Gupta-Bleuler quantization for minimally coupled scalar fields in de Sitter space

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Abstract

We present in this paper a fully covariant quantization of the minimally-coupled massless field on de Sitter space, thanks to a new representation of the canonical commutation relations. We thus obtain a formalism free of any infrared divergence. Our method is based on a rigorous group theoretical approach combined with a suitable adaptation (Krein spaces) of the Wightman-Gärding axiomatic for massless fields (Gupta-Bleuler scheme). We make explicit the correspondence between unitary irreducible representations of the de Sitter group and the field theory on de Sitter space-time. The minimally-coupled massless field is associated with a representation which is the lowest term of the discrete series of unitary representations of the de Sitter group. In spite of the presence of negative norm modes in the theory, no negative energy can be measured: expressions as \( \langle n_{k_1} n_{k_2} \ldots | T_{00} | n_{k_1} n_{k_2} \ldots \rangle \) are always positive.

1 Introduction

We present in this paper a fully covariant construction of the minimally-coupled quantum field on the de Sitter space-time. We specially emphasize the covariance aspect which should be understood in terms of the action of the de Sitter group SO\(_{0}(1, 4)\). The starting point of the field construction is, within the context of canonical quantization, the adoption of a new representation of the canonical commutation relations (ccr). This field, of the Gupta-Bleuler type, is free of infra-red as well as ultra-violet divergences (our approach yields a covariant renormalization of the stress tensor). The construction is of course \textit{not} coordinate dependent.

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However, we shall make use of a specific choice of global coordinates through this paper, namely those ones which are called conformal, in order to make the construction explicit.

The last decades have seen considerable interest in the quantization of fields in curved space-times of different types [BD, I2]. It is here not needed to detail the various physical motivations among which inflationary scenarii and quantum gravity take up a central position; see for instance a recent discussion by Lesgourgues, Polarski, and Starobinsky [LPS]. In this respect, the two de Sitter spaces, namely de Sitter and anti-de Sitter, have been intensively studied, owing to their constant curvature $H$ directly related to a non-zero cosmological constant $\Lambda = 3H^2$, and to the fact that both are of maximal symmetry. Their respective isometry (or relativity) groups are indeed the only two deformations [LN, BLL] of the Poincaré group. It was thus appealing to attempt a covariant construction of quantum field theories on such space-times. We refer in particular to a recent rigorous formulation of such a theory for the scalar “massive” fields on de Sitter space and its subsequent thermic interpretation, ([BGM] and references therein). Here we use quotes for the term massive since, in both de Sitter relativities, the concept of mass does not exist by itself as a conserved quantity. The mass is a galilean or minkowskian physical quantity. It can be measured through well-established procedures by galilean or minkowskian observers. The expression “massive” field in de Sitter space refers to an object which has a non ambiguous massive limit as space-time becomes flat.

Group representation theory and its Wigner interpretation in terms of elementary system allow one to control this limiting process through contraction of group representation [MN, Do]. As a matter of fact, “massive” representations of the de Sitter group, i.e. representations which contract to massive representations of the Poincaré group, are those of the principal series of representations of the de Sitter group $SO_o(1,4)$, whereas massive representations of the anti-de Sitter group are those of the (holomorphic) discrete series of the anti-de Sitter group $SO_o(2,3)$. We call massive fields the fields which transform under these representations. We here employ the standard terminology of representation theory for semi-simple Lie group [K].

The situation is different for “massless” field. A reasonable physical requirement one has to impose on the concept of massless quantity is light-cone propagation [F2G]. It is thus natural to call massless these representations of $SO_o(1,4)$ and $SO_o(2,3)$ which naturally extend to the conformal group $SO_o(2,4)$ [AF2S]. Hence, associated de Sitter fields really deserve the name of (conformally coupled) massless fields.

Applying this criterium, one finds that the AdS scalar massless field transforms under a representation which lies at the lower limit of the (holomorphic) discrete series for the anti-de Sitter group $SO_o(2,3)$. On the other hand, the de Sitter scalar massless field transforms under a specific representation of the complementary series of $SO_o(1,4)$. Once massive and massless de Sitter fields are well identified in terms of group representation theory, corresponding covariant quantum field theories can be envisaged along the lines proposed by Wightman and Gärding in their seminal paper [WG].

Within this context, the specific case of massless spin-two fields in de Sitter space, i.e. de Sitterian gravitational fields in their linear approximation, involves a specific scalar field, called minimally coupled massless field. Although the word massless is traditional in this case, this field is not really massless in the above sense: the field equation is not conformally invariant and the involved representation does not extend naturally to a representation of the
conformal group. Moreover this representation has no Poincaré limit in the sense we have proposed in the above. For various reasons, among which quantum gravity or more generally quantum cosmology occupy a central position, a large amount of literature has been devoted to the quantization problem for this field [AF, KG]. Here we wish to contribute to this long quest toward a satisfactory mathematical setting. One of us recently gave [DBR] a rigorous covariant treatment of the quantization problem for the minimally-coupled massless field in $1 + 1$ dimensional de Sitter space-time. It is natural (and almost straightforward) to extend the method to the $3 + 1$ dimensional case.

Therefore we address in this paper the question of constructing, for the minimally coupled massless equation in $3+1$ dimensional de Sitter space, a quantum field theory which be covariant according to criteria adapted from the original Wightman-Gårding paper. In this sense, a quantum field is, roughly speaking, a distribution $\varphi$ on space-time, solution of the field equation, with values in a set of symmetric operators in some inner product space and verifying some physically reasonable properties. (In the following, we underline the second-quantized spaces and the corresponding group action, in order to make a clear distinction between the states and the elements of the one-particle sector.)

- **[Covariance]** There exists a unitary representation $U$ of de Sitter group on the space of states, and the field is covariant

$$U_g \varphi(x) U_g^{-1} = \varphi(g \cdot x)$$

for any $g$ in de Sitter group and $x$ in space-time.

- **[Existence of the vacuum]** There exists a normalized state $|0\rangle$ called the vacuum, which is invariant under the representation $U$

$$U_g |0\rangle = |0\rangle \quad \text{for all } g \in G,$$

and which is unique, up to physical equivalence.

- **[Causality]** A local commutativity property holds:

$$[\varphi(x_1), \varphi(x_2)] = 0$$

as far as the points $x_1$ and $x_2$ are not causally connected.

This is usually achieved through a Fock construction based upon a suitable one-particle sector Hilbert space. However Allen [A] has proved that such a field does not exist on de Sitter space-time. The crucial point in Allen’s proof is the following. Any set of normalisable modes, complete in the sense that they, together with the conjugate modes, span the constant functions, cannot be de Sitter invariant. The existence of a constant solution is at the origin of the problem. Allen has noted that this constant function can be considered as the real part of a so-called zero mode. This zero mode has positive norm, but it cannot be part of the Hilbertian structure of the one particle sector. Indeed, the action of the de Sitter group on this mode generates all the negative frequency solutions (with respect to the conformal time) of the field equation.
Other approaches based on two-point functions (see [L] for instance) also failed in this case. Indeed, for the same reason, there is no covariant two-point function $G_1$ of the positive type, i.e. obeying
\[ \int G_1(x, x') u^*(x) u(x') dx \geq 0 \]
for any solution $u$. The usual procedure for overcoming this difficulty is to adopt a restrictive version of covariance by considering fields which are covariant with respect to a subgroup of the de Sitter group: this is the so-called “symmetry breaking”.

Our approach is different. We instead require full covariance as well as causality, but the Allen’s statement shows that we have to give up something. Before discussing this point, let us remark that there is a deep analogy between the zero mode problem and the quantization of the electromagnetic field. The Lagrangian of the minimally coupled field
\[ \mathcal{L} = \sqrt{|g|} \partial_\mu \phi \partial^\mu \phi \]
is invariant under the global transformation $\phi \mapsto \phi + \lambda$ which is like a gauge transformation. As it is well known, the correct procedure for quantizing electromagnetism does not consist in weakening the covariance. One has to adopt the Gupta-Bleuler quantization, and this is precisely what we do here for the minimally coupled field.

At this point, let us precise what we mean by Gupta-Bleuler formalism. In electrodynamics the Gupta-Bleuler triplet $V_g \subset V \subset V'$ is defined as follows [BFH][G]. The space $V_g$ is the space of longitudinal photon states or “gauge states”, the space $V$ is the space of positive frequency solutions of the field equation verifying the Lorentz condition, and $V'$ is the space of all positive frequency solutions of the field equation, containing non-physical states. The Klein-Gordon inner product defines an indefinite inner product on $V'$ which is Poincaré invariant. All three spaces carry representations of the Poincaré group but $V_g$ and $V$ are not invariantly complemented. The quotient space $V/V_g$ of states up to a gauge transformation is the space of physical one-photon states. The quantized field acts on the Fock space built on $V'$, which is not a Hilbert space, but is instead an indefinite inner product space.

We proceed in a similar manner for the minimally coupled field. The set $\mathcal{N}$ of constant functions will play the role of $V_g$. We also obtain a physical space $\mathcal{K}$ carrying a unitary representation of the de Sitter group. However, this space is not a Hilbert space: the Klein-Gordon inner product is degenerate (although positive), and there is of course no contradiction with the Allen’s result. Moreover the representation of the de Sitter group is not irreducible (although indecomposable). As discussed at length in [DBR], the field must be written on a nondegenerate inner product space. As a consequence we must introduce as a total space $\mathcal{H}$ a much larger space. The latter contains auxiliary states which can be of negative norm for the usual Klein-Gordon inner product. Nevertheless this does not mean that negative energies could be attainable in terms of observable measurements. Indeed, expressions like $\langle n_{k_1} n_{k_2} \ldots | T_{00} | n_{k_1} n_{k_2} \ldots \rangle$ are positive for any physical state $|n_{k_1} n_{k_2} \ldots \rangle$. Moreover this construction yields an automatic and covariant renormalization of the stress tensor: the above expression is free of any infinite term. This clearly indicates the crucial role played by the negative modes: they allow one to overcome in a totally covariant way the zero mode problem.
Again, we emphasize the fact that our minimally coupled field is defined on a space which is not a Hilbertian Fock space, and there is no contradiction with the result of Allen. This is due to the fact that the one-particle sector itself is not a Hilbert space (the inner product is not positive). The physical space stricto sensu is the quotient space $\mathcal{K}/\mathcal{N}$. This is a Hilbert space carrying a unitary irreducible representation of the de Sitter group. Nevertheless, such a quotient space is an abstract space and any attempt to realize it as a space of solutions of the field equation requires to invert the above quotient map. There are many ways to do this. None is natural (i.e. covariant). Any naive approach relies on such a construction (explicitly or not), and the consequence is a symmetry breaking in the theory. A frequently adopted manner to achieve this unnatural implementation of the Hilbert space structure in the theory is to write down the massive theory and then to put $(m^2_H + \xi R) \to 0$ in (4). Infinite divergences appear in this computation, from which it is often claimed that the vacuum state is not normalisable. On the contrary, our approach is to start from the minimally coupled framework (equation and its set of solutions), and no divergence exists. Indeed, all the states are of finite norm with respect to the natural inner product (see (6) below): in particular the norm of a global gauge state vanishes and no infra-red divergence appears.

We shall first present, in the next section, the de Sitter machinery. By this we mean a set of definitions and notations concerning geometry and wave equations on one hand, and the relevant group-theoretical material on the other hand. We shall especially insist on the terminology in use in representation theory in order to make the reader more familiar with a complete classification of unitary irreducible representations of $SO_o(1,4)$ and the respective physical meaning of the latter. Section 3 is devoted to the description of the space of solutions of involved scalar wave equations. In Section 4, we make explicit the Gupta-Bleuler structure lying behind the minimally-coupled massless field. Indeed, the interesting one-particle sector in the space of solutions of $\Box \phi = 0$ in de Sitter can be structured into a so-called Krein space $= Hilbert \oplus$ anti-Hilbert, of which an invariant subspace is made of constant functions. In section 5 we present the new representation of the ccr from which the quantum field is obtained. The Fock space carrying this representation is based on the Krein space. In order to control to what extent our quantization scheme is physically well-founded, we compute in section 6 the mean values of the stress tensor in our vacuum (we find zero!), and in excited states (we find positive values, as it should be reasonably expected even though the representation of the canonical commutation relations (ccr) involves negative norm solutions in order to preserve de Sitter covariance). After a brief comment on the extension of our method to massive fields (section 7), we finally conclude in section 8.

2 Presentation of the de Sitter machinery

The de Sitter space is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space

$$M_H = \{ X \in \mathbb{R}^5 | \ X^2 = \eta_{\alpha\beta}X^\alpha X^\beta = X_\alpha X^\alpha = -H^{-2} \},$$

where $\eta^{\alpha\beta} = diag(1, -1, -1, -1, -1)$. The (pseudo-)sphere $M_H$ is obviously invariant under five-dimensional Lorentz transformation. Therefore de Sitter space has a ten-parameter group
of isometries, the de Sitter group $O(1, 4)$. We only consider the connected component of the identity $SO_0(1, 4)$. We are in particular interested by the Poincaré limit of the latter through the group contraction $H \to 0$, i.e. when the curvature tends toward 0. The ten infinitesimal generators $M_{\alpha\beta}$ in some unitary representation of the de Sitter group obey the following well-known commutation rules (with $\hbar = 1$)

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -i(\eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\alpha\delta}M_{\beta\gamma} + \eta_{\beta\delta}M_{\alpha\gamma} - \eta_{\beta\gamma}M_{\alpha\delta}).$$  \hspace{1cm} (1)

In this work, we shall make use of a system of bounded global coordinates $(x^\mu$, $\mu = 0, 1, 2, 3)$ well-suited to describe a compactified version of $dS$, namely $S^3 \times S^1$ (Lie sphere). This system is given by

$$\begin{cases}
X^0 = H^{-1} \tan \rho \\
X^1 = (H \cos \rho)^{-1} (\sin \alpha \sin \theta \cos \phi), \\
X^2 = (H \cos \rho)^{-1} (\sin \alpha \sin \phi), \\
X^3 = (H \cos \rho)^{-1} (\sin \alpha \cos \theta), \\
X^4 = (H \cos \rho)^{-1} (\cos \alpha),
\end{cases}
$$

where $-\pi/2 < \rho < \pi/2$, $0 \leq \alpha \leq \pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. The coordinate $\rho$ is timelike and plays the role of a conformal time. The closure of the $\rho$-interval is actually involved when dealing with conformal action on compactified space-time. The de Sitter metrics now reads

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \frac{1}{H^2 \cos^2 \rho} (d\rho^2 - d\alpha^2 - \sin^2 \alpha d\theta^2 - \sin^2 \alpha \sin^2 \theta d\phi^2). \hspace{1cm} (2)$$

In the scalar representation carried by functions on $M_H$, the infinitesimal generators (1) are given by [CT]:

$$M_{ab} = -i \left( X_b \frac{\partial}{\partial X_a} - X_a \frac{\partial}{\partial X_b} \right) a, b = 0, 1, 2, 3, 4.$$

With our choice of coordinates the six generators of the compact $SO(4)$ subgroup, contracting to the Lorentz subalgebra when $H \to 0$, read as follows.

$$\begin{align*}
M_{12} &= -i \frac{\partial}{\partial \phi}, \\
M_{32} &= -i (\sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi}), \\
M_{31} &= -i (\cos \phi \frac{\partial}{\partial \theta} + \cot \theta \sin \phi \frac{\partial}{\partial \phi}), \\
M_{41} &= -i (\sin \theta \cos \phi \frac{\partial}{\partial \alpha} + \cot \alpha \cos \theta \cos \phi \frac{\partial}{\partial \theta} - \cot \alpha \sin \phi \frac{\partial}{\sin \theta \partial \phi}), \\
M_{42} &= -i (\sin \theta \sin \phi \frac{\partial}{\partial \alpha} + \cot \alpha \cos \theta \sin \phi \frac{\partial}{\partial \theta} + \cot \alpha \cos \phi \frac{\partial}{\sin \theta \partial \phi}), \\
M_{43} &= -i (\cos \theta \frac{\partial}{\partial \alpha} - \cot \alpha \sin \theta \frac{\partial}{\partial \theta}).
\end{align*}$$

6
The four generators contracting to the space-time translations when \( H \to 0 \) read as follows.

\[
M_{01} = -i (\cos \rho \sin \alpha \sin \theta \cos \phi \frac{\partial}{\partial \rho} + \sin \rho \cos \alpha \sin \theta \cos \phi \frac{\partial}{\partial \alpha} + \frac{\sin \rho \cos \theta \cos \phi}{\sin \alpha} \frac{\partial}{\partial \theta} - \frac{\sin \rho \sin \phi}{\sin \alpha \sin \theta} \frac{\partial}{\partial \phi}),
\]

\[
M_{02} = -i (\cos \rho \sin \alpha \sin \theta \sin \phi \frac{\partial}{\partial \rho} + \sin \rho \cos \alpha \sin \theta \sin \phi \frac{\partial}{\partial \alpha} + \frac{\sin \rho \cos \theta \sin \phi}{\sin \alpha} \frac{\partial}{\partial \theta} + \frac{\sin \rho \cos \phi}{\sin \alpha \sin \theta} \frac{\partial}{\partial \phi}),
\]

\[
M_{03} = -i (\cos \rho \sin \alpha \cos \theta \frac{\partial}{\partial \rho} + \sin \rho \cos \alpha \cos \theta \frac{\partial}{\partial \alpha} - \frac{\sin \rho \sin \theta}{\sin \alpha} \frac{\partial}{\partial \theta}),
\]

\[
M_{04} = -i (\cos \rho \cos \alpha \frac{\partial}{\partial \rho} - \sin \rho \sin \alpha \frac{\partial}{\partial \alpha}).
\]

The \( O(1,4) \)-invariant measure on \( M_H \) is

\[
d\mu = \sqrt{-g} dx^0 dx^1 dx^2 dx^3 = (\cos \rho)^{-4} d\rho d\Omega,
\]

(3)

where \( d\Omega = \sin^2 \alpha \sin \theta \, d\alpha \, d\theta \, d\phi \) is the \( O(4) \)-invariant measure on \( S^3 \). The Laplace-Beltrami operator on de Sitter space is given by

\[
\Box = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} g^{\mu \nu} \partial_\nu = H^2 \cos^4 \rho \left( \frac{\partial}{\partial \rho} \right)^2 - H^2 \cos^2 \rho \Delta_3,
\]

where

\[
\Delta_3 = \frac{\partial^2}{\partial \alpha^2} + 2 \cot \alpha \frac{\partial}{\partial \alpha} + \frac{1}{\sin^2 \alpha} \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{1}{\sin^2 \alpha} \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \alpha \sin^2 \theta} \frac{\partial^2}{\partial \phi^2}
\]

is the Laplace operator on the hypersphere \( S^3 \).

The wave equation for scalar fields \( \phi \) propagating on de Sitter space can be derived from variational principle on the action integral (\( \hbar = 1 \))

\[
S(\phi) = \frac{1}{2} \int_{M_H} \left[ g^{\mu \nu} \partial_\mu \phi \partial_\nu \phi - (m_H^2 + \xi R) \phi^2 \right] d\mu,
\]

(4)

where \( m_H \) is a “mass”, \( R = 12H^2 \) is the Ricci (or curvature) scalar, and \( \xi \) is a positive gravitational coupling with the de Sitter background. The variational principle applied to (4) leads to the field equation

\[
\Box_H (m_H^2 + \xi R) \phi(x) = 0.
\]

(5)

The Klein-Gordon inner-product is defined for any \( \phi, \psi \) solutions of (5) by

\[
\langle \phi, \psi \rangle = i \int_\Sigma \phi^* \left( \overrightarrow{\partial}_\mu - \overleftarrow{\partial}_\mu \right) \psi d\sigma^\mu \equiv i \int_\Sigma \phi^* \overrightarrow{\partial}_\mu \psi d\sigma^\mu,
\]

(6)

where \( \Sigma \) is a Cauchy surface, \( i.e. \) a space-like surface such that the Cauchy data on \( \Sigma \) define uniquely a solution of (5), and \( d\sigma^\mu \) is the area element vector on \( \Sigma \). This product is de Sitter
invariant and independent of the choice of $\Sigma$. In accordance with our choice of global coordinate system, the Klein-Gordon inner product (6) reads

$$\langle \phi, \psi \rangle = \frac{i}{H^2} \int_{\rho=0} \phi^*(\rho, \Omega) \overrightarrow{\partial}_\rho \psi(\rho, \Omega) d\Omega,$$

where $d\Omega = \sin^2 \alpha \sin \theta \ d\alpha \ d\theta \ d\phi$ is the invariant measure on $S^3$.

As we now explain, the equation (5) has a clear group-theoretical content. Let us recall that the Casimir operator $Q_0$ is defined by

$$Q_0 = -\frac{1}{2} M^{\alpha\beta} M_{\alpha\beta}.$$  \hfill (8)

This operator commutes with the action of the group generators and, as a consequence, it is constant on each unitary irreducible representation (UIR). As a matter of fact, the scalar UIR’s can be classified using the eigenvalues of $Q_0$. This allows to identify the scalar UIR associated to each scalar field on de Sitter space time because the Laplace-Beltrami operator and the Casimir operator are proportional:

$$\Box_H = -H^2 Q_0.$$  

Rewritten in these terms, (5) reads

$$Q_0 \phi = \kappa \phi,$$

with

$$\kappa = \langle Q_0 \rangle = \left( \frac{m_H}{H^2} \right)^2 + 12 \xi.$$  \hfill (9)

We consider only the positive values of $\kappa$ and we denote by $U^{(\kappa)}$ the scalar UIR corresponding to the value $\kappa$ of the Casimir operator. The classification of these scalar representations is the following [Di].

1. For

$$\kappa = \langle Q_0 \rangle \in \left[ \frac{9}{4}, +\infty \right[, $$

the corresponding UIR’s $U^{(\kappa)}$ are known as elements of the principal series of representations. They are written $\nu_{0,\kappa-2}$ in [Di].

2. For

$$\kappa = \langle Q_0 \rangle \in \left] 0, \frac{9}{4} \right[, $$

the corresponding UIR’s $U^{(\kappa)}$ are known as elements of the complementary series of representations. They are written $\nu_{0,\kappa-2}$ in [Di].

3. For

$$\kappa = \langle Q_0 \rangle = 0, $$

the corresponding UIR $U^{(0)}$ is known as the first term of the scalar discrete series of representations. It is written $\pi_{1,0}$ in [Di].
The physical content of each one from the point of view of a minkowskian observer (limit $H = 0$) is the following.

- First, we consider the “massive” case, i.e. the values of $\langle Q_0 \rangle$ corresponding to the principal series of representation. In order to obtain the contraction of group representations, the number $\kappa$, by which representations are labelled, goes to infinity in such a way that $H^2 \kappa \rightarrow m^2$. It has been proven that the principal series UIR’s $U^{(\kappa)}$, $\kappa \geq 0$, contracts toward the direct sum of two massive Poincaré UIR’s $\mathcal{P}(\pm m)$ with negative and positive energies respectively [MN]:

$$U^{(\kappa)} \xrightarrow{\kappa \to \infty} \mathcal{P}(-m) \oplus \mathcal{P}(m). \quad (10)$$

Note that the constraint $H^2 \kappa \rightarrow m^2$ and the equation (9) imply that the quantity $m_H$, supposed to depend on $H$, tends to the classical mass $m$ when the curvature goes to zero.

- Second we consider the massless case. As explained in the introduction, we select the representation having a natural extension to the conformal group. This representation is $U^{(2)}$, an element of the complementary series (this corresponds to $m_H = 0$ and $\xi = 1/6$). The representation $U^{(2)}$ extends to an UIR $C_0^+$ of the conformal group $SO_0(2,4)$ [BB]. In contrast to the massive case, the contraction process involves only one representation. The representation involved for each value of $H$, including $H = 0$ is equivalent to $C_0^+ \oplus C_0^-$ [AF2S]. The following diagram illustrates these connections:

$$C_0^- \oplus C_0^+ \xrightarrow{H \to 0} C_0^- \oplus C_0^+ \xrightarrow{\mathcal{P}^-(0)} \mathcal{P}^-(0) \oplus \mathcal{P}^+(0) \quad (11)$$

where the symbol $\sqcup$ means that the upper representation is an extension of the lower one and $\mathcal{P}^{\pm}(0)$ are the massless Poincaré UIR’s with positive and negative energies respectively.

- The minimally coupled field ($\langle Q_0 \rangle = 0$) has no minkowskian counterpart but it is interesting partly because this field appears when treating the spin-two field. The involved UIR is the first term $U^{(0)}$ of the discrete series of representations.

3 Space of solutions

Equation (5) can be solved by separation of variable [CT, KG]. We put

$$\phi(x) = \chi(\rho)D(\Omega),$$

where $\Omega \in S^3$, and obtain

$$[\Delta_3 + C]D(\Omega) = 0, \quad (12)$$

$$\left(\cos^4 \rho \frac{d}{d\rho} \cos^{-2} \rho \frac{d}{d\rho} + C \cos^2 \rho + \left(\frac{m_H}{H}\right)^2 + 12\xi\right)\chi(\rho) = 0. \quad (13)$$
We begin with the angular part problem (12). For $C = L(L + 2), \ L \in \mathbb{N}$ we find the hyperspherical harmonics $D = Y_{Llm}$ which are defined by

$$Y_{Llm}(\Omega) = \frac{1}{2\pi^2(L+l+1)!} \left( \frac{(L+1)(2l+1)(L-l)!}{2l!} \right)^\frac{1}{2} \frac{2^{l!} (\sin \alpha)^l C_{L-l}^{l+1}(\cos \alpha) Y_{lm}(\theta,\phi)},$$

for $(L,l,m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}$ with $0 \leq l \leq L$ and $-l \leq m \leq l$. In this equation the $C_n^\lambda$ are Gegenbauer polynomials [Tal] and $Y_{lm}$ are ordinary spherical harmonics:

$$Y_{lm}(\theta,\phi) = (-1)^m \left( \frac{(l-m)!}{(l+m)!} \right)^\frac{1}{2} P_l^m(\cos \theta) e^{im\phi},$$

where $P_l^m$ are the associated Legendre functions. With this choice of constant factors, the $Y_{Llm}$’s obey the orthogonality (and normalization) conditions:

$$\int_{S^3} Y_{Llm}^*(\Omega) Y_{Ll'm'}(\Omega) d\Omega = \delta_{LL'} \delta_{ll'} \delta_{mm'}.$$

We now come to the radial-part problem (13). Let $\lambda$ be defined by

$$\lambda = \sqrt{\frac{9}{4} - \kappa} \text{ when } \frac{9}{4} \geq \kappa \geq 0,$$

$$\lambda = i \sqrt{\kappa - \frac{9}{4}} \text{ when } \frac{9}{4} \leq \kappa. \quad (14)$$

Following [KG], we obtain the solutions

$$\chi_{\lambda L}(\rho) = A_L(\cos \rho)^{3/2} \left[ P_{L+\frac{3}{2}}^\lambda(\sin \rho) - \frac{2i}{\pi} Q_{L+\frac{3}{2}}^\lambda(\sin \rho) \right]. \quad (15)$$

Here $P_n^\lambda$ and $Q_n^\lambda$ are the Legendre functions on the cut, and $A_L$ is given by

$$A_L = H \frac{\sqrt{\pi}}{2} \left( \frac{\Gamma(L-\lambda+\frac{3}{2})}{\Gamma(L+\lambda+\frac{3}{2})} \right)^\frac{1}{2}.$$

We then obtain the complete set of modes

$$\phi_{Llm}^\lambda(x) = \chi_{\lambda L}(\rho) Y_{Llm}(\Omega), \ x = (\rho,\Omega) \in M_H, \quad (16)$$

for the field equation $(\Box + \kappa) \phi = 0$, where $\lambda$ is defined through (9) and (14), except for the minimally-coupled field $\kappa = 0, \lambda = 3/2$, for which the formulas break down. Note that this family of modes verify the orthogonality prescription:

$$\langle \phi_{Ll'm'}^\lambda, \phi_{Llm}^\lambda \rangle = \delta_{LL'} \delta_{ll'} \delta_{mm'} \quad \text{and} \quad \langle \phi_{Ll'm'}^\lambda, \phi_{Llm}^\lambda \rangle^* = 0.$$

This family can be used to define the euclidean vacuum in the standard terminology.
We now turn our attention to the singular case $\lambda = \frac{3}{2}$ corresponding to the minimally coupled field. For $L \neq 0$, we obtain the modes $\phi_{Llm}^{\frac{3}{2}}$ that we write $\phi_{Llm}$ for simplicity:

$$\phi_{Llm}(x) = \chi_L(\rho)Y_{Llm}(\Omega),$$  

with

$$\chi_L(\rho) = \frac{H}{2}[2(L + 2)(L + 1)L]^{-\frac{1}{2}} \left(Le^{-i(L+2)\rho} + (L+2)e^{-iL\rho}\right).$$  

The normalization constant $A_L$ breaks down at $L = 0$. This is the famous “zero-mode” problem. The space generated by the $\phi_{Llm}$ for $L \neq 0$ is not a complete set of modes. Moreover this set is not invariant under the action of the de Sitter group. Actually, an explicit computation gives

$$(M_{03} + iM_{04})\phi_{1,9,0} = -i \frac{4}{\sqrt{6}} \phi_{2,1,0} + \phi_{2,0,0} + \frac{3H}{4\pi \sqrt{6}},$$  

and the invariance is broken owing to the last term. As a consequence, canonical quantization applied to this set of modes yields a non covariant field, and this is due to the apparition of the last term in (19). Constant functions are of course solutions to the field equation. So one is led to deal with the space generated by the $\phi_{Llm}$’s and by a constant function denoted here by $\psi_g$, this is interpreted as a gauge state as announced in the introduction. This space, which is invariant under the de Sitter group, is the space of physical states as explained below. However, as an inner-product space equipped with the Klein-Gordon inner product, it is a degenerate space because the state $\psi_g$ is orthogonal to the whole space including itself. Due to this degeneracy, canonical quantization applied to this set of modes yields once again a non covariant field (see [DBR] for a detailed discussion of this fact).

Actually, for $L = C = \kappa = 0$, the equation (13) is easily solved. We obtain two independent solutions of the field equation, including the constant function discussed above:

$$\psi_g = \frac{H}{2\pi}$$ and $$\psi_s = -i \frac{H}{2\pi} \left[\rho + \frac{1}{2} \sin 2\rho\right].$$

Note that the constants of normalization are chosen in order to have $\langle \psi_g, \psi_s \rangle = 1$. One can now defines $\phi_{000} = \psi_g + \psi_s/2$. This is the “true zero mode” of Allen. We write $\phi_{000} = \phi_0$ in the following. With this mode, one obtains a complete set of strictly positive norm modes $\phi_{Lml}$ for $L \geq 0$, but the space generated by these modes is not de Sitter invariant. For instance, we have

$$(M_{03} + iM_{04})\phi_0 = (M_{03} + iM_{04})\psi_0 = -i \frac{\sqrt{6}}{4} \phi_{1,0,0} + \frac{\sqrt{6}}{4} \phi_{1,1,0} - \frac{\sqrt{6}}{4} \phi_{1,0,0}. \quad (20)$$

As a consequence the field obtained through canonical quantization and the usual representation of the ccr from the set of modes $\phi_{Lml}$ for $L \geq 0$ is not covariant. Nevertheless, the above space is $O(4)$ invariant, and with this set of modes one obtains by the usual construction a $O(4)$-covariant quantum field [A]. On the other hand, although our covariant and causal field is also obtained by canonical quantization:

$$\varphi(x) = \sum_k \phi_k(x)A_k + \phi_k^{*}(x)A_k^\dagger,$$  

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with \( A_k \) and \( A_k^\dagger \) satisfying the ccr:
\[
[A_k, A_{k'}^\dagger] = 2\delta_{kk'}, \quad [A_k, A_{k'}] = 0, \quad [A_k^\dagger, A_{k'}^\dagger] = 0,
\]
we use a different representation of the ccr in order to obtain the field as an operator valued distribution.

Note the appearance of negative norm modes in (20) which is the price to pay in order to obtain a fully covariant theory. The existence of these non physical states naturally leads us to adopt a kind of Gupta-Bleuler field quantization.

For later use, the non vanishing inner products between \( \psi_g, \psi_s \) and \( \phi_{Lml} \) and \((\phi_{Llm})^*\) for \( L > 0 \), read:
\[
\langle \phi_{Llm}, \phi_{Llm} \rangle = 1, \quad \langle \phi_{Llm}^*, \phi_{Llm}^* \rangle = -1, \quad L > 0 \quad \text{and} \quad \langle \psi_s, \psi_g \rangle = 1.
\] (21)

### 4 Gupta-Bleuler triplet

From now on we shall deal with the minimally coupled field for which we define the Gupta-Bleuler triplet \([BFH, G]\) in order to build a covariant quantum field. The field equation is given by
\[
\Box \phi = 0. \tag{22}
\]

In order to simplify the previous notations, let \( K \) be the set of indices for the positive norm modes, excluding the zero mode:
\[
K = \{(L, l, m) \in \mathbb{N} \times \mathbb{N} \times \mathbb{Z}; \; L \neq 0, \; 0 \leq l \leq L, \; -l \leq m \leq l\},
\]
and \( K' \) the same set including the zero mode:
\[
K' = K \cup \{0\}.
\]

As illustrated by (19), the set spanned by the \( \phi_k, k \in K \) is not invariant under the action of the de Sitter group. On the other hand, we obtain an invariant space by adding \( \psi_g \). More precisely, let us introduce the space,
\[
\mathcal{K} = \{c_g \psi_g + \sum_{k \in K} c_k \phi_k; \; c_g, c_k \in \mathbb{C}, \; \sum_{k \in K} |c_k|^2 < \infty\}.
\]

Equipped with the Klein-Gordon-like inner product (6), \( \mathcal{K} \) is a degenerate inner product space because the above orthogonal basis satisfies to
\[
\langle \phi_k, \phi_{k'} \rangle = \delta_{kk'} \quad \forall k, k' \in K, \quad \langle \phi_k, \psi_g \rangle = 0 \quad \forall k \in K, \quad \text{and} \quad \langle \psi_g, \psi_g \rangle = 0.
\]

It can be proved by conjugating the action (19) under the SO(4) subgroup that \( \mathcal{K} \) is invariant under the natural action of the de Sitter group. As a consequence, \( \mathcal{K} \) carries a unitary representation of the de Sitter group, this representation is indecomposable but not irreducible, and the null-norm subspace \( \mathcal{N} = \mathbb{C} \psi_g \) is an uncomplemented invariant subspace.
Let us recall that the Lagrangian
\[ L = \sqrt{|g|} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \]
of the free minimally coupled field is invariant when adding to \( \phi \) a constant function. As a consequence, in the “one-particle sector” of the field, the space of “global gauge states” is simply the invariant one dimensional subspace \( \mathcal{N} = \mathbb{C}\psi_g \). In the following, the space \( \mathcal{K} \) is called the (one-particle) physical space, but \textit{stricto sensu} physical states are defined up to a constant and the space of physical states is \( \mathcal{K}/\mathcal{N} \). The latter is a Hilbert space carrying the unitary irreducible representation of the de Sitter group \( U(0) \).

If one attempts to apply the canonical quantization starting from a degenerate space of solutions, then one inevitably breaks the covariance of the field [DBR]. Hence we must build a non degenerate invariant space of solutions \( \mathcal{H} \) admitting \( \mathcal{K} \) as an invariant subspace. Together with \( \mathcal{N} \), the latter are constituent of the so-called Gupta-Bleuler triplet \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{H} \). The construction of \( \mathcal{H} \) is worked out as follows.

We first remark that the modes \( \phi_k \) and \( \psi_g \) do not form a complete set of modes. Indeed, the solution \( \psi_s \) does not belong to \( \mathcal{K} \) nor \( \mathcal{K} + \mathcal{K}^* \) (where \( \mathcal{K}^* \) is the set of complex conjugates of \( \mathcal{K} \)): in this sense, it is not a superposition of the modes \( \phi_k \) and \( \psi_g \). One way to prove this is to note that \( \langle \psi_s, \psi_g \rangle = 1 \neq 0 \).

So we need a complete, non-degenerate and invariant inner-product space containing \( \mathcal{K} \) as a closed subspace. The smallest one fulfilling these conditions is the following. Let \( \mathcal{H}_+ \) be the Hilbert space spanned by the modes \( \phi_k \) together with the zero-mode \( \phi_0 \):
\[ \mathcal{H}_+ = \{ c_0 \phi_0 + \sum_{k \in \mathcal{K}} c_k \phi_k; \sum_{k \in \mathcal{K}} |c_k|^2 < \infty \}. \]
We now define the total space \( \mathcal{H} \) by
\[ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_+^*, \]
which is invariant, and we denote by \( U \) the natural representation of the de Sitter group on \( \mathcal{H} \) defined by : \( U g \phi(x) = \phi(g^{-1}x) \). Our Gupta-Bleuler triplet is precisely \( \mathcal{N} \subset \mathcal{K} \subset \mathcal{H} \). The space \( \mathcal{H} \) is defined as a direct sum of an Hilbert space and an anti-Hilbert space (a space with definite negative inner product) which proves that \( \mathcal{H} \) is a Krein space. Note that neither \( \mathcal{H}_+ \) nor \( \mathcal{H}_+^* \) carry a representation of the de Sitter group, so that the previous decomposition is not covariant, although it is \( O(4) \)-covariant. The following family is a pseudo-orthonormal basis for this Krein space:
\[ \phi_k, \phi_k^*, (k \in \mathcal{K}), \phi_0, \phi_0^*, \]
for which the non-vanishing inner products are
\[ \langle \phi_k, \phi_k \rangle = \langle \phi_0, \phi_0 \rangle = 1 \text{ and } \langle \phi_k^*, \phi_k^* \rangle = \langle \phi_0^*, \phi_0^* \rangle = -1. \]

Let us once more insist on the presence of non physical states in \( \mathcal{H} \). Some of them have negative norm, but, for instance, \( \phi_0 \) is not a physical state \( \phi_0 \notin \mathcal{K} \) in spite of the fact that \( \langle \phi_0, \phi_0 \rangle > 0 \): the condition of positivity of the inner product is not a sufficient condition for
selecting physical states. Moreover some non physical states go to negative frequency states when the curvature tends to 0. Nevertheless mean values of observables are computed on physical states and no negative energy appears.

The space $\mathcal{H}$ is connected to the propagator in the following way. Since de Sitter space-time is globally hyperbolic, there exist two elementary solutions of the field equation, $G^{\text{ret}}$ and $G^{\text{adv}}$, which are the unique ones verifying $[I_1]$

$$\Box_x G^{\text{adv}}(x, y) = \Box_x G^{\text{ret}}(x, y) = -\delta(x, y)$$

and for fixed $y$ the support in $x$ of $G^{\text{adv}}(x, y)$ (resp. $G^{\text{ret}}(x, y)$) lies in the future (resp. the past) of $y$. The so-called propagator $\tilde{G}$ is defined by

$$\tilde{G} = G^{\text{adv}} - G^{\text{ret}}.$$ 

This propagator is the reproducing kernel with respect to the Klein-Gordon inner product:

$$\phi(\rho, \Omega) = \frac{i}{H^2} \int_{\rho' = 0} (-i) \tilde{G} \left( (\rho, \Omega), (\rho', \Omega') \right) \phi(\rho', \Omega') d\Omega',$$

for any solution $\phi$ of the field equation. Using the above basis of $\mathcal{H}$, one obtains the following expression for $\tilde{G}$.

$$\tilde{G}(x, x') = \sum_{k \in K} \phi_k^*(x) \phi_k(x') + \phi_0^*(x) \phi_0(x') - \phi_0(x) \phi_0^*(x') - \sum_{k \in K} \phi_k(x) \phi_k^*(x')$$

$$= \sum_{k \in K} \frac{H}{2\pi} (\psi_s(x') - \psi_s(x)) - \sum_{k \in K} \phi_k(x) \phi_k^*(x').$$

(23)

This two-point function $\tilde{G}$ is linked to $\mathcal{H}$ in the following way. Since the Riesz representation theorem is valid in Krein spaces, for any continuous linear form $L$ there exists a unique element $\psi_L \in \mathcal{H}$ such that

$$L(\phi) = \langle \psi_L, \phi \rangle, \quad \forall \phi \in \mathcal{H}.$$ 

This representation theorem (see [DBR] for details) allows one to define for any real test function $f$ an element $p(f) \in \mathcal{H}$ such that

$$\langle p(f), \phi \rangle = \int_{M_H} f(x) \phi(x) d\mu(x).$$

(24)

This formula defines a $\mathcal{H}$-valued distribution $p$ on $M_H$. In the unsmeared form, (24) reads: $\langle p(x), \phi \rangle = \phi(x)$ for any $\phi \in \mathcal{H}$. Direct computation with the basis proves that

$$p(x) = \sum_{k \neq 0} \phi_k^*(x) \phi_k - \sum_{k \neq 0} \phi_k(x) \phi_k^* + \psi_s(x) \psi_s - \psi_s(x) \psi_s,$$

(25)

$$= \sum_{k \neq 0} \phi_k^*(x) \phi_k - \sum_{k \neq 0} \phi_k(x) \phi_k^* + \phi_0^*(x) \phi_0 - \phi_0(x) \phi_0^*.$$ 

(26)

Moreover, $-i\tilde{G}$ is the kernel of $p$, that is to say:

$$\langle p(x'), p(x) \rangle = -i\tilde{G}(x, x').$$

Note that from (24) one can prove immediately that $p$ commutes with the action of the de Sitter group: $U_g p(f) = p(U_g f)$. As a consequence, $\tilde{G}$ is invariant:

$$\tilde{G}(g \cdot x, g \cdot x') = \tilde{G}(x, x').$$
5 The Quantum Field

As explained at the end of section 3 the family \( \phi_k \) for \( k \in K \) together with \( \phi_0 \) is a complete set of mode. As a consequence, we obtain a quantum field through the usual formula:

\[
\varphi(x) = \sum_{k \in K} \phi_k(x) A_k + \phi_0(x) A_0 + \sum_{k \in K} \phi_k^*(x) A_k^\dagger + \phi_0^*(x) A_0^\dagger,
\]

where \( A_k \) and \( A_k^\dagger \) satisfy the \textit{ccr}:

\[
[A_k, A_{k'}^\dagger] = 2 \delta_{kk'}, \quad [A_k, A_{k'}] = 0, \quad [A_k^\dagger, A_{k'}^\dagger] = 0, \quad k, k' \in K' = K \cup \{0\}.
\]

So far this is the usual procedure, except for the insignificant factor 2. However, since the space generated by these modes is not closed under the action of de Sitter group, the usual representation of the \textit{ccr} yields a non covariant (although \textit{SO}(4)-invariant) field.

We now define a new representation of the \textit{ccr} leading to a covariant field. Let us first recall that we deal with a Gupta-Bleuler quantum field. It is a distribution the values of which are operators on the bosonic Fock space built on the total space \( \mathcal{H} \) (see [M] for the theory of Fock spaces on Krein spaces). As usual in a Gupta-Bleuler construction, mean values of observables will be evaluated only with physical states. The physical states are the states obtained from the Fock vacuum by creation of one particle physical states, which means creation of elements of \( \mathcal{K} \). In a Fock space, creation and annihilation operators are defined for arbitrary states, not only for modes. More precisely, let \( \mathcal{H} \) be the Fock space on \( \mathcal{H} \), the annihilator of a solution \( \phi \) of the field equation is defined by:

\[
(a(\phi)\Psi)(x_1, \ldots, x_{n-1}) = \sqrt{n} \frac{i}{\hbar} \int_{\mathcal{P}=0} \phi^*(\rho, \Omega) \bar{\partial}_\rho \Psi((\rho, \Omega), x_1, \ldots, x_{n-1}) d\Omega,
\]

for any square-integrable \( n \)-symmetric function \( \Psi \). The creator is defined as usual by

\[
(a^\dagger(\phi)\Psi)(x_1, \ldots, x_{n+1}) = \frac{1}{\sqrt{n+1}} \sum_{i=1}^{n+1} \phi(x_i)\Psi(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1}).
\]

One can easily check that these operators obey the usual commutation rules.

\[
[a(\phi), a(\phi')] = 0, \quad [a^\dagger(\phi), a^\dagger(\phi')] = 0, \quad [a(\phi), a^\dagger(\phi')] = \langle \phi, \phi' \rangle,
\]

and also

\[
U_g a^\dagger(\phi) U_g^* = a^\dagger(U_g\phi), \quad \text{and} \quad U_g a(\phi) U_g^* = a(U_g\phi),
\]

where \( U \) is the natural representation of the de Sitter group on \( \mathcal{H} \) and \( U \) its extension to the Fock space.

Let \( a_k, a_0, b_0 \) and \( b_k \) be the annihilators of the modes \( \phi_k, \phi_0, \phi_0^* \) and \( \phi_k^* \) respectively. We define

\[
A_k = a_k - b_k^\dagger, \quad \text{and} \quad A_k^\dagger = a_k^\dagger - b_k \quad \text{for} \quad k \in K' = K \cup \{0\}.
\]
The field now reads
\[
\varphi(x) = \sum_{k \in K} \phi_k(x) a_k + \phi_0(x) a_0 - \sum_{k \in K} \phi_k^*(x) b_k - \phi_0^*(x) b_0 \\
+ \sum_{k \in K} \phi_k^*(x) a_k^\dagger - \sum_{k \in K} \phi_k(x) b_k^\dagger + \phi_0^*(x) a_0^\dagger - \phi_0(x) b_0^\dagger,
\]
(30)

The non vanishing commutation relations between the operators in (30) are for \( k \in K' = K \cup \{0\} \):
\[
[a_k, a_k^\dagger] = 1, \quad [b_k, b_k^\dagger] = -1.
\]
(31)

Note the minus sign which follows from the formulas above and \( \langle \varphi_k^*, \varphi_k^* \rangle = -1 \). Note also that this field is clearly real as the sum of an operator and its conjugate.

**Remark** For later use we can rewrite the field \( \varphi(x) \) in terms of the operators \( a_s = a(\psi_s) \) and \( a_g = a(\psi_g) \):
\[
\varphi(x) = \sum_{k \in K} \phi_k(x) a_k - \sum_{k \in K} \phi_k^*(x) b_k + \frac{H}{2\pi} a_s + \psi_s(x) a_g \\
+ \sum_{k \in K} \phi_k^*(x) a_k^\dagger - \sum_{k \in K} \phi_k(x) b_k^\dagger + \frac{H}{2\pi} a_g^\dagger - \psi_s(x) a_g^\dagger.
\]
(32)

We claim that this field is covariant:
\[
U_g \varphi(x) U_g^{-1} = \varphi(g \cdot x).
\]

This is due to the fact that \( \mathcal{H} \) is closed under the action of the de Sitter group, although this is not the case for \( \mathcal{H}_+ \). In order to prove this statement, we firstly give a more synthetic expression of the field, directly issued from (25) and (30):
\[
\varphi(x) = a(p(x)) + a^\dagger(p(x)).
\]
(33)

It is then straightforward to check that the covariance of \( \varphi \) follows from (29) and the covariance of \( p \). Note also that the formula (33) is a coordinate free definition, hence our field does not depend on any choice of coordinates.

The causality of the field is also immediate from the values of the commutator
\[
[\varphi(x), \varphi(x')] = 2 \langle p(x), p(x') \rangle = -2i \tilde{G}(x, x').
\]
(34)

We see indeed that the field is causal since \( \tilde{G} \) vanishes when \( x \) and \( x' \) are space-like separated.

The Gupta-Bleuler vacuum is precisely the Fock vacuum characterized by
\[
a_k |0 \rangle = b_k |0 \rangle = 0, \quad \text{for} \ k \in K' = K \cup \{0\},
\]
and it is trivially invariant under the action of de Sitter group. At this point, let us emphasize the differences between the usual point of view in QFT and ours. The set of modes that we have used in our construction is exactly the one used in [A, AF] in order to obtain the O(4)
and not SO(1, 4)) invariant vacuum. Nevertheless our theory is SO(1, 4) invariant. This is due
to the new representation of the ccr. Since our field is different from the usual one, we think
that comparing vacua is misleading.

So we have obtained a quantum field verifying the Wightman axioms. Some non-physical
states are present in the construction, they are really needed in order to assure the de Sitter
covariance. Now we must define the observables of the theory. We also have to verify
that non-physical states do not yield any trouble like the appearing of negative energies.
This is the content of the next section in which we prove that the stress tensor is an observ-
able and that it is computed directly without any renormalization. Moreover, expressions like
\[ \langle n_{k_1}n_{k_2} \ldots | T_{00} | n_{k_i}n_{k_2} \ldots \rangle \]
are positive for any physical state \(| n_{k_1}n_{k_2} \ldots \rangle\), and no negative energy
can be observed.

\section{The stress tensor}

As explained in [DBR], the global gauge change \( \gamma^\lambda \) (which would be local if we were dealing
with QED! [BFH], [G]) can be implemented in the theory by
\[ \gamma^\lambda = \exp \left( -{\pi \lambda \over H} (a_{g}^\dagger - a_{g}) \right), \]
from which one verifies that
\[ \gamma^{-\lambda} \varphi(x) \gamma^\lambda = \varphi(x) + \lambda \text{Id}. \]

We now define the (second-quantized) physical space \( \mathcal{K} \) as the space generated from the
Fock vacuum by creating elements of \( \mathcal{K} \), the set of one particle physical states: \( \mathcal{K} \) is the space
generated by the \((a_{g}^\dagger)^{n_0}(a_{k_1}^\dagger)^{n_1} \ldots (a_{k_l}^\dagger)^{n_l} | 0 \rangle\). We call \( \mathcal{N} \) the subspace of \( \mathcal{K} \) orthogonal to \( \mathcal{K} \).
\[ \Psi \in \mathcal{N} \text{ iff } \Psi \in \mathcal{K} \text{ and } \langle \Psi, \Phi \rangle = 0 \forall \Phi \in \mathcal{K}. \] (35)

Note that, when restricted to \( \mathcal{K} \), the operator \( a_{g} \) is the null operator and that for any
physical state \( \Psi \), the state \( a_{g}^\dagger \Psi \in \mathcal{N} \). As a consequence, for any physical state \( \Psi \) and any real \( \lambda \), the states \( \Psi \) and \( \gamma^\lambda \Psi \) are equal up to an element of \( \mathcal{N} \). This is the motivation for defining
elements of \( \mathcal{N} \) as our second quantized set of global gauge states, and we have obtained our
second-quantized Gupta-Bleuler triplet:
\[ \mathcal{N} \subset \mathcal{K} \subset \mathcal{H}, \]
which is clearly invariant under the action of the de Sitter group. Consistently, two physical
states are said to be physically equivalent when they differ from a global gauge state and a
gauge change transforms a physical state into an equivalent state.

Remark (Quasi-uniqueness of the vacuum): The space of the de Sitter invariant states of \( \mathcal{H} \) is
\( \mathcal{N} \) the space generated from the vacuum by \( a_{g}^\dagger \). This space is an infinite dimensional subspace
of \( \mathcal{N} \), hence the Fock vacuum is not the unique de Sitter invariant state. Nevertheless one can
easily see that all these states are physically equivalent to an element of the one dimensional
space generated by the vacuum state. In this sense we can say that the vacuum is unique.
We now have to define the observables of the theory, under the condition that a global gauge change must not be observed. An observable $A$ is a symmetric operator on $\mathcal{H}$ such that, when $\Psi$ and $\Psi'$ are equivalent physical states (elements of $\mathcal{K}$ such that $\Psi - \Psi'$ belongs to $\mathcal{N}$), we must have

$$\langle \Psi | A | \Psi \rangle = \langle \Psi' | A | \Psi' \rangle.$$ 

One can easily verify that the field $\varphi$ is not an observable. This is due to the presence of the terms $a_s$ and $a_s^\dagger$ (32), these terms disappear in $\partial_\mu \varphi$ and this is the reason for which the stress tensor $T_{\mu\nu}$ is an observable.

At this point, one can understand the reason why the approach through two-point functions is not relevant for this field. In fact since the field is not an observable, quantities like Wightman or Hadamard functions

$$G(x, x') = \langle 0 | \varphi(x) \varphi(x') | 0 \rangle, \ \ G^{(1)}(x, x') = \frac{1}{2} \langle 0 | \varphi(x) \varphi(x') + \varphi(x') \varphi(x) | 0 \rangle,$$

are not gauge invariant. Hence any definition a priori of such a function in order to obtain a field cannot yield a covariant theory. (If one computes $G^{(1)}$ for our field one finds 0, but once again, this result has no physical significance). Actually, there exists no non trivial covariant two-point function of positive type, this is nothing but another formulation of Allen’s theorem. The only two-point function which naturally appears is the commutator, but it is not of positive type and it does not allow to select physical states. Moreover, the usual classification of vacua is based on two-point functions and our vacuum does not fit this classification. We do insist on the fact that it is the field itself which is different in our construction and not only the vacuum.

The stress tensor, which in this case is the same as the improved stress tensor, is given by [BD]

$$T_{\mu\nu} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} g^{\rho\sigma} \partial_\rho \phi \partial_\sigma \phi.$$ 

Let us consider the excited physical state

$$|\tilde{k}⟩ = |k_1^{n_1} \cdots k_j^{n_j}⟩ = \frac{1}{\sqrt{n_1! \cdots n_j!}} (a_{k_1}^\dagger)^{n_1} \cdots (a_{k_j}^\dagger)^{n_j} |0⟩.$$ 

In order to compute $\langle \tilde{k} | T_{\mu\nu}(x) | \tilde{k} \rangle$, we begin with $\langle \tilde{k} | \partial_\mu \varphi(x) \partial_\nu \varphi(x) | \tilde{k} \rangle$. As mentioned before, the terms containing $a_s$ and $a_s^\dagger$ disappear in the derivation. Moreover $a_g$ and $a_g^\dagger$ commute with all the remaining operators including themselves, and so the corresponding terms vanish in the computation. We then obtain

$$\langle \tilde{k} | \partial_\mu \varphi(x) \partial_\nu \varphi(x) | \tilde{k} \rangle = \sum_{k \in K} \partial_\mu \phi_k(x) \partial_\nu \phi^*_k(x) - \partial_\mu \phi^*_k(x) \partial_\nu \phi_k(x) + 2 \sum_{i=1}^l n_i \mathcal{R} (\partial_\mu \phi^*_k(x) \partial_\nu \phi_k(x)).$$ 

The first and third terms are those one obtains in the usual computation, the first one carries infinite terms which have to be renormalized in the usual theory. The unusual second term, with the minus sign, is due to the presence of $b_k$ and $b_k^\dagger$ in the field. Thanks to this term, there is no need to renormalize the stress tensor because one obtains immediately that

$$\langle \tilde{k} | \partial_\mu \varphi(x) \partial_\nu \varphi(x) | \tilde{k} \rangle = 2 \sum_{i=1}^l n_i \partial_\mu \phi^*_k(x) \partial_\nu \phi_k(x).$$
A direct consequence of this formula is the positivity of the energy, more precisely, one can see at once that

\[ \langle \tilde{k} | T_{\mu0} | \tilde{k} \rangle \geq 0, \]

for any physical state \( | \tilde{k} \rangle \) and that this quantity vanishes if and only if \( | \tilde{k} \rangle = | 0 \rangle \). This is not in contradiction with the existence of other states for which the energy can be negative, but of course these states are not physical. One can see that the non-physical states do not play any role for the free field. In the interacting case, the situation would be different. One can then expect the appearing of some virtual particles like for QED in presence of charges (see [MS] for instance).

**Remarks on the renormalization:** The present renormalization is fully covariant and has nothing to do neither with the choice of modes nor with the presence of zero modes. It is totally different from other existing renormalization procedures. This is due to \( \langle \partial_{\mu} p(x), \partial_{\mu} p(x) \rangle = 0 \) which implies that

\[ [a(p(x)), a^\dagger(p(x))] = 0. \]

Moreover, this renormalization eliminates infra-red as well as ultra-violet divergence. Actually, both divergences are carried by the Hadamard function \( G^{(1)} \) and the latter vanishes here. Finally, this renormalization fulfills the so-called Wald axioms.

1. The stress tensor is covariant and causal since the field is.
2. The computation above shows that it furnishes the usual (i.e. formal) results for physical states.
3. The corner stone of the above computation is the following:

\[ [b_k, b_k^\dagger] = -1, \]

which implies that

\[ a_k a_k^\dagger + a_k^\dagger a_k + b_k b_k^\dagger + b_k^\dagger b_k = 2a_k a_k^\dagger + 2b_k b_k^\dagger. \]

One can see that this is equivalent to reordering when applied to physical states (on which \( b_k \) vanishes).

In conclusion, we have introduced auxiliary states (states which do not belong to \( \mathcal{K} \)) for constructing a covariant quantization of the massless minimally coupled scalar field. But the effect of these auxiliary states appears in the physics of the problem by allowing an automatic renormalization of the stress tensor, and, once again, the auxiliary states do not yield any measurable negative energy.

### 7 Back to the massive field

As explained in the above, the crucial point about the minimally coupled field is the fact that there does not exist a covariant decomposition

\[ \mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-, \]

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where $\mathcal{H}_+$ (resp. $\mathcal{H}_-$) is a Hilbert space (resp. anti-Hilbert space). This was the reason for which our space of states contains negative frequency solutions. It is not the case for the scalar massive field for which such a decomposition exists, where $\mathcal{H}_+$ is the usual physical states space and $\mathcal{H}_- = \mathcal{H}_+^*$. Nevertheless one can define a $\mathcal{H}$-valued distribution $p$ as in (24) and a field $\varphi$ as in (33), except that the above decomposition of $\mathcal{H}$ yields a covariant decomposition $p = p_+ + p_-$ with $p_\epsilon(x) \in \mathcal{H}_\epsilon$ and a decomposition of the field into two parts

$$\varphi = \varphi_+ + \varphi_-.$$  

The positive frequency part $\varphi_+$, written in terms of annihilators and creators is exactly the usual field. Moreover, for $\Psi$ and $\Psi'$ physical states, we have

$$\langle \Psi | \varphi(x) | \Psi' \rangle = \langle \Psi | \varphi_+(x) | \Psi' \rangle.$$  

However, this does not mean that $\varphi$ and $\varphi_+$ are the same object, as operators they are different and quantities like

$$\langle \Psi | \varphi(x) \varphi(x') | \Psi' \rangle \quad \text{and} \quad \langle \Psi | \varphi_+(x) \varphi_+(x') | \Psi' \rangle$$  

are different. In particular, the energy-momentum tensor computation presented in the previous section can be easily transposed to the present massive scalar field with the following issue. Using the Gupta-Bleuler quantization for the massive field provides an automatic and covariant renormalization of the vacuum energy divergence. Note that the conformal massless case is in some sense a particular case of the massive one and our construction supplies a field which is covariant and conformally covariant in a strong sense. As a consequence, the trace anomaly does not appear. This is not surprising because, after all, trace anomaly can appear only by breaking the conformal invariance.

In a future work [G²R], we show that this quantization explains the appearance of negative frequency terms in the flat limit of de Sitter space-time.

As a final remark, relevant to both massive and massless case, let us discuss the Bogolioubov transformations in our Gupta-Bleuler framework. First of all, let us point out that from our point of view, there is a unique vacuum, the (Krein-)Fock vacuum, which is invariant and normalizable. This does not mean that the Bogolioubov transformations are no longer valid in Gupta-Bleuler quantization. Any element like $\tilde{\phi}_k = A_k \phi_k + B_k \phi_k^*$ belongs to $\mathcal{H}$, and a Bogolioubov transformation is just a change of physical states. The new physical space is $\tilde{\mathcal{H}}_+ = \text{span}(\tilde{\phi}_k)$, for which there corresponds a new $\tilde{\varphi}_+$. If one wants to characterize the new physical space by some two-point function, one can compute

$$\tilde{G}_+^{(1)} = \langle 0 | \tilde{\varphi}_+(x) \tilde{\varphi}_+(x') + \tilde{\varphi}_+(x') \tilde{\varphi}_+(x) | 0 \rangle.$$  

This gives exactly the same family of function as the expression (2.14) of [A].

### 8 Conclusion and outlook

Any consistent approach to quantization of fields in de Sitter space-time has to deal with the negative-energy problem from a minkowskian point of view (see (10)). This problem of
“negative-frequency” modes from a curved space-time point of view is also present in the manipulation of the zero-modes.

Different ways to go round this problem have been proposed in order to reach a point in the theory where only positive-energy states are taken into account: restriction to a subgroup [A], analyticity constraint (massive case) [BGM], modification of the vacuum definition [AF, KG].

Another difficulty appears when dealing with fields involving a gauge invariance. The Gupta-Bleuler formalism has been created in order to manage both covariance and gauge invariance in quantum electrodynamics. It is not surprising that an analogous construction accomplishes the same task for the minimally-coupled field on de Sitter space-time. We have here presented a new proposal which is a continuation of [DBR]. The guideline is the covariance of the full theory under the full SO_{0}(1, 4) in the spirit of the Wightman-G"arding approach. The fact that our Krein-Gupta-Bleuler quantization gives the correct sign for the energy pleads in favor of an extension of this method to the massless spin-2 case on de Sitter space-time. The corresponding field is indeed built up from two copies of minimally coupled fields [Tak] and we hope that the present paper will open the way to a satisfactory covariant quantization of this field.

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References


