Conformally Coupled Induced Gravity with Gradient Torsion

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Abstract

It is found that conformally coupled induced gravity with gradient torsion gives a dilaton gravity in Riemann geometry. In the Einstein frame of the dilaton gravity the conformal symmetry is hidden and a non-vanishing cosmological constant is not plausible due to the constraint of the conformal coupling.

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I. INTRODUCTION

Before the success of Weinberg-Salam model, the weak interaction was characterized by the dimensional Fermi’s coupling constant, \( G_F = (300 Gev)^{-2} \), far below the electro-weak scale. But later it turns out that the dimensional coupling constant is the low energy effective coupling which is determined by the dimensionless electro-weak coupling constant and the vacuum expectation value of Higgs scalar field through the spontaneous symmetry breaking. The weakness of the weak interaction is originated from the large vacuum expectation value of Higgs field [1].

From this lesson, it is suspected that gravity may be also characterized by a dimensionless coupling constant \( \xi \) with the gravitational constant \( G_N \) given by the inverse square of the vacuum expectation value of a scalar field. The weakness of the gravity can be associated with a symmetry breaking at a very high energy scale. It has been independently proposed by Zee [2], Smolin [3], and Adler [4] that the Einstein-Hilbert action can be replaced by the induced gravity action

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \xi \phi^2 R + \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) \right\},
\]

(1)

where the coupling constant \( \xi \) is dimensionless. The potential \( V(\phi) \) is assumed to have its minimum value at \( \phi = \sigma \), then the above action is reduced to the well known Einstein-Hilbert action with gravitational constant \( G_N = \frac{1}{8\pi \xi \sigma^2} \).

In the analogy of the SU(2) \( \times U(1) \) symmetry of the electro-weak interactions, we can consider a symmetry which may be broken through a spontaneous symmetry breaking in the gravitational interactions. The most attractive candidate symmetry is the Weyl’s conformal symmetry which rejects the Einstein-Hilbert action, but admits the induced gravity action Eq.(1) with a specific conformal coupling, \( \xi = \frac{1}{6} \).

In Riemann space, the conformal coupling is unique with \( \xi = \frac{1}{6} \). However, introducing the vector torsion, an extended conformal coupling is possible in induced gravity [5,6] because the vector torsion plays the role of a conformal gauge field in Riemann-Cartan space [3,7,8].
It is found that induced gravity at conformal coupling should have conformal invariance for consistency [6,9]. We investigate the conformal coupling in induced gravity with a gradient torsion.

II. CONFORMAL COUPLINGS IN INDUCED GRAVITY

The induced gravity action Eq.(1) is invariant under the conformal transformation,

\[ g'_{\mu\nu}(x) = \exp(2\rho)g_{\mu\nu}(x), \quad \phi'(x) = \exp(-\rho)\phi(x), \]

(2)

at the conformal coupling \( \xi = \frac{1}{6} \) for a conformally invariant scalar potential.

In Riemann-Cartan space, an extension of conformal coupling with the torsion in induced gravity is possible. It is found that the minimal extension to Riemann-Cartan space is sufficient for our purpose.

The conformal transformation of the affine connections \( \Gamma^\gamma_{\beta\alpha} \) is determined from the invariance of the tetrad postulation,

\[ D_\alpha e^i_\beta \equiv \partial_\alpha e^i_\beta + \omega^j_\beta e^i_\gamma - \Gamma^\gamma_{\beta\alpha} e^i_\gamma = 0, \]

(3)

under the following tetrads \( e^i_\alpha \) transformations;

\[ (e^i_\alpha)' = \exp(\rho)e^i_\alpha. \]

(4)

The spin connections \( \omega^j_\beta \) are conformally invariant like other gauge fields.

The affine connections and the torsions which are the antisymmetric components of the affine connections transform as follows;

\[ (\Gamma^\gamma_{\beta\alpha})' = \Gamma^\gamma_{\beta\alpha} + \delta^\gamma_{\beta}\partial_\alpha \rho, \quad (T^\gamma_{\beta\alpha})' = T^\gamma_{\beta\alpha} + \delta^\gamma_{\beta}\partial_\alpha \rho - \delta^\gamma_{\alpha}\partial_\beta \rho. \]

(5)

Therefore, the trace of the torsion \( T^\gamma_{\beta\alpha} \) effectively plays the role of a conformal gauge field.

In general, the torsion can be decomposed into three components;

\[ T^\alpha_{\beta\gamma} = \Sigma^\alpha_{\beta\gamma} + A^\alpha_{\beta\gamma} - \delta^\alpha_{\gamma}S_\beta + \delta^\alpha_{\beta}S_\gamma, \]

(6)
where \( \Sigma_{\alpha\beta\gamma} = 0 \), \( \Sigma^\alpha_{\alpha\gamma} = 0 \), \( A_{\alpha\beta\gamma} = T_{\alpha\beta\gamma} \). The traceless part of torsion \( C^\alpha_{\beta\gamma} \equiv \Sigma^\alpha_{\beta\gamma} + A^\alpha_{\beta\gamma} \) is conformally invariant.

\[
(S_\alpha)' = S_\alpha + \partial_\alpha \rho, \quad (C^\alpha_{\beta\gamma})' = C^\alpha_{\beta\gamma};
\]

(7)

Because the minimal extension to Riemann-Cartan space is sufficient for our purpose, we impose the conformally invariant torsionless condition;

\[
C^\alpha_{\beta\gamma} \equiv 0.
\]

(8)

This condition is the conformally invariant extension of the torsionless condition in Riemann space, \( T^\alpha_{\beta\gamma} = 0 \). For this minimally extended Riemann-Cartan space, the affine connection can be written in terms of \( g_{\mu\nu} \) and \( S_\alpha \);

\[
\Gamma^\alpha_{\beta\gamma} = \{^\alpha_{\beta\gamma}\} + S^\alpha g_{\beta\gamma} - S_\beta \delta^\alpha_{\gamma}.
\]

(9)

Introducing the conformally covariant derivative \( D_\alpha \) for scalar field \( \phi \),

\[
D_\alpha \phi \equiv \partial_\alpha \phi + S_\alpha \phi,
\]

(10)

we have an extended conformal coupling of induced gravity up to total derivatives as follow;

\[
I = \int d^4x \sqrt{-g} \left\{ \frac{\xi}{2} R(\Gamma) \phi^2 + \frac{1}{2} D_\alpha \phi D^\alpha \phi - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} - V(\phi) \right\},
\]

(11)

where we have excluded the curvature square terms. Now, the coupling \( \xi \) is a dimensionless arbitrary constant. Using Eq.(9) we can rewrite this action in terms of Riemann curvature scalar \( R(\{}\));

\[
I = \int d^4x \sqrt{-g} \left\{ \frac{\xi}{2} R(\{} \phi^2 + \frac{1}{2} \partial_\alpha \phi \partial^\alpha \phi + (1 - 6\xi) S^\alpha (\partial_\alpha \phi) \phi \\
+ \frac{1}{2} (1 - 6\xi) S_\alpha S^\alpha \phi^2 - \frac{1}{4} H_{\alpha\beta} H^{\alpha\beta} - V(\phi) \right\}.
\]

(12)

The more general form of induced gravity action can be considered [10,11], but we restrict the couplings to be conformal. In the limit of \( \xi \rightarrow \frac{1}{6} \), this extended conformal coupling is reduced to the ordinary conformal coupling in Riemann space decoupled from the vector torsion.
III. DILATON GRAVITY FROM INDUCED GRAVITY WITH GRADIENT TORSION

Analyzing the equations of motion for the action Eq.(12) with an effective potential $V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})$ which depends on metric and torsion in general, we obtain the following two equations of motion and a constraint for the scalar potential;

$$\nabla_\mu H^{\mu\nu} = -(1 - 6\xi)\{[\partial^\nu \phi]\phi + S^\nu \phi^2\} + \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial S_\nu},$$

(13)

$$\xi \phi^2 G^{\mu\nu} = (H^{\mu\alpha}H_\nu^\alpha - \frac{1}{4}g^{\mu\nu}H^{\alpha\beta}H_\alpha^\beta) - (\partial_\mu \phi \partial_\nu \phi - \frac{1}{2}g^{\mu\nu}\partial_\alpha \phi \partial_\alpha \phi) - (1 - 6\xi)\phi^2(S_\mu S_\nu - \frac{1}{2}g_{\mu\nu}S^\alpha S_\alpha)

- (1 - 6\xi)(S_\mu \phi \partial_\nu \phi + S_\nu \phi \partial_\mu \phi - g_{\mu\nu}S^\alpha \phi \partial_\alpha \phi) + \xi\{\nabla_\mu(\phi \partial_\nu \phi) + \nabla_\nu(\phi \partial_\mu \phi) - g_{\mu\nu}\Box \phi^2\}

-g_{\mu\nu}V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma}) + 2\frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial g^{\mu\nu}},$$

(14)

$$4V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma}) - \phi \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial \phi} = 2\frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial g^{\mu\nu}}g^{\mu\nu} + \nabla_\nu \frac{\partial V_{\text{eff}}(\phi; S_\alpha, g_{\beta\gamma})}{\partial S_\nu},$$

(15)

where all covariant derivatives are in Riemann space with the Christoffel connections [6,9].

The constraint Eq.(15) requires that the metric independent bare potential should be quartic in the scalar field, $V_o(\phi) = \frac{\lambda}{4!}\phi^4$, and the deviation of the radiatively corrected effective potential from the quartic form is only allowed with the compensation by the metric and vector torsion dependencies of the effective potential. Because this constraint comes from the assumption that the bare action is conformally invariant except the potential term, if we consider non-conformal coupling in kinetic and interacting terms, such a constraint would not appear.

Let us consider a reduction of the system. If the effective potential does not have the vector torsion dependency, i.e. $V_{\text{eff}}(\phi; g_{\beta\gamma})$, Eq.(13) allows the following conformally invariant reduction;

$$D_\alpha \phi = 0.$$

(16)
This implies that the vector torsion is a gradient form;

\[ S_\alpha = - \partial_\alpha \ln (\phi/\phi_o) = - \partial_\alpha \sigma, \quad \phi \equiv \phi_o e^\sigma, \quad (17) \]

where \( \phi_o \) is a dimensional constant, and the field strength of the vector torsion vanishes 
\( H_{\alpha\beta} = 0 \), which is consistent with Eq.(13). In this reduction, the bare action of Eq.(11) becomes

\[ I = \int d^4x \sqrt{-g} \phi_o^2 e^{2\sigma} \{ \frac{\xi}{2} R(\{\}) + 3\xi \partial_\alpha \sigma \partial^\alpha \sigma - \frac{\lambda}{4!} \phi_o^2 e^{2\sigma} \}. \quad (18) \]

This is the form of conformal factor theory of dilaton gravity [12,13]. If the antisymmetric torsion term is \( \frac{1}{12} C_{\alpha\beta\gamma} C^{\alpha\beta\gamma} \) included, this action is the form of string gravity with some redefinition of fields except the quartic potential term [14].

For the reduction of the effective action, the potential term is replaced by \( e^{-2\sigma} V_{eff}(\phi_o e^\sigma, g_{\alpha\beta}) \). In this reduction, a dimensional constant \( \phi_o \) is introduced, but the action is invariant under the global scaling \( dx^\mu \rightarrow adx^\mu, \phi_o \rightarrow \phi_o/a \) if no conformal anomaly is introduced in the effective action. However, the appearance of the conformal anomaly in the conformally induced gravity is not allowed due to the constraint Eq.(15) which is the requirement of conformal invariance [9] from the consistency of equations of motion for the bare and the effective action in the conformally induced gravity [6]. In this reduction, the constraint which effective potential should satisfy is

\[ 4V_{eff}(\phi; g_{\beta\gamma}) - \phi \frac{\partial V_{eff}(\phi; g_{\beta\gamma})}{\partial \phi} = 2 \frac{\partial V_{eff}(\phi; g_{\beta\gamma})}{\partial g^{\mu\nu}} g^{\mu\nu}. \quad (19) \]

Therefore, only metric independent quartic potential with effective coupling \( \lambda_{eff} \) seems to be allowed. But, in general, the quantum effect by quantum fluctuation of scalar field on the classical metric and torsion background [12,13] gives the following corrections to the scalar mass \( m^2 \), the coupling \( \lambda \) and the cosmological constant \( \Lambda \) respectively [15,16].

\[ \delta m^2 \propto \lambda m^2, \quad \delta \lambda \propto \lambda^2, \quad \delta \Lambda \propto a m^2 + b \lambda. \quad (20) \]

Therefore, \( \lambda = 0 \) would be the only solution of Eq.(19) as a trivial fixed point of the renormalization group. However, the definite claim about \( \lambda = 0 \) is possible only after the
consideration of full quantum effects of the theory, which is of course far beyond of our scope yet.

Redefining the metric,

\[ g_{\alpha\beta}e^{2\sigma} \rightarrow g_{\alpha\beta}, \tag{21} \]

the above dilaton gravity action can be written in the following standard Einstein action;

\[ I = \int d^4x \sqrt{-g}\left\{ \frac{\xi}{2}\phi_o^2 R(\{}\{-) - \frac{\lambda}{4!}\phi_o^4 \right\}, \tag{22} \]

where the gravitational constant \( G_N = \frac{1}{8\pi\xi\phi_o^2} \), and the cosmological constant \( \Lambda = \frac{\lambda}{4!}\left(\frac{1}{8\pi\xi G_N}\right)^2 \).

If the effective potential deviates from the quartic form, then there would remain explicit \( \sigma \) dependency in the action after this redefinition of metric even though it is not plausible due to the constraint Eq.(19).

In this Einstein frame the conformal symmetry is hidden and the cosmological constant term has the origin of the quartic potential term in the original induced gravity action. Reminding the constraint Eq.(19) which the effective potential should satisfy, the non-zero coupling \( \lambda \) can be hardly expected after the radiative correction by the scalar field in the original induced gravity action [15,16]. Therefore, we can say that if the Einstein gravity have the root in the conformally induced gravity, the non-zero cosmological constant is not plausible due to the constraint from the conformal coupling.

In this discussion, we have not considered torsions generated by matter fields because vector torsion couplings are not expected in the standard minimal action for Dirac and gauge fields [11].

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