatly formalizes these multi-photon interference phenomena.

**FIG. 2.** BLUBBeR equivalence: Here we illustrate the equivalence between pair of squeezers ($S$) combined at a 50:50 beam-splitter (BS) and a single two-mode down-converter ($D_2$).

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**FIG. 3.** Polarization entanglement without loss of which-way information. Here we illustrate the equivalence between an entangling four-mode down-converter ($E_4$) with a pair of non-entangling two-mode down-converters ($D_2$) which are randomly creating photon pairs with opposite polarizations $\circlearrowright$. The polarization dependent beam-splitters (PBS) direct all photons to the upper paths. BLUBBeR reduction shows that is impossible to (randomly) create such entangled states with only a single pass through a single non-entangling two-mode down-converter.

Similarly, BLUBBeR reduction applied to the entan-
gling four-mode down-converter (E₄) of Eq. (13) shows that four squeezers are required in reduced form. Thus, a random polarization entangled state cannot be formed from a single pass through a single non-entangling down-converter [D₂, Eq. (12)]. Nonetheless, it may be made easily enough with two such devices. In Fig. 3 we give just such an equivalence. This particular construction is all the more surprising since it produces entanglement without erasing the which-way information about the photons. It should be noted that this scheme is very different (in terms of the irreducible resources used) than the entanglement swapping scheme of Zukowski et al [12] which starts with a pair of entangling down-converters.

As a final application for the BLUBeR reduction theorem we consider constructing optimal optical circuits using as little squeezing as possible. Consider the ideal quantum non-demolition (QND) coupling between a pair of optical modes

\[
\hat{b}_1 = \hat{a}_1 - \frac{1}{2} \hat{a}_2 + \frac{1}{2} \hat{a}_2^\dagger,
\]

\[
\hat{b}_2 = \frac{1}{2} \hat{a}_1 + \hat{a}_2 + \frac{1}{2} \hat{a}_1^\dagger.
\]

The relevant decomposition is given by

\[
A = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sin \theta & -i \cos \theta \\
\cos \theta & i \sin \theta
\end{pmatrix} \begin{pmatrix}
\sqrt{\frac{r-1}{r}} & 0 \\
0 & \sqrt{\frac{r}{r+1}}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -i \sin \theta \\
\sin \theta & i \cos \theta
\end{pmatrix}^\dagger,
\]

\[
B = \begin{pmatrix}
0 & \frac{1}{2} \\
\frac{1}{2} & 0
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\sin \theta & -i \cos \theta \\
\cos \theta & i \sin \theta
\end{pmatrix} \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix}
\begin{pmatrix}
\cos \theta & -i \sin \theta \\
\sin \theta & i \cos \theta
\end{pmatrix}^T,
\]

where \( \theta = \frac{1}{2} \sin^{-1}(2/\sqrt{5}) \approx 31.72^\circ \). The circuit consists of a pair of squeezers with equal squeezing parameters of \( r = \ln[(1 + \sqrt{5})/2] \) (corresponding to roughly 4.18 dB) and a pair of unequal unbalanced beam-splitters with energy transmission coefficients of 27.64% and 72.36%.

In fact, this circuit is equivalent to one derived by Yurke [13], however, BLUBeR guarantees its optimality. We can improve on it further by noting that the singular value eigenvalues in (17) are degenerate and so the decomposition is not unique; a construction with much simpler 50:50 beam-splitters is given by

\[
U = \frac{1}{\sqrt{2}} \begin{pmatrix}
ie^{i\theta} & ie^{-i\theta} \\
e^{-i\theta} & -ie^{i\theta}
\end{pmatrix}
\]

\[
V = \frac{1}{\sqrt{2}} \begin{pmatrix}
-e^{-i\theta} & e^{i\theta} \\
-ie^{-i\theta} & -ie^{i\theta}
\end{pmatrix},
\]

with \( \theta \) as above. We note that the QND coupling (16) has recently been used in error correction codes for quantum optical fields [14,15].

In conclusion, we have derived a reduction theorem for general bosonic linear unitary Bogoliubov transformations (BLUBeR). We have shown the equivalence between a number of elementary sources of weak random states, including a simple scheme to randomly generate polarization entanglement without a loss of which-way information. When supplemented by detection of a single photon we have shown that superpositions of macroscopically distinct states cannot be created out of vacuum using linear optics and down-converters, squeezers, etc (all corresponding to linear Bogoliubov transformations). Finally, we used BLUBeR reduction to study the construction of minimal optical circuits. Although we have concentrated on applications for photonic modes in quantum optics the BLUBeR reduction theorem holds for all bosonic modes.

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