DIMENSIONAL REDUCTION OF 4D HETEROTIC STRING BLACK HOLES

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Abstract

We perform the spherical symmetric dimensional reduction $4d \to 2d$ of heterotic string theory. We find a class of two-dimensional (2d) dilaton gravity models that gives a general description of the near-horizon, near-extremal behavior of four-dimensional (4d) heterotic string black holes. We show that the duality group of the 4d theory is realized in two dimensions in terms of Weyl transformations of the metric. We use the 2d dilaton gravity theory to compute the statistical entropy of the near-extremal 4d, $a = 1/\sqrt{3}$, black hole.

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1 Introduction

The black hole solutions of four-dimensional (4d) heterotic string theory are very interesting from several points of view [1]. They give a general description of the whole set of charged black hole solutions of string theory (and general relativity) in four dimensions. Heterotic, type IIA and type IIB strings in four dimensions are connected one with the other by a web of dualities [2, 3], so that one can use the black hole solutions of 4d heterotic string theory as representative for the whole set of 4d black hole solutions in string theory. Moreover, using truncated models, it has been realized that there are only four classes of black holes, which can be described by effective 4d dilaton gravity models with dilaton coupling $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$ [4, 5].

The universal character of these solutions has found support both from the compositeness idea, according to which the $a = 1, 1/\sqrt{3}, 0$ solutions can be considered as bound states of the $a = \sqrt{3}$ elementary solution [5] and from the interpretation of them as intersection of D-branes [6].

Each class of 4d heterotic string black holes is characterized by its spacetime structure, its singularities and its thermodynamical behavior. Particularly remarkable is the behavior of the $a = 0$ black hole (essentially the Reissner-Nordstrom black hole of general relativity) compared with that of the $a = 1, \sqrt{3}, 1/\sqrt{3}$, cases. The extremal limit of the $a = 0$ black hole is a zero-temperature ground state with non vanishing entropy, whereas for $a = 1, \sqrt{3}, 1/\sqrt{3}$ at the extremal limit we have zero temperature and entropy. The previous features can be understood using different approaches, even though a complete and satisfactory picture of the all subject is still missing. For instance, the existence of a ground state with the previously described behavior can be traced back to the presence of a mass gap. This mass gap can be explained both using string theory [7] or just general features of the 4d effective dilaton gravity theory [8].

Until now the various attempts to give a general description and to clarify the subject have mainly approached it from “above”, i.e using higher dimensional, $d \geq 4$, models. The recently proposed AdS$_d$ (anti-de Sitter)/ CFT$_{d-1}$ (conformal field theory) correspondence [9] from one side and the discovery of dualities between four- and two-dimensional black holes [10] from the other side, have changed slightly the perspective. For $d = 2, 3$ the AdS/CFT duality has been used to compute the statistical entropy of 2d and 3d black holes [11, 12]. On the other hand, it is known that near the horizon the geometry of the 4d, magnetically charged, $a = 0, 1, 1/\sqrt{3}$ black hole factorizes as the product of two spaces of constant curvature, $\mathcal{M}^2 \times S^2$, where $S^2$ is the two-sphere and $\mathcal{M}^2$ is 2d Minkowski space for $a = 1$ and $\mathcal{M}^2 = \text{AdS}_2$ for $a = 0, 1/\sqrt{3}$ [3]. It is therefore natural to try to extend the computation of Ref. [12] of the statistical entropy to the 4d case. The hope is that the explanation of the entropy in terms of microstates will also help to explain the different behavior of the various black hole solutions.

Motivated by the previous arguments, in this paper we perform a generic $4d \rightarrow 2d$, spherical symmetric dimensional reduction of 4d effective heterotic string theory. We find that the near-horizon, near-extremal behavior of 4d heterotic string black holes are described by a class of 2d dilaton gravity models. We show that these 2d models encode all the relevant information about the 4d theory but in a much simpler form. In particular, we show that the duality group of the 4d theory is realized in the 2d theory in terms of Weyl transformations...
of the metric and we use the 2d dilaton gravity model to compute the statistical entropy of
the near-extremal 4d, $a = 1/\sqrt{3}$, black hole.

In Sect. 2 we describe the dimensional reduction $4d \to 2d$ of heterotic string theory and
the single-scalar field truncation that produces the effective 2d dilaton gravity models. In
Sect. 3 we study the realization of the 4d duality symmetry in the 2d context, showing
that it corresponds to Weyl transformations of the metric. In Sect. 4 we use the 2d dilaton
gravity model together with the results of Ref. [12] to compute the statistical entropy of
the near-extremal, 4d, $a = 1/\sqrt{3}$, black hole. Finally in Sect. 5 we draw our conclusions.

2 Dimensional reduction of 4d heterotic string theory

In the string frame the bosonic action for heterotic string theory compactified on a six-
torus [13, 14] can be written as follows:

$$A_H = \frac{1}{16\pi} \int d^4x \sqrt{-g} \ e^{-2\phi} \left\{ R + 4(\partial \phi)^2 - 2 \left[ (\partial \sigma)^2 + (\partial \rho)^2 \right] \right\} - \frac{1}{4} \left\{ e^{-2\sigma-2\rho} F_1^2 + e^{-2\sigma+2\rho} F_2^2 + e^{2\sigma+2\rho} F_3^2 + e^{2\sigma-2\rho} F_4^2 \right\}.$$  \hspace{1cm} (1)

In the bosonic action of heterotic string theory toroidally compactified to $d = 4$, we have
set to zero the axion fields and all the $U(1)$ field strengths but four, two Kaluza-Klein fields
$F_1, F_2$ and two winding modes $F_3, F_4$. In the action (1) and throughout this paper we set the
4d Newton constant $G = 1$. The scalar fields $\phi, \rho, \sigma$ are related to the standard definitions of
the string coupling, Kähler form and complex structure of the torus, through the equations

$$e^{-2\phi} = \text{Im} S, \quad e^{-2\sigma} = \text{Im} T, \quad e^{-2\rho} = \text{Im} U.$$  \hspace{1cm} (2)

The extremal, Bogomol'ny-Prasad-Sommerfield (BPS), solutions of the action (1) are given by (see for instance Ref. [15])

$$ds^2 = -\psi_1 \psi_3 \ dt^2 + (\chi_2 \chi_4)^{-1} (dr^2 + r^2 d\Omega_2^2),$$

$$e^{4\phi} = \frac{\psi_1 \psi_3}{\chi_2 \chi_4}, \quad e^{4\sigma} = \frac{\psi_1 \chi_4}{\chi_2 \psi_3}, \quad e^{4\rho} = \frac{\psi_1 \chi_2}{\psi_3 \chi_4},$$

$$F_1 = \pm d\psi_1 \wedge dt, \quad F_2 = \pm d\chi_2 \wedge dt, \quad F_3 = \pm d\psi_3 \wedge dt, \quad F_4 = \pm d\chi_4 \wedge dt.$$  \hspace{1cm} (3)

where $(\psi_1)^{-1}, (\psi_3)^{-1}, (\chi_2)^{-1}, (\chi_4)^{-1}$ are harmonic functions, $d\Omega_2^2$ is the metric of the two-
sphere and $F_2 = e^{-2(\phi+\sigma-\rho)} F_2, \ F_4 = e^{-2(\phi-\sigma+\rho)} F_4$ ($^*$ denotes the Hodge dual).

Motivated by the fact that the action (1) admits solutions that are the direct product
of two two-dimensional spaces of constant curvature $\mathcal{M}^2 \times S^2$ [3], we study the general,
spherical symmetric, dimensional reduction $4d \to 2d$ of the theory.

Let us first fix our notation. Greek letters from the middle of the alphabet denote 4d
spacetime indices, $\mu, \nu \ldots = 0, 1, 2, 3$. Greek letters from the beginning of the alphabet
denote 2d spacetime indices, $\alpha, \beta \ldots = 0, 1$. The capital latin letters $I, J \ldots = 2, 3$ label the
coordinates of the internal two-sphere, $S^2$. The lower-case latin letters $i, j \ldots = 1, 2, 3, 4$ are
used to label the four $U(1)$ field strengths $F_{(i)}$ whereas $a, b \ldots = 1, 2, 3$ label the moduli,
$\eta_a = (\phi, \sigma, \rho)$.  

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The dimensional reduction of the 4d theory is performed by splitting the metric and the $U(1)$ field strengths into their 2d parts, using the ansatz

$$d s^2_{(4)} = d s^2_{(2)} + Q^2 e^{2\psi} d \Omega^2_2,$$

$$F_{(i)\mu\nu} = \{F_{(i)\alpha\beta}; F_{(i)IJ}\}, \quad (4)$$

where $Qe^\psi$ is the radius of the two-sphere, $Q$ is a constant that is related to the $U(1)$ charges (see Eq. (7) below) and the scalar fields $\eta_a, \psi$ depend only on the 2d spacetime coordinates.

The 4d field equations together with the ansatz (4) constrain strongly the form of the fields strengths $F_{(i)}$, we find

$$F_{(i)IJ} = p_i \epsilon_{IJ}, \quad F_{(i)\alpha\beta} = q_i e^{2\eta b_i - 2\psi} \epsilon_{\alpha\beta}. \quad (5)$$

In the previous equations, and in the following, the dot means scalar product in the moduli space, $b_i$ are the vectors:

$$b_1 = (1, 1, 1), \quad b_2 = (1, 1, -1), \quad b_3 = (1, -1, -1), \quad b_4 = (1, -1, 1), \quad (6)$$

whereas $p_i, q_i$ are respectively the magnetic and electric charge-vectors that characterize the particular dimensional reduction. For sake of simplicity, we will consider only vectors of the form,

$$p_i = Q\hat{p}_i, \quad q_i = Q\hat{q}_i, \quad (7)$$

where $\hat{p}_i$ and $\hat{q}_i$ are vectors with entries 0 or 1.

The form of $\hat{p}_i$ and $\hat{q}_i$ determines the 4d background on which we are performing the dimensional reduction. In general different charge-vectors will give rise to different 2d models. The 4d solutions are connected one with the other by $O(3,Z)$ duality transformations, some of them change the spacetime structure of the solutions, others leave it invariant [3]. The most efficient way to organize the spectrum of the 4d solutions is to use the $O(3,Z)$ duality symmetries together with the compositeness idea [3].

It looks therefore very natural to use the same procedure of Ref. [3] to classify the 2d models deriving from the dimensional reduction of the action (1). The 4d solutions can be classified in multiplets, labeled by $N$, of a given number (up to four) of elementary constituent, on which the duality symmetry $O(3,Z)$ acts in a natural way [3]. Moreover, the multiplets with $N = 1, 2, 3, 4$ can be put in correspondence with the solutions of the single-scalar, single $U(1)$ field strength model [4, 5],

$$A = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left\{ R - 2(\partial i\hat{\phi})^2 - \frac{1}{4} e^{-2a\phi} F^2 \right\}, \quad (8)$$

with coupling $a$ given respectively by $a = \sqrt{3}, 1, 1/\sqrt{3}, 0$.

It turns out that 4d solutions characterized by the same number of magnetic and electric elementary constituents produce after dimensional reduction the same 2d model. For this reason the various dimensional reductions (or equivalently the corresponding 2d models) can be classified by giving, apart form $N$, the numbers $n, m$ of electric, respectively, magnetic elementary constituents, with the obvious constrain $N = n + m$. In this way one obtains 8 different 2d models, which can be put in correspondence with BPS states of the 4d model (1),
\[ N = 1 \text{ multiplet, } a = \sqrt{3}, \]
\[ n = 1, \ m = 0, \ \hat{q}_i = (1, 0, 0, 0), \ \hat{p}_i = 0; \]
\[ n = 0, \ m = 1, \ \hat{q}_i = 0, \ \hat{p}_i = (0, 1, 0, 0). \]  

(9)

\[ N = 2 \text{ multiplet, } a = 1, \]
\[ n = 2, \ m = 0, \ \hat{q}_i = (1, 0, 1, 0), \ \hat{p}_i = 0; \]
\[ n = 0, \ m = 2, \ \hat{q}_i = 0, \ \hat{p}_i = (0, 1, 0, 1); \]
\[ n = 1, \ m = 1, \ \hat{q}_i = (1, 0, 0, 0), \ \hat{p}_i = (0, 0, 0, 1). \]  

(10)

\[ N = 3 \text{ multiplet, } a = \frac{1}{\sqrt{3}}, \]
\[ n = 2, \ m = 1, \ \hat{q}_i = (1, 0, 1, 0), \ \hat{p}_i = (0, 1, 0, 0); \]
\[ n = 1, \ m = 2, \ \hat{q}_i = (1, 0, 0, 0), \ \hat{p}_i = (0, 1, 0, 1). \]  

(11)

\[ N = 4 \text{ multiplet, } a = 0, \]
\[ n = 2, \ m = 2, \ \hat{q}_i = (1, 0, 1, 0), \ \hat{p}_i = (0, 1, 0, 1). \]  

(12)

In the previous equations we have given only one representative element for each model, characterized by \( \hat{p}_i \) and \( \hat{q}_i \). There are, of course, different values of the \( U(1) \) charge-vectors that give rise to the same 2d model. For instance, the model with \( N = 1, n = 1, m = 0 \) can be obtained not only with the values of \( \hat{q}_i \) given in Eq. (9), but also when \( \hat{p}_i = 0 \) and \( \hat{q}_i \) has one entry equal to one, the others being zero. This degeneracy is related with the duality symmetries of the theory and will be discussed in detail in the next section.

In the following we will use \( S(N, m) \) to denote a dimensional reduction associated with a state with \( N \) elementary constituents, of which \( m \) are magnetic \((n = N - m)\). We can now perform explicitly the dimensional reduction of the model (1) defined by the ansatz (4). The duality symmetry group \( O(3, \mathbb{Z}) \) contains off-shell dualities, for this reason it is convenient to perform the dimensional reduction at the level of the equation of motion instead of directly dimensionally reducing the action (1). Using the ansatz (4) and the equations (5) in the 4d field equations stemming from the action (1), we get the following 2d field equations,

\[ R + 4\nabla^2 \Phi - 2(\partial \Phi)^2 - 2(\partial \psi)^2 - 2(\partial \eta) \cdot (\partial \eta) + \]
\[ \frac{1}{2Q^4} e^{2\Phi} \left\{ \sum_i q_i^2 e^{2\eta_{b_i}} - e^{-4\psi} \sum_i p_i^2 e^{-2\eta_{b_i}} \right\} + \frac{2}{Q^2} e^{-2\psi} = 0, \]  

(13)

\[ 4\nabla_\alpha \left( e^{-2\Phi} \nabla^\alpha \eta_a \right) - \frac{1}{Q^4} \left\{ \sum_i b_{ia} q_i^2 e^{2\eta_{b_i}} - e^{-4\psi} \sum_i b_{ia} p_i^2 e^{-2\eta_{b_i}} \right\} = 0, \ a = 2, 3, \]  

(14)

\[ 4\nabla_\alpha \left( e^{-2\Phi} \nabla^\alpha \psi \right) + \frac{2}{Q^4} e^{-4\psi} \sum_i p_i^2 e^{-2\eta_{b_i}} - \frac{4}{Q^4} e^{-2(\psi + \Phi)} = 0, \]  

(15)
\[
\n\nabla_a \nabla_\beta \Phi + \partial_a \Phi \partial_\beta \Phi - (\partial_\alpha \eta) \cdot (\partial_\alpha \eta) - \partial_a \psi \partial_\beta \psi +
\]
\[-g_{\alpha \beta} \left\{ \nabla^2 \Phi - \frac{1}{2} \left[ (\partial \Phi)^2 + (\partial \psi)^2 + (\partial \eta)^2 \right] - \frac{1}{8Q^4} e^{2\Phi} \sum_i q_i^2 e^{2\eta_b} +
\]
\[-g_{\alpha \beta} \left( \frac{1}{2} \frac{8Q^4}{2Q^2} e^{-2\Phi} \right) + \frac{1}{2Q^2} e^{-2\Phi} \right\} = 0,
\]
(16)

where the curvature \( R \) and all the differential operators are 2d quantities, the field \( \Phi \) is the 2d dilaton related, as usual, to the 4d dilaton \( \phi \) by
\[
\phi = \Phi - \psi,
\]
(17)

and the vector \( \eta_a \) is now defined in terms of \( \Phi \), \( \eta_a = (\Phi, \sigma, \rho) \).

The field equations (13)-(16) are rather complicated. They assume a much simpler form by considering a consistent, single-scalar field truncation that reduces them to those of 2d gravity coupled to the dilaton \( \Phi \). These models have been widely investigated in recent years under the name of 2d dilaton gravity. The single-scalar field truncation is obtained using an ansatz expressing the fields \( \psi, \sigma, \rho \) in terms of the dilaton \( \Phi \) in a way that is consistent with the field equations (13)-(16),
\[
\begin{align*}
\ e^{2\psi} &= A^2 \exp \left( \frac{4(m - 2)}{a^2 N} \Phi \right), \\
\ e^{2\sigma} &= A^{-1} \exp \left( \frac{2C_\sigma}{a^2 N} \Phi \right), \\
\ e^{2\rho} &= A^{-1} \exp \left( \frac{2C_\rho}{a^2 N} \Phi \right),
\end{align*}
\]
(18)

where \( A = \sqrt{3/2} \) for \( N = 1 \), \( A = 1 \) for \( N = 2, 3, 4 \). \( C_\sigma = -1 \) for the states \( S(1, 1), S(2, 1), S(3, 2) \); \( C_\sigma = 0 \) for \( S(2, 2), S(2, 0), S(4, 2) \); \( C_\sigma = 1 \) for \( S(1, 0), S(3, 1) \). \( C_\rho = -1 \) for the states \( S(1, 1), S(1, 0), S(3, 2), S(3, 1) \); \( C_\rho = 0 \) for \( S(2, 2), S(2, 0), S(2, 1), S(4, 2) \).

After inserting Eqs. (9)-(12) and (18) into the field equations (13)-(16), we find that, for \( N = 1, 2, 3 \), they are equivalent to those derived from a class of 2d dilaton gravity models whose action has the form,
\[
A_2 = \frac{1}{2} \int d^2 x \ \sqrt{-g} \ e^{-2\Phi} \left[ R + \Omega (\partial \Phi)^2 + \lambda^2 V(\Phi) \right],
\]
(19)

where \( \lambda^2 = 1/Q^2 \) for \( N = 1, 2 \), and \( \lambda^2 = 1/(2Q^2) \) for \( N = 3 \). The kinetic coefficient \( \Omega \) and the dilaton potential \( V \) are completely determined in terms of \( n, m, a, N \),
\[
\begin{align*}
\Omega &= 8 \left( \frac{1 - n}{a^2 N} \right), \\
V(\Phi) &= \exp \left[ 4 \left( \frac{2 - m}{a^2 N} \right) \Phi \right].
\end{align*}
\]
(20)

For \( N = 4 \) the field equations (13)-(16) become equivalent to those derived from the action,
\[
A = \frac{1}{2} \int d^2 x \ \sqrt{-g} \ e^{-2\Phi} \left[ R + 4(\partial \Phi)^2 - \frac{1}{2} F^2 + \lambda^2 \right],
\]
(21)
which describes the heterotic string in 2d target-space [16].

The class of dilaton gravity models defined by the action (19) contains, as particular cases, models that have been already investigated in the past. For \( N = 2, m = 2 \) the action describes the Callan-Giddings-Harvey-Strominger (CGHS) model [17]. For \( N = 2, m = 1 \) we get the Weyl rescaled CGHS model investigate in Ref. [18]. Finally, for \( N = 3, m = 2 \) we obtain the Jackiw-Teitelboim (JT) model [19].

In general the 2d dilaton gravity model (19) give a near-horizon description of the, near-extremal, 4d black hole solution of the action (1). This near-horizon description corresponds to setting to zero the constant term in the harmonic functions appearing in Eq. (3). On the other hand, one can easily verify that this is exactly the way to make the ansatz (18) consistent the 4d extremal solutions (3).

3 Dualities and Weyl transformations

In the canonical frame \( g_C = e^{-2\phi} g_S \), the 4d field equations following from the action (1) are invariant under the action the \( O(3, Z) \) duality group that acts on the moduli \( \phi, \sigma, \rho \) and on the \( U(1) \) field strengths but leaves the 4d metric unchanged [3]. The \( O(3, Z) \) duality group can be generated using three \( S - T - U \) duality transformations \( \tau_S, \tau_T, \tau_U \) together with the permutation group \( P_3 \) acting on the moduli \( \phi, \sigma, \rho \) and on the \( U(1) \) field strengths \( F_i \) (see Ref.[3]). After passing to the string metric \( g_S \) and performing the dimensional reduction described in the previous section, the duality group \( O(3, Z) \) becomes a symmetry of the 2d field equations (13)-(16). Because the 4d action (1) is written in terms of the string metric the duality transformation will act also on the 2d metric \( g_{\alpha\beta} \) and, owing to Eq. (5), also on the charge-vectors \( p_i, q_i \). One can easily verify that the \( O(3, Z) \) duality transformations leave invariant the 2d dilaton field \( \Phi \), whereas in terms of the remaining 2d fields \( \psi, \rho, \sigma, g_{\alpha\beta} \) and of the charge vectors \( p_i, q_i \) it is realized as follows:

\[
\begin{align*}
\tau_S : & \quad \psi \rightarrow -\psi - 2\Phi, \quad g_{\alpha\beta} \rightarrow e^{-4(\Phi+\psi)} g_{\alpha\beta}, \quad q_1 \rightarrow p_3, \quad q_3 \rightarrow p_1, \quad q_2 \rightarrow p_4, \\
\tau_T : & \quad \sigma \rightarrow -\sigma, \quad q_1 \leftrightarrow q_4, \quad q_2 \leftrightarrow q_3; \\
\tau_U : & \quad \rho \rightarrow -\rho, \quad q_1 \leftrightarrow q_2, \quad q_3 \leftrightarrow q_4; \\
P_3 : & \quad \sigma \leftrightarrow \rho, \quad q_2 \leftrightarrow q_4; \\
\psi \rightarrow & \quad \sigma - \Phi, \quad \sigma \rightarrow \psi + \Phi, \quad g_{\alpha\beta} \rightarrow e^{2(\sigma - \Phi - \psi)} g_{\alpha\beta}, \quad q_3 \rightarrow p_4, \quad q_4 \rightarrow p_3; \\
\psi \rightarrow & \quad \sigma - \Phi, \quad \sigma \rightarrow \rho, \quad \rho \rightarrow \Phi + \psi, \quad g_{\alpha\beta} \rightarrow e^{2(\sigma - \Phi - \psi)} g_{\alpha\beta}, \quad q_4 \rightarrow q_2, \\
q_2 \rightarrow & \quad p_3, \quad q_3 \rightarrow p_4; \\
\psi \rightarrow & \quad \rho - \Phi, \quad \rho \rightarrow \Phi + \psi, \quad g_{\alpha\beta} \rightarrow e^{2(\rho - \Phi - \psi)} g_{\alpha\beta}, \quad q_2 \rightarrow p_3, \quad q_3 \rightarrow p_2; \\
\psi \rightarrow & \quad \rho - \Phi, \quad \sigma \rightarrow \Phi + \psi, \quad \rho \rightarrow \sigma, \quad g_{\alpha\beta} \rightarrow e^{2(\rho - \Phi - \psi)} g_{\alpha\beta}, \quad q_2 \rightarrow q_4, \\
q_3 \rightarrow & \quad p_2, \quad q_4 \rightarrow p_3.
\end{align*}
\]  

The previous equations describe the action of the duality group on electric states \((p_i = 0)\), the action on magnetic states \((q_i = 0)\) can be easily obtained from Eqs. (22),(23), by interchanging \( q_i \leftrightarrow p_i \). The duality group generated by the transformations (22),(23)
becomes extremely simple once we perform the single-scalar field truncation described in Sect. 2. One can easily verify that the transformations \(\tau_T, \tau_U\) and the first transformation in Eq. (23) change the ansatz (18) but not the resulting 2d action (19), (21). This fact has a natural explanation. \(\tau_T, \tau_U\) and the first duality in Eq. (23) do not change the number \(n\) of electric (or magnetic \(m\)) elementary constituents of a state, so that they cannot change the 2d action because the latter is parametrized in terms of \(n\) and \(m\) only. On the other hand, using Eqs. (18) into Eqs. (22),(23), one finds that the remaining duality transformations (in particular the \(\tau_S\) duality) act on the 2d dilaton gravity models (19) and (21) as Weyl transformations of the 2d metric,

\[
g_{\alpha\beta} \rightarrow e^{P\Phi} g_{\alpha\beta}, \quad P = \frac{4}{a^2 N} (m - m'),
\]

which map one into the other models in Eqs. (19), (21) with the same value of \(N\) but with a number \(m\) and \(m'\) of magnetic elementary constituents.

Acting at \(N\) fixed, the Weyl transformations (24) connect one with the other models within a given multiplet. For instance, taking \(N = 2\) we have three models: \(S(2,2)\) (\(\Omega = 4, V(\phi) = 1\), the CGHS model), \(S(2,1)\) (\(\Omega = 0, V(\phi) = \exp(2\Phi)\), the Weyl-rescaled CGHS model of Ref. [18] and \(S(2,0)\) (\(\Omega = -4, V(\phi) = \exp(4\Phi)\)). These models are obtained one from the other using the Weyl transformations (24). \(g_{\alpha\beta} \rightarrow e^{2\Phi} g_{\alpha\beta}\), maps \(S(2,2) \rightarrow S(2,1)\) and \(S(2,1) \rightarrow S(2,0)\) whereas \(g_{\alpha\beta} \rightarrow e^{4\Phi} g_{\alpha\beta}\) maps \(S(2,2) \rightarrow S(2,0)\).

The \(\tau_S\) duality is essentially an electro/magnetic duality, so that the strong/weak coupling duality of the 4d theory (1) is translated, after dimensional reduction to two dimensions, at least for the 4d model under consideration, into a dilaton-dependent Weyl rescaling of the 2d metric.

It has been shown that dilaton-dependent Weyl transformations leave invariant the physical parameters (the mass and for 2d black hole solutions, the Hawking temperature and radiation flux) associated with the solutions of 2d dilaton gravity [20]. Also other physical parameters that can be expressed in terms of the mass and the temperature (e.g the entropy) are invariant under such transformations. The dimensional reduction seems to wash out most of the information about the magnetic or electric character of the 4d solution. At the 2d level the relevant information is encoded in the number \(N\), the number of elementary constituents, the actual number of magnetic and electric elementary constituents being irrelevant for the physical parameters of the solution. At the level of the 4d theory this implies the equivalence of electrically and magnetically charged configurations, as long as only excitations near extremality are concerned. Moreover, in the 2d context, the duality implies the equivalence of spacetime structure that behave rather differently. 2d spacetimes of constant curvature (e.g the solutions of the 2d model \(S(3,2)\)) are dual (i.e connected by Weyl transformations) to spacetimes with singularities (e.g the solutions of the 2d model \(S(3,1)\)).

Presently, we do not know if this is a peculiarity of the 4d heterotic string theory (1) or a general feature of 4d effective string theory. However, our results indicate that the dimensionally reduced 2d theory takes care only of the relevant physical properties of the 4d model and can therefore be used to give a universal classification of the near-extremal behavior of the 4d black hole solutions of string theory.
4 Statistical entropy of the $a=1/\sqrt{3}$ 4d black hole

The results of the previous sections together with those of Ref. [12] can be used to calculate, microscopically, the entropy of the near-extremal 4d, $a=1/\sqrt{3}$, black hole. The near-extremal, near-horizon behavior of this black hole is described by the 2d model of Eq. (19) with $N = 3, m = 2$, i.e by the JT dilaton gravity model. For the JT black hole a derivation of the statistical entropy has been given in Ref. [12], one can, therefore, use it to compute, microscopically, the entropy of the 4d, $a=1/\sqrt{3}$, black hole.

For $a=1/\sqrt{3}$ the single-scalar field model of Eq.(8) takes the form,

$$A = \frac{1}{16\pi} \int d^4 x \sqrt{-g} \left\{ R - 2(\partial \phi)^2 - \frac{1}{4} e^{-\sqrt{3} \phi} F^2 \right\}, \quad (25)$$

where the scalar field $\hat{\phi}$ is connected to the 4d dilaton $\phi$ through $\hat{\phi} = \sqrt{3} \phi$. As mentioned above, this model arises as single-scalar field, single $U(1)$ field strength truncation of the $N = 3$ composite solutions of the action (1) and as compactification of the five-dimensional (5d) Einstein-Maxwell theory. The general (non extremal) black hole solution of the model has the form [21],

$$ds^2 = -H^{-\frac{1}{2}} \left( 1 - \frac{\mu}{r} \right) dt^2 + H^{\frac{3}{2}} \left( 1 - \frac{\mu}{r} \right)^{-1} dr^2 + H^{\frac{3}{2}} r^2 d\Omega_2^2,$$

$$e^{2\phi} = H^{\frac{1}{2}}, \quad H = 1 + \frac{\mu \sinh^2 \alpha}{r}, \quad (26)$$

where for simplicity we have set the constant mode of the 4d dilaton $\phi_0 = 0$. The integration constants $\mu$ and $\alpha$ are related to the mass and charge of the solution by

$$M = \frac{1}{2} \mu \left( 1 + \frac{3}{2} \sinh^2 \alpha \right), \quad Q = \frac{1}{2} \sinh 2\alpha. \quad (27)$$

To be more precise, $Q$ is the (common) charge of the three $U(1)$ fields in the action (1). $Q$ is related to the charge $\tilde{Q}$ of the single $U(1)$ field appearing in the action (25) by $Q = (2/\sqrt{3})\tilde{Q}$. Using the area law we find that the Bekenstein-Hawking entropy of the hole is given by

$$S = \frac{\mathcal{A}}{4} = \sqrt{\mu} \left( \mu + \mu \sinh^2 \alpha \right)^{3/2}, \quad (28)$$

where $\mathcal{A}$ is the area of the event horizon.

The extremal black hole is obtained in the limit $\mu \to 0, \alpha \to \infty$, keeping $\mu \sinh^2 \alpha = Q$. In this limit the solution is given by Eq. (26) with $\mu = 0$ and $H = (1 + Q/r)$ whereas the mass is $M_{ex} = \frac{2}{3} Q$. Let us now consider small excitations near the extremal solution, the entropy of these configurations is given by

$$S_{(4)} = \pi Q^{3/2} \sqrt{\mu} + o \left( \mu^{3/2} \right). \quad (29)$$

We know that, when expressed in terms of the string metric, the near-horizon, near-extremal, magnetically charged, black hole solution (26) factorizes as $\mathcal{M}^2 \times S^2$, where $\mathcal{M}^2$ is the solution of the 2d dilaton gravity model (19) with $N = 3, m = 2, a = 1/\sqrt{3}$,

$$A = \frac{1}{2} \int d^2 x \sqrt{-g} \ e^{-2\phi} \left( R + 2\lambda^2 \right). \quad (30)$$
We have rescaled $\lambda^2 \rightarrow 2\lambda^2$ in order to match the conventions of Ref. [12].

The dilaton gravity model (30) admits solutions that can be interpreted as black holes in 2d AdS space [22],

$$ds^2 = -(\lambda^2 x^2 - b^2) dt^2 + (\lambda^2 x^2 - b^2)^{-1} dx^2, \quad e^{-2\Phi} = e^{-2\Phi_0} \lambda x. \quad (31)$$

The mass $M_{(2)}$ and the entropy $S_{(2)}$ of the 2d black hole are given in terms of the integration constants $b, \Phi_0$, by

$$M_{(2)} = \frac{1}{2} e^{-2\Phi_0} b^2 \lambda, \quad S_{(2)} = 4\pi \sqrt{\frac{e^{-2\Phi_0} M_{(2)}}{2\lambda}}. \quad (32)$$

If the 2d model (30) has to describe the near-extremal, near-horizon regime of the 4d black hole solutions (26) then the 2d expression for the entropy (32) should match the leading order of the corresponding 4d quantity in Eq. (29). To show this, we first write the 4d solution (26) in terms of the string metric, we expand the solution near extremality and near the horizon. After some manipulations we get

$$ds^2 = -(\lambda^2 x^2 - \lambda \mu) dt^2 + (\lambda^2 x^2 - \lambda \mu)^{-1} dx^2 + e^{2\phi_0} Q^2 d\Omega_2^2, \quad e^{-2\phi} = \sqrt{2\lambda} x. \quad (33)$$

where $\lambda = 1/(2Q)$.

As expected the 4d solution factorizes as the product of a 2d spacetime and a two-sphere of constant radius. Taking into account that the dimensional reduction $4d \rightarrow 2d$ implies the following relation between the 2d dilaton $\Phi$ and the 4d one $\phi$:

$$e^{-2\Phi} = \frac{1}{2} Q^2 e^{-2\phi}, \quad (34)$$

and comparing Eq. (33) with Eq. (31), we get

$$\mu = \frac{2M_{(2)}}{\lambda^2} e^{2\phi_0}, \quad e^{-2\Phi_0} = \frac{\sqrt{2}}{8\lambda^2}. \quad (35)$$

Using Eqs. (35) into the expression (29) for the 4d entropy and taking into account only the leading term, we obtain a complete agreement with the 2d results, i.e $S_{(4)} = S_{(2)}$.

Until now we have considered only 4d, magnetically charged, solutions, i.e the state $S(3, 2)$. The 4d electrically charged solution $S(3, 1)$ does not factorize as direct product of two 2d spaces. Near extremality it is described by the 2d model with $N = 3, m = 1$, which is dual to the model (30). Because the 2d entropy does not change under Weyl rescaling of the metric, it follows that the Eqs. (32) and the equality $S_{(4)} = S_{(2)}$ hold also for excitations near 4d extremal, electrically charged, solutions.

We have shown that the semiclassical dynamics of small excitations near extremality of the 4d black hole can be described by the 2d model (30) and that at the leading order, the 2d and 4d thermodynamical entropy is the same. One can therefore use the results of Ref. [12] as an indirect calculation of the statistical entropy of the 4d black hole (26) in the near-extremal regime.
In Ref. [12] we have found a mismatch of a $\sqrt{2}$ factor between the thermodynamical and the statistical entropy of the 2d black hole. This implies that also the statistical entropy of the 4d $a = 1/\sqrt{3}$ black hole agrees only up to a $\sqrt{2}$ factor with the thermodynamical result. A simple explanation of this $\sqrt{2}$ factor could be found when the 2d AdS black hole arises as compactification of 3d one [12]. The same arguments of Ref. [12] apply also in the case under consideration because the 4d black hole solution (26) arises as compactification of the solutions of 5d Einstein-Maxwell gravity that behave, near the horizon, as AdS$_3 \times$ S$^2$.

Let us consider the 5d Einstein-Maxwell action,

$$A = \int d^5x \sqrt{-g} \left\{ R - \frac{1}{4} F^2 \right\}. \quad (36)$$

Compactifying the fifth dimension $x_4$, using the ansatz

$$ds^2_{(5)} = e^{-\frac{4}{\sqrt{3}} \phi} dx_4^2 + e^{\frac{4}{\sqrt{3}} \phi} ds^2_{(4)},$$

$$F_{\mu,\nu} = \{ F_{\mu\nu}, F_{4\nu} \}, F_{4\nu} = 0, \quad \mu, \nu = 0\ldots 4, \quad (37)$$

we get the 4d action (25). The extremal 5d solutions (37) behave near the horizon as AdS$_3 \times$ S$^2$,

$$ds^2_{(3)} = \frac{r}{Q} (-dt^2 + dx_4^2) + \left( \frac{Q}{r} \right)^2 dr^2 + Q^2 d\Omega_2^2. \quad (38)$$

Hence, the explanation of the $\sqrt{2}$ factor of Ref. [12] can be immediately translated to the case under consideration.

Moreover, the expression (38) suggests that the discrepancy between thermodynamical and statistical entropy of the 4d, $a = 1/\sqrt{3}$, black hole could also have a geometrical explanation. The 3d part of the metric (38) describes a spacetime that is AdS$_3$ with a conical singularity. In fact, if in Eq. (38) $x_4 = Q \varphi$, with $0 \leq \varphi \leq 2\pi$, changing coordinates $r \rightarrow r^2/4, \varphi \rightarrow 2\varphi$, the 3d part of the metric (38) becomes

$$ds^2_{(3)} = - \frac{r^2}{4Q^2} dt^2 + \frac{4Q^2}{r^2} dr^2 + r^2 d\varphi^2, \quad (39)$$

but with $0 \leq \varphi \leq \pi$.

## 5 Conclusions

The dimensional reduction of 4d heterotic string theory presented in this paper has shown once again that 2d dilaton gravity models can be used as a simplified description that retain the relevant information about the 4d physics. The class of 2d dilaton gravity models we have derived gives a general description of excitations near the extremal 4d heterotic black hole. The geometrical structures, thermodynamical features and the duality symmetries of the 4d theory become much simpler when translated in the 2d context.

Particularly interesting are those 2d models that admit AdS$_2$ as solution. In this case one can use the AdS/CFT duality to compute the statistical entropy of the near-extreme 4d black hole. We have performed this calculation for the $a = 1/\sqrt{3}$ black hole but in principle the same should be possible for the $a = 0$ black hole. The near-horizon geometry factorizes
also for $a = 0$ as $\text{AdS}_2 \times S^2$ and the excitations near extremality are now described by the model (21). Differently from the $a = 1/\sqrt{3}$ case, where we have a linear varying $\exp(-2\Phi)$, for $a = 0$ the dilaton is constant. A constant dilaton makes a black hole interpretation of the solutions very difficult, at least from the 2d point of view. One cannot use the arguments of Ref. [12] to compute the statistical entropy of the black hole.

This difficulties of the $a = 0$ case (actually the most interesting case from the string point of view) are connected with a peculiarity, mentioned in the introduction, of the $a = 0$ case, namely the existence of a mass gap separating the extremal configuration from the continuous part of the spectrum. This implies that the finite-energy excitations near extremality are suppressed [23]. Probably, this behavior is related with other puzzling features of the $\text{AdS}_2/\text{CFT}_1$ correspondence [23, 24, 12].

References


