Covariant Hamiltonian field theory

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We study the relations between the equations of first order Lagrangian field theory on fiber bundles and the covariant Hamilton equations on the finite-dimensional polysymplectic phase space of covariant Hamiltonian field theory. The main peculiarity of these Hamilton equations lies in the fact that, for degenerate systems, they contain additional gauge fixing conditions. We develop the BRST extension of the covariant Hamiltonian formalism, characterized by a Lie superalgebra of BRST and anti-BRST symmetries.

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I. INTRODUCTION

As is well known, when applied to field theory, the familiar symplectic techniques of mechanics take the form of instantaneous Hamiltonian formalism on an infinite-dimensional phase space. The finite-dimensional covariant Hamiltonian approach to field theory is vigorously developed from the seventies in its multisymplectic and polysymplectic variants.\textsuperscript{1−3} Its final purpose is the covariant Hamiltonian quantization of field theory.

In the framework of this approach, one deals with the following types of PDEs: Euler–Lagrange and Cartan equations in the Lagrangian formalism, Hamilton–De Donder equations in multisymplectic Hamiltonian formalism, covariant Hamilton equations and restricted Hamilton equations in polysymplectic Hamiltonian formalism. If a Lagrangian is hyperregular, all these PDEs are equivalent. The present work addresses degenerate semiregular and almost regular Lagrangians. From the mathematical viewpoint, these notions of degeneracy are particularly appropriate in order to study the relations between the above-mentioned PDEs. From the physical one, Lagrangians of almost all field theories are of these types.

To formulate our results, let us recall briefly some notions. Given a fiber bundle $Y \rightarrow X$, coordinated by $(x^\lambda, y^i)$, a first order Lagrangian $L$ is defined as a horizontal density

$$ L = L\omega : J^1Y \rightarrow \wedge^n T^*X, \quad \omega = dx^1 \wedge \cdots dx^n, \quad n = \dim X, $$

(1)

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on the affine jet bundle $J^1Y \to Y$, provided with the adapted coordinates $(x^\lambda, y^i, y_\lambda^i)$. $J^1Y$ can be seen as a finite-dimensional configuration space of fields represented by sections of $Y \to X$.

Given a Lagrangian $L$, the associated Euler–Lagrange equations define both the equations of the variational problem on $Y$ for $L$ and the kernel of the Euler–Lagrange operator, which can be also introduced in an intrinsic way as a coboundary element of the variational cochain complex.

The Cartan equations characterize the variational problem on $J^1Y$ for the Poincaré–Cartan form $H_L$, which is a horizontal Lepagean equivalent of $L$ on $J^1Y \to Y$, i.e. $L = h_0(H_L)$, where $h_0$ is the horizontal projection (14). At the same time, the Cartan equations can be seen both as the kernel of the Euler–Lagrange–Cartan operator and the Hamilton equations of the Lagrangian polysymplectic structure on $J^1Y$. The Cartan equations are the Lagrangian counterpart of covariant Hamilton equations.

The Hamilton–De Donder equations and the covariant Hamilton equations are related to two different Legendre morphisms in the first order calculus of variations.

Firstly, every Poincaré–Cartan form $H_L$ yields the Legendre morphism $\widehat{H}_L$ of $J^1Y$ to the homogeneous Legendre bundle

$$Z_Y = J^1*Y = T^*Y \wedge (\wedge^{n-1}T^*X)$$

which is the affine $n+1$-valued dual of $J^1Y \to Y$, and is treated as a homogeneous finite-dimensional phase space of fields. $Z_Y$ is provided with the canonical exterior $n$-form $\Xi_Y$ (28) and the multisymplectic form $d\Xi_Y$. If $\widehat{H}_L(J^1Y)$ is an imbedded subbundle of $Z_Y \to Y$, the pull-back of $\Xi_Y$ yields the Hamilton–De Donder equations on $\widehat{H}_L(J^1Y)$. If a Lagrangian $L$ is almost regular, these equations are quasi-equivalent to the Cartan equations, i.e., there is a surjection of the set of solutions of the Cartan equations onto that of the Hamilton–De Donder equations.1

Secondly, every Lagrangian $L$ defines the Legendre map $\widehat{L}$ of $J^1Y$ to the Legendre bundle

$$\Pi = \wedge^nT^*X \otimes V^*Y \otimes TX,$$

provided with the holonomic coordinates $(x^\lambda, y^i, p_\lambda^i)$, which can be seen as a finite-dimensional momentum phase space of fields.5–9 The relationship between multisymplectic and polysymplectic phase spaces is given by the exact sequence

$$0 \longrightarrow \Pi \times \wedge^nT^*X \hookrightarrow Z_Y \longrightarrow \Pi \longrightarrow 0,$$

where

$$\pi_{Z\Pi} : Z_Y \to \Pi$$

2
is a 1-dimensional affine bundle. Given any section \( h \) of \( Z_Y \to \Pi \), the pull-back

\[
H = h^*\Xi_Y = p^*_\lambda dy^i \wedge \omega_\lambda - \mathcal{H}\omega
\]

is a polysymplectic Hamiltonian form on \( \Pi \).\(^2^4\) The Legendre \( \Pi \) is equipped with the canonical polysymplectic form \( \Omega_Y \) (33).\(^2^3\) This form differs from those in Refs. [4-6], and is globally defined. With the polysymplectic form \( \Omega_Y \), one introduces Hamiltonian connections and covariant Hamilton equations

\[
\begin{align*}
y^i_\lambda &= \partial_i \mathcal{H}, \\
p^\lambda_{\lambda i} &= -\partial_i \mathcal{H},
\end{align*}
\]

which are PDEs on the phase space \( \Pi \) defined by the kernel of the Hamilton operator \( \mathcal{E}_H \) (43). If \( X = \mathbb{R} \), covariant Hamiltonian formalism provides the adequate Hamiltonian formulation of time-dependent mechanics.\(^1^0^,\!^1^1\) In this case, \( Z_Y = T^*Y \) and \( \Pi = V^*Y \) are the homogeneous and momentum phase spaces of time-dependent mechanics, respectively.

It should be emphasized that a Hamiltonian form \( H \) (6) is the Poincaré–Cartan form of the Lagrangian

\[
L_H = h_0(H) = (p^\lambda_{\lambda i} y^i_\lambda - \mathcal{H})\omega
\]

on the jet manifold \( J^1\Pi \). It is the the Poincaré–Cartan form (17) of this Lagrangian. It follows that the Euler–Lagrange operator (19) for \( L_H \) is precisely the Hamilton operator \( \mathcal{E}_H \) (43) for \( H \) and, consequently, the Euler–Lagrange equations for \( L_H \) are equivalent to the Hamilton equations for \( H \). The Lagrangian \( L_H \) plays a prominent role in the path integral approach to quantization of Hamiltonian systems.\(^1^2^–^1^4\)

The results of this paper demonstrate that polysymplectic Hamiltonian formalism is not equivalent to the Lagrangian one, but can provide the adequate description of degenerate field systems which do not necessarily possess gauge symmetries.

We show that, if \( r : X \to \Pi \) is a solution of the Hamilton equations for a Hamiltonian form \( H \) associated with a semiregular Lagrangian \( L \) and if \( r \) lives in the Lagrangian constraint space \( \hat{L}(J^1Y) \), then the projection of \( r \) onto \( Y \) is a solution of the Euler–Lagrange equations for \( L \). The converse assertion is more intricate. One needs a complete set of associated Hamiltonian forms in order to exhaust all solutions of the Euler–Lagrange equations for a degenerate Lagrangian. It follows that the covariant Hamilton equations contain additional conditions in comparison with the Euler–Lagrange ones. In the case of almost regular Lagrangians, one can introduce the constrained Hamilton equations. They are weaker than the Hamilton equations restricted to the Lagrangian constraint space, are equivalent to the Hamilton–De Donder equations and, consequently, are quasi-equivalent to the Cartan equations.

The detailed analysis of degenerate quadratic Lagrangian systems in Section VI is appropriate for application to many physical models. We find a complete set of associated
Hamiltonian forms. The key point is the splitting of the configuration space $J^1Y$ into the dynamic sector and the gauge one coinciding with the kernel of the Legendre map $\hat{L}$. As an immediate consequence of this splitting, one can separate a part of the Hamilton equations independent of momenta which play the role of gauge-type conditions, while other equations restricted to the Lagrangian constraint space coincide with the constrained Hamilton equations, and are quasi-equivalent to the Cartan equations.

Thus, we observe that the main features in gauge theory are not directly related to the gauge invariance condition, but are common in all field models with degenerate quadratic Lagrangians. The important peculiarity of the Hamiltonian description of these models lies in the fact that, in comparison with a Lagrangian, an associated Hamiltonian form $H$ and the Lagrangian $L_H$ (8) contain gauge fixing terms. Therefore we will construct the BRST extension of the Hamiltonian form $H$ (6) and the Lagrangian $L_H$ (8) in order to provide them with symmetries which lead, e.g., to the corresponding Slavnov identities under quantization. This is a preliminary step towards the covariant Hamiltonian quantization of degenerate systems.

A natural idea of the covariant Hamilton quantizations is also to generalize the Poisson bracket in symplectic mechanics to multisymplectic or polysymplectic manifolds and then to quantize it. The main difficulty is that the bracket must be globally defined. Let us note that multisymplectic manifolds such as $(Z_Y, d\Xi_Y)$, look rather promising for algebraic constructions since multisymplectic forms are exterior forms. Nevertheless, the above mentioned $X = \mathbb{R}$ reduction of the covariant Hamiltonian formalism leads to time-dependent mechanics, but not conservative symplectic mechanics. In this case, the momentum phase space $V^*Y$, coordinated by $(t, y^i, p_i = \dot{y}_i)$, is endowed with the canonical degenerate Poisson structure given by the bracket

\[ \{ f, g \}_V = \partial^i f \partial_i g - \partial^i g \partial_i f, \quad f, g \in C^\infty(V^*Y). \] (9)

However, the Poisson bracket $\{ \mathcal{H}, f \}_V$ of a Hamiltonian $\mathcal{H}$ and functions $f$ on the momentum phase space $V^*Y$ fails to be a well-behaved entity because $\mathcal{H}$ is not a scalar with respect to time-dependent transformations. In particular, the equality $\{ \mathcal{H}, f \}_V = 0$ is not preserved under such transformations. As a consequence, the evolution equation in time-dependent mechanics is not reduced to a Poisson bracket. At the same time, the Poisson bracket (9) leads to the following current algebra bracket. Let $u = u^i \partial_i$ be a vertical vector field on $Y \rightarrow \mathbb{R}$, and $J_u = u^i p_i$ the corresponding symmetry current on $V^*Y$ along $u$. The symmetry currents $J_u$ constitute a Lie algebra with respect to the bracket

\[ [J_u, J_{u'}] = \{ J_u, J_{u'} \}_V = J_{[u, u']}. \]

This current algebra bracket can be extended to the general polysymplectic case as follows.
There is the canonical isomorphism
\[ \theta = p^\lambda dy^i \wedge \omega_\lambda : \Pi \to V^*Y \wedge (\wedge^{n-1} T^*X). \]

Let \( u = u^i \partial_i \) be a vertical vector field on \( Y \to X \). The corresponding symmetry current (70) is a horizontal exterior \((n-1)\)-form
\[ J_u = u^i p^\lambda_i \omega_\lambda \] (10)
on the Legendre bundle \( \Pi \) (3). The symmetry currents (10) constitute a Lie algebra with respect to the bracket
\[ [J_u, J_{u'}] \overset{\text{def}}{=} J_{[u, u']}. \] (11)

If \( Y \to X \) is a vector bundle and \( X \) is provided with a non-degenerate metric \( g \), the bracket (11) can be extended to any horizontal \((n-1)\)-forms \( \phi = \phi^\alpha \omega_\alpha \) on \( \Pi \) by the law
\[ [\phi, \sigma] = g_{\alpha \beta} g^{\mu \nu} (\partial^i_\mu \phi^\sigma \partial_i \sigma^\beta - \partial^i_\mu \sigma^\beta \partial_i \phi^\alpha) \omega_\nu. \]

Similarly, the bracket of horizontal 1-forms on \( \Pi \) is defined.\(^{11}\) The bracket (11) looks promising for the current algebra quantization of the covariant Hamiltonian formalism. We will use this bracket in order to construct the algebra of supercurrents in the BRST extended Hamiltonian formalism.

Note that, since the above mentioned Poisson bracket \{\( \mathcal{H}, f \}\}_V \) is not preserved under time-dependent transformations, the standard BRST technique, based on the Lie algebra of constraints, can not be applied in a straightforward manner to time-dependent mechanics and covariant Hamiltonian field theory. We generalize the BRST mechanics of E.Gozzi and M.Reuter\(^{11,12,19}\) in the terms of simple graded manifolds.

II. TECHNICAL PRELIMINARIES

All maps throughout the paper are smooth, while manifolds are real, finite-dimensional, Hausdorff, second-countable and connected. A base manifold \( X \) is oriented.

Given a fiber bundle \( Y \to X \) coordinatized by \((x^\lambda, y^i)\), the \( s \)-order jet manifold \( J^sY \) is endowed with the adapted coordinates \((x^\lambda, y^i)\), \( 0 \leq |\Lambda| \leq s \), where \( \Lambda \) is a symmetric multi-index \((\lambda_k...\lambda_1)\), \( |\Lambda| = k \). The repeated jet manifold \( J^1J^1Y \) is coordinatized by \((x^\lambda, y^i, \tilde{y}^\lambda_\lambda, y^i_\lambda, y^i_{\lambda\mu})\). There are the canonical morphisms
\[ \lambda = dx^\lambda \otimes (\partial_\lambda + y^i_\lambda \partial_i) : J^1Y \to T^*X \otimes TY; \] (12)
\[ S_1 = (\tilde{y}^\lambda_\lambda - y^i_\lambda)dx^\lambda \otimes \partial_i : J^1J^1Y \to T^*X \otimes VY. \] (13)
Exterior forms $\phi$ on a manifold $J^sY$, $s = 0, 1, \ldots$, are naturally identified with their pull-backs onto $J^{s+1}Y$. There is the exterior algebra homomorphism, called the horizontal projection,

$$h_0: \phi_\lambda dx^\lambda + \phi^i_A dy^i_A \mapsto \phi_\lambda dx^\lambda + \phi^i_A y^i_A dx^\lambda$$

(14)

which sends exterior forms on $J^sY$ onto the horizontal forms on $J^{s+1}Y \to X$, and vanishes on the contact forms $\theta^\lambda_i = dy^i - y^i_\lambda + \Lambda^i dx^\lambda$. Note that the horizontal projection $h_0$ and the pull-back operation with respect to bundle morphisms over $X$ mutually commute. Recall also the operators of the total derivative $d_\lambda = \partial_\lambda + y^i_\lambda \partial_i + \partial_j \Gamma^i_\lambda \partial_j y^i + \cdots$,

and the horizontal differential $d_H \phi = dx^\lambda \wedge d_\lambda \phi$ such that $h_0 \circ d = d_H \circ h_0$.

We regard a connection on a fiber bundle $Y \to X$ as a global section

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i)$$

(15)

of the affine jet bundle $\pi_1^1: J^1Y \to Y$. Sections of the underlying vector bundle $T^*X \otimes VY \to Y$ are called soldering forms. Every connection $\Gamma$ on a fiber bundle $Y \to X$ gives rise to the connection

$$V\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i + \partial_j \Gamma^i_\lambda \partial_j y^i \frac{\partial}{\partial y^i})$$

(16)

on the fiber bundle $VY \to X$.

### III.LAGRANGIAN DYNAMICS

We follow the first variational formula of the calculus of variations.\(^3\) Given a Lagrangian $L$ and its Lepagean equivalent $H_L$, this formula provides the canonical decomposition of the Lie derivative of $L$ along a projectable vector field $u$ on $Y$ in accordance with the variational problem. We restrict our consideration to the Poincaré–Cartan form

$$H_L = L\omega + \pi^i_\lambda \theta^i \wedge \omega^\lambda, \quad \pi^i_\lambda = \partial^i_\lambda L, \quad \omega^\lambda = \partial_\lambda [\omega].$$

(17)

In contrast with other Lepagean equivalents, $H_L$ is a horizontal form on the affine jet bundle $J^1Y \to Y$. Moreover, it is the Lagrangian counterpart of polysymplectic Hamiltonian forms (see the relations (51) and (53) below). The first variational formula reads

$$L_{J^1u}L = u_V [E_L + d_H h_0(u] H_L),$$

(18)

where $u_V = (u | \theta^i) \partial_i$ and

$$E_L = (\partial_i - d_\lambda \partial^i_\lambda) L \theta^i \wedge \omega: J^2Y \to T^*Y \wedge (\Lambda^i T^*X)$$

(19)
is the Euler–Lagrange operator associated with $L$. The kernel of $\mathcal{E}_L$ defines the Euler–Lagrange equations on $Y$ given by the coordinate relations

\[(\partial_i - d\lambda \partial^\lambda)\mathcal{L} = 0.\] (20)

Solutions of these equations are critical sections of the variational problem for the Lagrangian $L$.

Remark 1: The first variational formula (18) also provides the Lagrangian conservation laws.\textsuperscript{3,21} On-shell, we have the weak identity

\[L_{J^1u}L \approx d_H h_0(u\mid H_L),\]

and, if $L_{J^1u}L = 0$, the weak conservation law

\[0 \approx d_H h_0(u\mid H_L) = -d\lambda T^\lambda \omega\] (21)

of the symmetry current

\[T = -h_0(u\mid H_L) = T^\lambda \omega_\lambda = -[\pi_\lambda^i (u^\mu y^\mu_i - u^i) - u^\lambda \mathcal{L}] \omega_\lambda\] (22)

along the vector field $u$.

Instead of the variational problem on $Y$ for a Lagrangian $L$, one can consider that on $J^1Y$ for the Poincaré–Cartan form $H_L$ (17). Critical sections $s: X \to J^1Y$ of this variational problem satisfy the relation

\[s^*(u\mid dH_L) = 0\] (23)

for all vertical vector fields $u$ on $J^1Y \to X$. This relation defines the Cartan equations on $J^1Y$. We regain these equations in another way.\textsuperscript{3}

Let us consider the above-mentioned Legendre map

\[\hat{L}: J^1Y \to \Pi, \quad p_\lambda^\lambda \circ \hat{L} = \pi_\lambda^\lambda.\]

The Legendre bundle $\Pi$ (3) is equipped with the canonical tangent-valued Liouville form $\theta$ (32). Its pull-back on $J^1Y$ by $\hat{L}$ is

\[\theta_L = \hat{L}^* \theta = -\pi_\lambda^i dy^i \wedge \omega \otimes \partial_\lambda.\]

We construct the reduced Lagrangian

\[\overline{\mathcal{L}} = L - S_1 \theta_L = (\mathcal{L} + (\tilde{y}_\lambda^i - y_\lambda^i) \pi_\lambda^i) \omega\] (24)
on $J^1 J^1 Y$ (see the notation (13)). The associated Euler–Lagrange operator, called the Euler–Lagrange–Cartan operator for $L$, reads

$$
\mathcal{E}_L : J^1 J^1 Y \to T^* J^1 Y \wedge (\wedge^n T^* X),
$$

\begin{align*}
\mathcal{E}_L &= \left[ (\partial_i \mathcal{L} - \tilde{d}_i \pi^i \lambda + \partial_i \pi_{\lambda}^i (\tilde{y}_j^i - y_j^i)) \, dy^i + \partial_i \pi^i_\mu (\tilde{y}_j^\mu - y_j^\mu) \, dy^i \right] \wedge \omega,
\end{align*}

$$
\tilde{d}_\lambda = \partial_\lambda + \tilde{y}_\lambda^i \partial_i + y_{\lambda \mu}^i \partial_\mu.
$$

This is the Lagrangian counterpart of the polysymplectic Hamilton operator (see the relation (55) below). Its kernel $\text{Ker} \mathcal{E}_L \subset J^1 J^1 Y$ is given exactly by the Cartan equations

\begin{align*}
\partial_{\lambda} \pi^i_\mu (\tilde{y}_j^\mu - y_j^\mu) &= 0, \\
\partial_i \mathcal{L} - \tilde{d}_i \pi^i \lambda + (\tilde{y}_j^i - y_j^i) \partial_i \pi^i_\lambda &= 0.
\end{align*}

Since $\mathcal{E}_L |_{J^2 Y} = \mathcal{E}_L$, the Cartan equations (26a) – (26b) are equivalent to the Euler–Lagrange equations (20) on integrable sections of $J^1 Y \to X$. These equations are equivalent in the case of regular Lagrangians.

With the Poincaré–Cartan form $H_L (17)$, we have the Legendre morphism

$$
\bar{H}_L : J^1 Y \to Z_Y, \quad (p_i^\mu, p) \circ \bar{H}_L = (\pi_i^\mu, \mathcal{L} - \pi_i^\mu y_i^\mu),
$$

where the fiber bundle $Z_Y (2)$ is endowed with holonomic coordinates $(x^\lambda, y^i, p_i^\lambda, p)$. It is readily observed that

$$
\bar{L} = \pi_{Z \Pi} \circ \bar{H}_L.
$$

Owing to the monomorphism $Z_Y \hookrightarrow \wedge^n T^* Y$, the bundle $Z_Y$ is equipped with the pull-back

$$
\bar{\Xi}_Y = p \omega + p_i^\lambda dy^i \wedge \omega_\lambda
$$

of the canonical form $\Theta$ on $\wedge^n T^* Y$ whose exterior differential $d\Theta$ is the $n$-multisymplectic form in the sense of Martin.22

Let $Z_L = \bar{H}_L (J^1 Y)$ be an imbedded subbundle $i_L : Z_L \hookrightarrow Z_Y$ of $Z_Y \to Y$. It is provided with the pull-back De Donder form $i_L^* \Xi_Y$. We have

$$
H_L = \bar{H}_L^* \Xi_L = \bar{H}_L^* (i_L^* \Xi_Y).
$$

By analogy with the Cartan equations (23), the Hamilton–De Donder equations for sections $r$ of $Z_L \to X$ are written as

$$
r^* (u \bigwedge d \Xi_L) = 0
$$

where $u$ is an arbitrary vertical vector field on $Z_L \to X$. To obtain an explicit form of these equations, one should substitute solutions $(y^{\lambda}_i, \mathcal{L})$ of the equations

$$
p_i^\lambda = \pi_i^\lambda, \quad p = \mathcal{L} - \pi_i^\lambda y_i^\lambda
$$

(31)
in the Cartan equations. However, if a Lagrangian $L$ is degenerate, the equations (31) may admit different solutions or no solution at all. Something more is said in the following theorem.\footnote{Theorem 1: Let the Legendre morphism $\tilde{H}_L \colon J^1 Y \rightarrow Z_L$ be a submersion. Then a section $\bar{s}$ of $J^1 Y \rightarrow X$ is a solution of the Cartan equations (23) if and only if $\tilde{H}_L \circ \bar{s}$ is a solution of the Hamilton–De Donder equations (30), i.e., Cartan and Hamilton–De Donder equations are quasi-equivalent.}

**IV. COVARIANT HAMILTONIAN DYNAMICS**

Given a fiber bundle $Y \rightarrow X$, let $\Pi$ be the Legendre bundle (3). Holonomic coordinates $(x^\lambda, y^i, p_\lambda^i)$ on $\Pi$ are compatible with its composite fibration

$$\pi_{\Pi X} = \pi \circ \pi_{\Pi Y} : \Pi \rightarrow Y \rightarrow X.$$ We have the canonical bundle monomorphism

$$\theta = -p_\lambda^i dy^i \wedge \omega \otimes \partial_\lambda : \Pi \leftarrow_{Y}^{n + 1} T^* Y \otimes T X,$$

(32)
called the tangent-valued Liouville form on $\Pi$. It should be emphasized that the exterior differential $d$ can not be applied to the tangent-valued form (32). At the same time, there is a unique $T X$-valued $(n + 2)$-form $\Omega_Y$ on $\Pi$ such that the relation

$$\Omega_Y \cdot \phi = -d(\theta \cdot \phi)$$

holds for any exterior 1-form $\phi$ on $X$.\footnote{This form, called the polysymplectic form, is given by the coordinate expression}

$$\Omega_Y = dp_\lambda^i \wedge dy^i \wedge \omega \otimes \partial_\lambda.$$

(33)

As was mentioned above, every section $h$ of the fiber bundle (5) defines the pull-back (6) of the canonical form $\Xi_Y$ (28), called a Hamiltonian form on the Legendre bundle $\Pi$.

**Proposition 2:** Hamiltonian forms on $\Pi$ constitute a non-empty affine space modelled over the linear space of horizontal densities $\tilde{H} = \tilde{H}_\omega$ on $\Pi \rightarrow X$.

**Proof:** This is an immediate consequence of the fact that (5) is an affine bundle modelled over the pull-back vector bundle $\Pi \times_{X}^{\wedge} T^* X \rightarrow \Pi$.

**Lemma 3:**\footnote{Every connection $\Gamma$ (15) on $Y \rightarrow X$ yields the splitting}

$$h_\Gamma : \bar{dy}^i \mapsto dy^i - \Gamma_\lambda^i dx^\lambda$$
of the exact sequence (4) and, as a consequence, defines the Hamiltonian form

\[ H_\Gamma = h^-_r \Xi_{\nabla} = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Gamma^i_\lambda \omega. \]  

(34)

Proposition 2 and Lemma 3 lead to the following

**Corollary 4:** Given a connection \( \Gamma \) on \( Y \to X \), every Hamiltonian form \( H \) admits the decomposition

\[ H = H_\Gamma - \tilde{H}_\Gamma = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Gamma^i_\lambda \omega - \tilde{H}_\Gamma \omega. \]  

(35)

**Remark 2:** The physical meaning of the splitting (35) is illustrated by the fact that, in the case of \( X = \mathbb{R} \), \( \tilde{H}_\Gamma \) is exactly the energy of a mechanical system with respect to the reference frame \( \Gamma \).10,11

We will mean by a Hamiltonian map any bundle morphism

\[ \Phi = dx^\lambda \otimes (\partial_\lambda + \Phi^i_\lambda \partial_i) : \Pi \to J^1Y. \]  

(36)

In particular, let \( \Gamma \) be a connection on \( Y \to X \). Then, the composition

\[ \hat{\Gamma} = \Gamma \circ \pi_{IV} = dx^\lambda \otimes (\partial_\lambda + \Gamma^i_\lambda \partial_i) : \Pi \to Y \to J^1Y, \]  

(37)

is a Hamiltonian map. Conversely, every Hamiltonian map \( \Phi \) yields the associated connection \( \Gamma_\Phi = \Phi \circ \hat{0} \) on \( Y \to X \), where \( \hat{0} \) is the global zero section of the Legendre bundle \( \Pi \to Y \). In particular, we have \( \Gamma_{\hat{\Gamma}} = \Gamma \). The following two facts will be used in the sequel.

**Proposition 5:** Every Hamiltonian form \( H \) (6) yields the Hamiltonian map \( \hat{H} \) such that

\[ y^i_\lambda \circ \hat{H} = \partial_\lambda \mathcal{H}. \]  

(38)

**Proposition 6:** Every Hamiltonian map (36) defines the Hamiltonian form

\[ H_\Phi = \Phi \circ \theta = p^i_\lambda dy^i \wedge \omega_\lambda - p^i_\lambda \Phi^i_\lambda \omega. \]  

Proof: Given an arbitrary connection \( \Gamma \) on the fiber bundle \( Y \to X \), the corresponding Hamiltonian map (37) defines the form \( \hat{\Gamma} \circ \theta \) which is exactly the Hamiltonian form \( H_\Gamma \) (34). Since \( \Phi - \hat{\Gamma} \) is a \( VY \)-valued basic 1-form on \( \Pi \to X \), \( H_\Phi - H_\Gamma \) is a horizontal density on \( \Pi \). Then the result follows from Proposition 2. Note that \( H = H_\tilde{H} \) iff \( H = H_\Gamma \) (34).
Let $J^1\Pi$ be the first order jet manifold of $\Pi \to X$. It is equipped with the adapted coordinates $(x^\lambda, y^i, p^\lambda_i, y_\mu^i, p^\lambda_\mu_i)$ such that $y_\mu^i \circ J^1\pi \Pi = p_\mu^i$. A connection
\[ \gamma = dx^\lambda \otimes (\partial_\lambda + \gamma^i_\lambda \partial_i + \gamma^\mu_\lambda_i \partial_\mu) \] (39)
on $\Pi \to X$ is called a Hamiltonian connection if the exterior form $\gamma]\Omega_Y$ is closed. A Hamiltonian connection $\gamma$ is said to be associated with a Hamiltonian form $H$ if it obeys the condition
\[ \gamma]\Omega_Y = dH, \] (40)
\[ \gamma^\lambda_\lambda = \partial_\lambda H, \quad \gamma^\lambda_\lambda = -\partial_\lambda H. \] (41)

**Theorem 7:** For every Hamiltonian connection $\gamma$, there exists a local Hamiltonian form $H$ on a neighbourhood of any point $q \in \Pi$ such that the equation (40) holds.

**Proof:** If $\gamma]\Omega_Y$ is closed, there is a contractible neighbourhood $U$ of a point $q \in \Pi$ which belongs to a holonomic coordinate chart $(x^\lambda, y^i, p^\lambda_i)$ and where the local form $\gamma]\Omega_Y$ is exact. We have
\[ \gamma]\Omega_Y = dH = dp^\lambda_i \wedge dy^i \wedge \omega_\lambda - (\gamma^\lambda_\lambda dp^\lambda_i - \gamma^\lambda_\lambda dy^i) \wedge \omega \] (42)
on $U$. It is readily observed that the second term in the right-hand side of this equality is also an exact form on $U$. By virtue of the relative Poincaré lemma, it can be brought into the form $dH \wedge \omega$ where $H$ is a local function on $U$. Then the form $H$ in the expression (42) reads
\[ H = p^\lambda_i dy^i \wedge \omega_\lambda - H \omega. \]

Using Corollary 4, one can easily show that this is a Hamiltonian form on $U$.

**Theorem 8:** Every Hamiltonian form has an associated Hamiltonian connection.

**Proof:** Given a Hamiltonian form $H$, let us consider the first order differential operator
\[ \mathcal{E}_H : J^1\Pi \to T^*\Pi \wedge (\lambda T^*X), \]
\[ \mathcal{E}_H = dH - \lambda]\Omega_Y = [(y_\lambda_\lambda - \partial_\lambda H)dp^\lambda_i - (p^\lambda_\lambda + \partial_\lambda H)dy^i] \wedge \omega, \] (43)
on $\Pi$ where $\lambda$ is the canonical monomorphism (12). It is called the Hamilton operator associated with $H$. A glance at the expression (43) shows that this operator is an affine morphism over $\Pi$ of constant rank. It follows that its kernel is an affine closed imbedded subbundle of the jet bundle $J^1\Pi \to \Pi$. This subbundle has a global section $\gamma$ which is a connection on $\Pi \to X$. This connection obeys the equation (40).
It should be emphasized that, if \( n > 1 \), there is a set of Hamiltonian connections associated with the same Hamiltonian form \( H \). They differ from each other in soldering forms \( \sigma \) on \( \Pi \to X \) which obey the equation \( \sigma [\Omega_Y] = 0 \).

**Proposition 9:** Every Hamiltonian connection \( \gamma \) associated with a Hamiltonian form \( H \) satisfies the relation

\[
J^1\pi_{\Pi Y} \circ \gamma = \tilde{H}.
\]  

**Proof:** The proof is based on the expressions (38) and (41).

Being a closed subbundle of the jet bundle \( J^1\Pi \to X \), the kernel of the Hamilton operator \( E_H \) (43) defines first order Hamilton equations (7a) – (7b) on the Legendre bundle \( \Pi \). Every integral section \( J^1r = \gamma \circ r \) of a Hamiltonian connection \( \gamma \) associated with a Hamiltonian form \( H \) is obviously a solution of the Hamilton equations (7a) – (7b). Conversely, a solution of the Hamilton equations (7a) – (7b) is a section \( r \) of \( \Pi \to X \) such that its jet prolongation \( J^1r \) lives in \( \text{Ker} E_H \). If \( r : X \to \Pi \) is a global solution, there exists an extension of the local section \( J^1r : r(X) \to J^1\Pi \) to a Hamiltonian connection which has \( r \) as an integral section. Substituting \( J^1r \) in (44), we obtain the equality

\[
J^1(\pi_{\Pi Y} \circ r) = \tilde{H} \circ r,
\]

which is the coordinate-free form of the Hamilton equations (7a). Nevertheless, it may happen that the Hamilton equations have no solution through a given point \( q \in \Pi \).

**Remark 3:** The Hamilton equations can be introduced without appealing to the Hamilton operator. As was for the Cartan equations (23), they are equivalent to the condition

\[
r^*(u_j dH) = 0
\]

for any vertical vector field \( u \) on \( \Pi \to X \).

**V. LAGRANGIAN AND HAMILTONIAN DEGENERATE SYSTEMS**

Let us study the relations between Hamilton and Euler–Lagrange equations when a Lagrangian is degenerate. Their main peculiarity lies in the fact that there is a set of Hamiltonian forms associated with the same degenerate Lagrangian.

**Remark 4:** Let us recall the coordinate expressions

\[
(y^{\mu}_i, \tilde{y}^i_{\lambda}, y_{\lambda\mu}) \circ J^1\tilde{H} = (\partial^{\mu}_i \mathcal{H}, y^{\lambda}_i, d_\lambda \partial^i_\mu \mathcal{H}),
\]

\[
(p^\lambda_i, y^{\mu}_i, p^\mu_i) \circ J^1\tilde{L} = (\pi^{\lambda}_i, \tilde{y}^i_{\mu}, \tilde{d}_\mu \pi^i_\lambda).
\]
In particular, if $\gamma$ is a Hamiltonian connection for a Hamiltonian form $H$, we obtain from (45) and (47) that the composition $J^1\tilde{H} \circ \gamma$ takes its values into the sesquiholonomic jet bundle $\tilde{J}^2Y$.

A Hamiltonian form $H$ is said to be associated with a Lagrangian $L$ if $H$ satisfies the relations

$$\tilde{L} \circ \tilde{H} \circ \tilde{L} = \tilde{L},$$  \hspace{1cm} (49a) \\
$$H = H_{\tilde{H}} + \tilde{H}^*L.$$  \hspace{1cm} (49b)

A glance at the relation (49a) shows that $\tilde{L} \circ \tilde{H}$ is the projector

$$p_i^\mu(q) = \partial_\mu^i L(x^\mu, y^i, \partial_j^\lambda \mathcal{H}(q)), \hspace{1cm} q \in N_L,$$  \hspace{1cm} (50)

from $\Pi$ onto the Lagrangian constraint space $N_L = \tilde{L}(J^1Y)$. Accordingly, $\tilde{H} \circ \tilde{L}$ is the projector from $J^1Y$ onto $\tilde{H}(N_L)$.

**Lemma 10:** Any Hamiltonian form $H$ associated with a Lagrangian $L$ obeys the relation

$$H \mid_{N_L} = \tilde{H}^*L \mid_{N_L},$$  \hspace{1cm} (51)

where $H_L$ is the Poincaré–Cartan form (17).

**Proof:** The relation (49b) takes the coordinate form

$$\mathcal{H} = p_i^\mu \partial_\mu^i \mathcal{H} - \mathcal{L}(x^\mu, y^i, \partial_j^\lambda \mathcal{H}).$$  \hspace{1cm} (52)

Substituting (50) and (52) in (6), we obtain the relation (51).

Something more can be said in the case of semiregular Lagrangians. A Lagrangian $L$ is called semiregular if the pre-image $\tilde{L}^{-1}(q)$ of any point $q \in N_L$ is a connected submanifold of $J^1Y$. The following fact will be used in the sequel.

**Lemma 11:** The Poincaré–Cartan form $H_L$ for a semiregular Lagrangian $L$ is constant on the connected pre-image $\tilde{L}^{-1}(q)$ of any point $q \in N_L$.

**Proof:** Let $u$ be a vertical vector field on the affine jet bundle $J^1Y \to Y$ which takes its values into the kernel of the tangent map $T\tilde{L}$ to $\tilde{L}$. Then $L_u H_L = 0$.

An immediate consequence of this fact is the following assertion.

**Proposition 12:** All Hamiltonian forms associated with a semiregular Lagrangian $L$ coincide with each other on the Lagrangian constraint space $N_L$, and the Poincaré–Cartan form $H_L$ (17) for $L$ is the pull-back

$$H_L = \tilde{L}^*H,$$  \hspace{1cm} (53) \\
$$(\pi_3^\lambda y^i_j - \mathcal{L})\omega = \mathcal{H}(x^\mu, y^i, \pi_j^\mu)\omega,$$
of any such a Hamiltonian form $H$.

**Proof:** Given a vector $v \in T_q \Pi$, the value $T\hat{H}(v)|H_L(\hat{H}(q))$ is the same for all Hamiltonian maps $\hat{H}$ satisfying the relation (49a). Then the results follow from the relation (51).

Proposition 12 enables us to connect Euler–Lagrange and Cartan equations for a semiregular Lagrangian $L$ with the Hamilton equations for Hamiltonian forms associated with $L$.

**Theorem 13:** Let a section $r$ of $\Pi \to X$ be a solution of the Hamilton equations (7a) – (7b) for a Hamiltonian form $H$ associated with a semiregular Lagrangian $L$. If $r$ lives in the constraint space $N_L$, the section $s = \pi_{\Pi Y} \circ r$ of $Y \to X$ satisfies the Euler–Lagrange equations (20), while $\hat{s} = \hat{H} \circ r$ obeys the Cartan equations (26a) – (26b).

**Proof:** Acting by the exterior differential on the relation (53), we obtain the relation

$$
(y^i - \partial_i^\mu \mathcal{H} \circ \hat{L}) d\pi^\mu_i \wedge \omega - (\partial^i \mathcal{L} + \partial_i (\mathcal{H} \circ \hat{L})) dy^i \wedge \omega = 0 \tag{54}
$$

which is equivalent to the system of equalities

$$
\partial^i \pi^\mu_j (y^j - \partial^j \mu \mathcal{H} \circ \hat{L}) = 0,
\partial_i \pi^\mu_j (y^j - \partial^j \mu \mathcal{H} \circ \hat{L}) - (\partial^i \mathcal{L} + (\partial_i \mathcal{H}) \circ \hat{L}) = 0.
$$

Using these equalities and the expression (48), one can easily see that

$$
\mathcal{E}_T = (J^1 \hat{L})^* \mathcal{E}_H, \tag{55}
$$

where $\mathcal{E}_T$ is the Euler–Lagrange–Cartan operator (25). Let $r$ be a section of $\Pi \to X$ which lives in the Lagrangian constraint space $N_L$, and $\hat{s} = \hat{H} \circ r$. Then we have

$$
r = \hat{L} \circ \hat{s}, \quad J^1 r = J^1 \hat{L} \circ J^1 \hat{s}.
$$

If $r$ is a solution of the Hamilton equations, the exterior form $\mathcal{E}_H$ vanishes on $J^1 r(X)$. Hence, the pull-back form $\mathcal{E}_T = (J^1 \hat{L})^* \mathcal{E}_H$ vanishes on $J^1 \hat{s}(X)$. It follows that $\hat{s}$ obeys the Cartan equations (26a) – (26b). We obtain from the equality (45) that $\hat{s} = J^1 s$, $s = \pi_{\Pi Y} \circ r$. Hence, $s$ is a solution of the Euler–Lagrange equations.

The same result can be obtained from the relation

$$
\mathcal{L} = (J^1 \hat{L})^* L_H \tag{56}
$$

where $\mathcal{L}$ is the Lagrangian (24) on $J^1 J^1 Y$ and $L_H$ is the Lagrangian (8) on $J^1 \Pi$.

**Theorem 14:** Given a semiregular Lagrangian $L$, let a section $\hat{s}$ of the jet bundle $J^1 Y \to X$ be a solution of the Cartan equations (26a) – (26b). Let $H$ be a Hamiltonian form associated with $L$, and let $H$ satisfy the relation

$$
\hat{H} \circ \hat{L} \circ \hat{s} = J^1 (\pi^1_0 \circ \hat{s}). \tag{57}
$$
Then, the section \( r = \hat{L} \circ \pi \) of the Legendre bundle \( \Pi \to X \) is a solution of the Hamilton equations (7a) – (7b) for \( H \).

**Proof:** The Hamilton equations (7a) hold by virtue of the condition (57). Substituting \( \hat{L} \circ \pi \) in the Hamilton equations (7b) and using the relations (54) and (57), we come to the Cartan equations (26b) for \( \pi \) as follows:

\[
\hat{d}_\lambda \pi^\lambda \circ \pi + (\partial_i H) \circ \hat{L} \circ \pi = \hat{d}_\lambda \pi^\lambda \circ \pi + (\pi^j_\mu - \partial^j_\mu H \circ \hat{L} \circ \pi) \partial_i \pi^\mu_j \circ \pi - \partial_i \mathcal{L} \circ \pi = \hat{d}_\lambda \pi^\lambda \circ \pi - (\partial_\mu \pi^j - \pi^j_\mu) \partial_i \pi^\mu_j \circ \pi - \partial_i \mathcal{L} \circ \pi = 0.
\]

**Remark 5:** Since \( \hat{H} \circ \hat{L} \) in Theorem (14) is a projection operator, the condition (57) implies that the solution \( \pi \) of the Cartan equations is actually an integrable section \( \pi = J^1 s \) where \( s \) is a solution of the Euler–Lagrange equations. Theorems 13 and 14 show that, if a solution of the Cartan equations provides a solution of the covariant Hamilton equations, it is necessarily a solution of the Euler–Lagrange equations. In fact, the relation (55) gives more than it is needed for proving Theorem 13. Using this relation, one can justify that, if \( \gamma \) is a Hamiltonian connection for a Hamiltonian form \( H \) associated with a semiregular Lagrangian \( L \), then the composition \( J^1 \hat{H} \circ \gamma \circ \hat{L} \) takes its values in \( \text{Ker} \mathcal{E}_\gamma \cap J^2 \hat{Y} \) (see Remark 4), i.e., this is a local sesquiholonomic Lagrangian connection on \( \hat{H}(N_L) \). A converse of this assertion, however, fails to be true in the case of semiregular Lagrangians. Let a Lagrangian \( L \) be hyperregular, i.e., the Legendre map \( \hat{L} \) is a diffeomorphism. Then \( \hat{L}^{-1} \) is a Hamiltonian map, and there is a unique Hamiltonian form

\[
H = H_{\hat{L}^{-1}} + \hat{L}^{-1} * L
\]

associated with \( L \). In this case, both the relation (55) and the converse one

\[
\mathcal{E}_H = (J^1 \hat{H})^* \mathcal{E}_\gamma
\]

hold. It follows that the Euler–Lagrange equations for \( L \) and the Hamilton equations for \( H \) (58) are equivalent.

We will say that a set of Hamiltonian forms \( H \) associated with a semiregular Lagrangian \( L \) is complete if, for each solution \( s \) of the Euler-Lagrange equations, there exists a solution \( r \) of the Hamilton equations for a Hamiltonian form \( H \) from this set such that \( s = \pi_{HY} \circ r \).

By virtue of Theorem 14 and Remark 5, a set of associated Hamiltonian forms is complete if, for every solution \( s \) on \( X \) of the Euler-Lagrange equations for \( L \), there is a Hamiltonian form \( H \) from this set which fulfills the relation

\[
\hat{H} \circ \hat{L} \circ J^1 s = J^1 s.
\]
As for the existence of complete sets of associated Hamiltonian forms, we refer to the following theorem. A Lagrangian $L$ is said to be almost regular if (i) $L$ is semiregular, (ii) the Lagrangian constraint space $N_L$ is a closed imbedded subbundle $i_N : N_L \hookrightarrow \Pi$ of the Legendre bundle $\Pi \rightarrow Y$ and (iii) the Legendre map
\[ \tilde{L} : J^1Y \rightarrow N_L \] is a submersion, i.e., a fibred manifold.

**Proposition 15.** Let $L$ be an almost regular Lagrangian. On an open neighbourhood in $\Pi$ of each point $q \in N_L$, there exist local Hamiltonian forms associated with $L$ which constitute a complete set.

In the case of an almost regular Lagrangian $L$, we can say something more on the relations between Lagrangian and Hamiltonian systems as follows. Let us assume that the fibred manifold (60) admits a global section $\Psi$. Let us consider the pull-back
\[ H_N = \Psi^* H_L, \] called the constrained Hamiltonian form. By virtue of Lemma 11, it is uniquely defined for all sections of the fibred manifold $J^1Y \rightarrow N_L$, and $H_L = \tilde{L}^* H_N$. For sections $r$ of the fiber bundle $N_L \rightarrow X$, we can write the constrained Hamilton equations
\[ r^*(u_N \rfloor_{N_L} dH_N) = 0 \] (62)
where $u_N$ is an arbitrary vertical vector field on $N_L \rightarrow X$. These equations possess the following important properties.

**Theorem 16:** For any Hamiltonian form $H$ associated with an almost regular Lagrangian $L$, every solution $r$ of the Hamilton equations which lives in the Lagrangian constraint space $N_L$ is a solution of the constrained Hamilton equations (62).

**Proof:** Such a Hamiltonian form $H$ defines the global section $\Psi = \tilde{H} \circ i_N$ of the fibred manifold (60). Due to the relation (53), $H_N = i_N^* H$ and the constrained Hamilton equations can be written as
\[ r^*(u_N \rfloor_{N_L} di_N^* H) = r^*(u_N \rfloor_{N_L} dH |_{N_L}) = 0. \] (63)
Note that they differ from the Hamilton equations (46) restricted to $N_L$ which read
\[ r^*(u \rfloor_{N_L} dH |_{N_L}) = 0, \] (64)
where $r$ is a section of $N_L \rightarrow X$ and $u$ is an arbitrary vertical vector field on $\Pi \rightarrow X$. A solution $r$ of the equations (64) satisfies obviously the weaker condition (63).
Theorem 17: The constrained Hamilton equations (62) are equivalent to the Hamilton–De Donder equations (30).

Proof: In accordance with the relation (27), the projection $\pi_{Z\Pi}$ (5) yields a surjection of $Z_L$ onto $N_L$. Given a section $\Psi$ of the fibred manifold (60), we have the morphism

$$\widehat{H}_L \circ \Psi : N_L \rightarrow Z_L.$$ 

By virtue of Lemma (11), this is a surjection such that

$$\pi_{Z\Pi} \circ \widehat{H}_L \circ \Psi = \text{Id}_{N_L}.$$ 

Hence, $\widehat{H}_L \circ \Psi$ is a bundle isomorphism over $Y$ which is independent of the choice of a global section $\Psi$. Combining (29) and (61) gives

$$H_N = (\widehat{H}_L \circ \Psi)^* \Xi_L$$

that leads to the desired equivalence.

The above proof gives more. Namely, since $Z_L$ and $N_L$ are isomorphic, the Legendre morphism $H_L$ fulfills the conditions of Theorem 1. Then combining Theorem 1 and Theorem 17, we obtain the following theorem.

Theorem 18: Let $L$ be an almost regular Lagrangian such that the fibred manifold (60) has a global section. A section $\bar{s}$ of the jet bundle $J^1 Y \rightarrow X$ is a solution of the Cartan equations (23) iff $\hat{L} \circ \bar{s}$ is a solution of the constrained Hamilton equations (62).

Theorem 18 is also a corollary of the following Lemma 19. The constrained Hamiltonian form $H_N$ (61) defines the constrained Lagrangian

$$L_N = h_0(H_N) = (J^1 i_N)^* H_H$$

on the jet manifold $J^1 N_L$ of the fiber bundle $N_L \rightarrow X$.

Lemma 19: There are the relations

$$\overline{L} = (J^1 \hat{L})^* L_N, \quad L_N = (J^1 \Psi)^* \overline{L},$$

where $\overline{L}$ is the Lagrangian (24).

Proof: The first of the relations (66) is an immediate consequence of the relation (56). The latter follows from the expression (47) and the relation (50) if we put $\Psi = \widehat{H} \circ i_N$ for some Hamiltonian form $H$ associated with the almost regular Lagrangian $L$.

The Euler–Lagrange equation for the constrained Lagrangian $L_N$ (65) are equivalent to the constrained Hamilton equations and, by virtue of Lemma 19, are quasi-equivalent to the
Cartan equations. At the same time, Cartan equations of degenerate Lagrangian systems contain an additional freedom in comparison with the restricted Hamilton equations (see the next Section).

The correspondence between Lagrangian and covariant Hamiltonian dynamics of classical fields can be extended to symmetry currents and conservation laws as follows. Given a projectable vector field $u$ on a fiber bundle $Y \to X$ and its lift

$$\tilde{u} = u^\mu \partial_\mu + u^i \partial_i + (-\partial_i u^\mu p^\lambda_j - \partial_\mu u^\mu p^\lambda_i + \partial_\mu u^\lambda p^\mu_i)\partial^\lambda_i$$

onto the Legendre bundle $\Pi$, we obtain

$$L_{\tilde{u}}H = L_{\tilde{u}}L_H,$$

i.e., the Hamiltonian form $H$ (6) and the Lagrangian $L_H$ (8) have the same symmetries. If the Lie derivatives (68) vanish, the corresponding symmetry current

$$J_u = h_0(\tilde{u}|H)|_{y_\lambda^i = \partial^\lambda_i H}$$

is conserved on-shell. In particular, if $u$ is a vertical vector field, we have

$$J_u = \tilde{u}|H = u|H.$$  

Proposition 20: Let a Hamiltonian form $H$ be associated with a semiregular Lagrangian $L$. Let $r$ be a solution of the Hamilton equations for $H$ which lives in the Lagrangian constraint space $N_L$. Let $s = \pi_{\Pi Y} \circ r$ be the corresponding solution of the Euler–Lagrange equations for $L$ so that the relation (59) holds. Then, for any projectable vector field $u$ on the fiber bundle $Y \to X$, we have

$$J_u(r) = T_u(\tilde{H} \circ r), \quad J_u(\tilde{L} \circ J^1 s) = T_u(s),$$

where $T$ is the current (22) on $J^1 Y$ and $J$ is the current (69) on $\Pi$.

It follows that the constrained Hamilton equations have symmetries of the Euler–Lagrange equations. At the same time, given a Hamiltonian form $H$ associated with a degenerate Lagrangian $L$, the Lagrangian $L_H$ (8) contains gauge fixing terms in comparison with $L$ and the Lagrangian $L_N$ (65) (see the next Section).

VI. QUADRATIC DEGENERATE SYSTEMS

This Section is devoted to the physically important case of almost regular quadratic Lagrangians. Given a fiber bundle $Y \to X$, let us consider a quadratic Lagrangian $L$ which has the coordinate expression

$$L = \frac{1}{2} a_{ij}^\lambda y^i_\lambda y^j_\mu + b_i^\lambda y^i_\lambda + c,$$
where $a$, $b$ and $c$ are local functions on $Y$. This property is coordinate-independent due to the affine transformation law of the coordinates $y_i^\lambda$. The associated Legendre map

$$p_i^\lambda \circ \hat{L} = a^\lambda_{ij} y_j^i$$

is an affine morphism over $Y$. It defines the corresponding linear morphism

$$\mathcal{L} : T^*X \otimes VY \to \Pi, \quad p_i^\lambda \circ \mathcal{L} = a^\lambda_{ij} \overline{y}_\mu^i,$$

where $\overline{y}_\mu^i$ are bundle coordinates on the vector bundle $T^*X \otimes VY$.

**Lemma 21:** The Lagrangian (71) is semiregular.

**Proof:** Solutions $y_\mu^i$ of the pointwise linear algebraic equations (72) form an affine space modelled over the linear space of solutions of the linear algebraic equations $a^\lambda_{ij} \overline{y}_\mu^i = 0$. At each point of $N_L$, these spaces are obviously connected.

Let the Lagrangian $L$ (71) be almost regular, i.e., the matrix function $a^\lambda_{ij}$ is of constant rank. Then the Lagrangian constraint space $N_L$ (72) is an affine subbundle of the Legendre bundle $\Pi \to Y$, modelled over the vector subbundle $\overline{N}_L$ (73) of $\Pi \to Y$. Hence, $N_L \to Y$ has a global section. For the sake of simplicity, let us assume that it is the canonical zero section $\hat{0}(Y)$ of $\Pi \to Y$. Then $\overline{N}_L = N_L$. Accordingly, the kernel of the Legendre map (72) is an affine subbundle of the affine jet bundle $J^1Y \to Y$, modelled over the kernel of the linear morphism $\mathcal{L}$ (73). Then there exists a connection

$$\Gamma : Y \to \text{Ker} \hat{L} \subset J^1Y,$$

$$a^\lambda_{ij} \Gamma_j^\mu + b_i^\lambda = 0,$$

on $Y \to X$. Connections (74) constitute an affine space modelled over the linear space of soldering forms $\phi$ on $Y \to X$ satisfying the conditions

$$a^\lambda_{ij} \phi_{\mu}^j = 0$$

and, as a consequence, the conditions $\phi_{\mu}^j b_i^\lambda = 0$. If the Lagrangian (71) is regular, the connection (74) is unique.

**Lemma 22:** There exists a linear bundle map

$$\sigma : \Pi \to T^*X \otimes VY, \quad \overline{y}_\lambda^i \circ \sigma = \sigma_{\lambda}^i \mu p_\mu^i,$$

such that $\mathcal{L} \circ \sigma \circ i_N = i_N$.

**Proof:** The map (77) is a solution of the algebraic equations

$$a^\lambda_{ij} \sigma_{\mu}^k a^\mu_{kb} = a^\lambda_{db}.$$
After pointwise diagonalization, the matrix $a$ has some non-vanishing components $a^{AA}$, $A \in I$. Then a solution of the equations (78) takes the form

$$\sigma_{AA} = (a^{AA})^{-1}, \quad \sigma_{AA'} = 0, \quad A \neq A', \quad A, A' \in I,$$

while the remaining components $\sigma_{BC}$, $B \notin I$, are arbitrary. In particular, there is a solution with

$$\sigma_{BC} = 0, \quad B \notin I. \quad (79)$$

It satisfies the particular relation

$$\sigma = \sigma \circ \hat{L} \circ \sigma. \quad (80)$$

Further on, we will take $\sigma$ to be the solution (79). If the Lagrangian (71) is regular, the linear map (77) is uniquely determined by the equations (78).

The following theorem is the key point of our consideration.

**Theorem 23:** There are the splittings

$$J^1 Y = S(J^1 Y) \oplus \mathcal{F}(J^1 Y) = \text{Ker} \hat{L} \oplus \text{Im}(\sigma \circ \hat{L}), \quad (81a)$$

$$y^i_\lambda = S^i_\lambda + F^i_\lambda = [y^i_\lambda - \sigma^i_\lambda (a_{kj} y^j_\mu + b^k_\mu)] + [\sigma^i_\lambda (a_{kj} y^j_\mu + b^k_\mu)], \quad (81b)$$

$$\Pi = \mathcal{R}(\Pi) \oplus \mathcal{P}(\Pi) = \text{Ker} \sigma \oplus N_L, \quad (82a)$$

$$p^\lambda_i = \mathcal{R}^\lambda_i + \mathcal{P}^\lambda_i = [p^\lambda_i - a^\lambda_\mu \sigma^{jk}_\mu p^j_k] + [a^\lambda_\mu \sigma^{jk}_\mu p^j_k]. \quad (82b)$$

**Proof:** The proof follows from a direct computation by means of the relations (75), (78) and (80).

It is readily observed that, with respect to the coordinates $S^i_\lambda$ and $F^i_\lambda$ (81b), the Lagrangian (71) reads

$$\mathcal{L} = \frac{1}{2} a^{\lambda\mu}_{ij} F^i_\lambda F^j_\mu + c'. \quad (83)$$

Note that, in gauge theory, we have the canonical splitting (81a) where $2\mathcal{F}$ is the strength tensor. The Yang–Mills Lagrangian of gauge theory is exactly of the form (83) where $c' = 0$. The Lagrangian of Proca fields is also of the form (83) where $c'$ is the mass term. This is an example of a degenerate Lagrangian system without gauge symmetries.

Given the linear map $\sigma$ (77) and a connection $\Gamma$ (74), let us consider the affine Hamiltonian map

$$\Phi = \hat{\Gamma} + \sigma : \Pi \rightarrow J^1 Y, \quad \Phi^i_\lambda = \Gamma^i_\lambda + \sigma^{ij}_\nu p^\mu_j, \quad (84)$$

20
and the Hamiltonian form

\[ H = H_\Phi + \Phi^* L = p_\lambda dy^i \wedge \omega_\lambda - [\Gamma_\lambda^i (p_\lambda^i - \frac{1}{2} b_\lambda^i) + \frac{1}{2} \sigma_\lambda^{ij} p_\lambda^i p_\mu^j - c] \omega = \]

\[ (R^\lambda_i + \mathcal{P}^\lambda_i) dy^i \wedge \omega_\lambda - [(R^\lambda_i + \mathcal{P}^\lambda_i) \Gamma_\lambda^i + \frac{1}{2} \sigma_\lambda^{ij} p_\lambda^i p_\mu^j - c'] \omega. \]  

(85)

Theorem 24: The Hamiltonian forms (85) spanned by connections \( \Gamma \) (74) are associated with the Lagrangian (71) and constitute a complete set.

Proof: By the very definitions of \( \Gamma \) and \( \sigma \), the Hamiltonian map (84) satisfies the condition (49a). A direct computation shows that \( \Phi = \bar{H} \). Then the relation (49b) also holds and, if \( \Gamma \) is a connection (74), the Hamiltonian form \( H \) (85) is associated with the Lagrangian (71). Let us write the corresponding Hamilton equations (7a) for a section \( r \) of the Legendre bundle \( \Pi \rightarrow X \). They are

\[ J^1 s = (\bar{\Gamma} + \sigma) \circ r, \quad s = \pi_{\Pi Y} \circ r. \]  

(86)

Due to the surjections \( \mathcal{S} \) and \( \mathcal{F} \) (81a), the Hamilton equations (86) break in two parts

\[ \mathcal{S} \circ J^1 s = \Gamma \circ s, \]  

(87)

\[ \partial_\lambda r^i - \sigma_{\lambda\alpha}^{ik} (a_{kj} \partial_\mu r^j + b_\lambda^k) = \Gamma_\lambda^i \circ s, \]

\[ \mathcal{F} \circ J^1 s = \sigma \circ r, \]  

(88)

\[ \sigma_{\lambda\alpha}^{ik} (a_{kj} \partial_\mu r^j + b_\lambda^k) = \sigma_{\lambda\alpha i}^{ck}. \]

Let \( s \) be an arbitrary section of \( Y \rightarrow X \), e.g., a solution of the Euler–Lagrange equations. There exists a connection \( \Gamma \) (74) such that the relation (87) holds, namely, \( \Gamma = \mathcal{S} \circ \Gamma' \) where \( \Gamma' \) is a connection on \( Y \rightarrow X \) which has \( s \) as an integral section. It is easily seen that, in this case, the Hamiltonian map (84) satisfies the relation (59) for \( s \). Hence, the Hamiltonian forms (85) constitute a complete set.

Of course, this complete set is neither minimal nor unique. Hamiltonian forms \( H \) (85) of this set differ from each other in the term \( \phi^i_\lambda R^\lambda_i \) where \( \phi \) are the soldering forms (76). If follows from the splitting (82a) that this term vanishes on the Lagrangian constraint space. The corresponding constrained Hamiltonian form \( H_\mathcal{N} = \iota^* \mathcal{N} H \) and the constrained Hamilton equations (62) can be written. In the case of quadratic Lagrangians, we can improve Theorem 16 as follows.

Theorem 25: For every Hamiltonian form \( H \) (85), the Hamilton equations (7b) and (88) restricted to the Lagrangian constraint space \( \mathcal{N} \) are equivalent to the constrained Hamilton equations.
Proof: Due to the splitting (82a), we have the corresponding splitting of the vertical tangent bundle $V_Y \Pi$ of the Legendre bundle $\Pi \rightarrow Y$. In particular, any vertical vector field $u$ on $\Pi \rightarrow X$ admits the decomposition

\[ u = [u - u_{TN}] + u_{TN}, \]

\[ u_{TN} = u^i \partial_i + a^{\lambda \mu \sigma \mu \kappa}_{ij} u^i \partial_{\lambda}, \]

such that $u_N = u_{TN} |_{N_L}$ is a vertical vector field on the Lagrangian constraint space $N_L \rightarrow X$. Let us consider the equations

\[ r^*(u_{TN}) dH = 0 \]  

(89)

where $r$ is a section of $\Pi \rightarrow X$ and $u$ is an arbitrary vertical vector field on $\Pi \rightarrow X$. They are equivalent to the pair of equations

\[ r^*(a^{\lambda \mu \sigma \mu \kappa}_{ij} \partial_{\lambda}) dH = 0, \]  

(90a)

\[ r^*(\partial_{\lambda} dH) = 0. \]  

(90b)

The equations (90b) are obviously the Hamilton equations (7b) for $H$. Bearing in mind the relations (75) and (80), one can easily show that the equations (90a) coincide with the Hamilton equations (88). The proof is completed by observing that, restricted to the Lagrangian constraint space $N_L$, the equations (89) are exactly the constrained Hamilton equations (63).

Note that, in Hamiltonian gauge theory, the restricted Hamiltonian form and the restricted Hamilton equations are gauge invariant.

Theorem 25 shows that, restricted to the Lagrangian constraint space, the Hamilton equations for different Hamiltonian forms (85) associated with the same quadratic Lagrangian (71) differ from each other in the equations (87). These equations are independent of momenta and play the role of gauge-type conditions as follows.

By virtue of Theorem 18, the constrained Hamilton equation are quasi-equivalent to the Cartan equations. A section $\bar{s}$ of $J^1 Y \rightarrow X$ is a solution of the Cartan equations for an almost regular quadratic Lagrangian (71) iff $r = \hat{L} \circ \bar{s}$ is a solution of the Hamilton equations (7b) and (88). In particular, let $\bar{s}$ be such a solution of the Cartan equations and $\bar{s}_0$ a section of the fiber bundle $T^* X \otimes V Y \rightarrow X$ which takes its values into Ker $\hat{L}$ (see (73)) and projects onto the section $s = \pi_0^1 \circ \bar{s}$ of $Y \rightarrow X$. Then the affine sum $\bar{s} + \bar{s}_0$ over $s(X) \subset Y$ is also a solution of the Cartan equations. Thus, we come to the notion of a gauge-type freedom of the Cartan equations for an almost regular quadratic Lagrangian $L$. One can speak of the gauge classes of solutions of the Cartan equations whose elements differ from each other in the above-mentioned sections $\bar{s}_0$. Let $z$ be such a gauge class whose elements project onto a section $s$ of $Y \rightarrow X$. For different connections $\Gamma$ (74), we consider the condition

\[ S \circ \bar{s} = \Gamma \circ s, \quad \bar{s} \in z. \]  

(91)
Proposition 26: (i) If two elements \( \mathfrak{s} \) and \( \mathfrak{s}' \) of the same gauge class \( z \) obey the same condition (91), then \( \mathfrak{s} = \mathfrak{s}' \). (ii) For any solution \( \mathfrak{s} \) of the Cartan equations, there exists a connection (74) which fulfills the condition (91).

Proof: (i) Let us consider the affine difference \( \mathfrak{s} - \mathfrak{s}' \) over \( s(X) \subset Y \). We have \( \mathcal{S}(\mathfrak{s} - \mathfrak{s}') = 0 \) iff \( \mathfrak{s} = \mathfrak{s}' \). (ii) In the proof of Theorem 24, we have shown that, given \( s = \pi_0^1 \circ \mathfrak{s} \), there exists a connection \( \Gamma \) (74) which fulfills the relation (87). Let us consider the affine difference \( \mathcal{S}(\mathfrak{s} - J^1 s) \) over \( s(X) \subset Y \). This is a local section of the vector bundle \( \text{Ker} \mathcal{L} \rightarrow Y \) over \( s(X) \). Let \( \phi \) be its prolongation onto \( Y \). It is easy to see that \( \Gamma + \phi \) is the desired connection.

Due to the properties in Proposition 26, one can treat (91) as a gauge-type condition on solutions of the Cartan equations. The Hamilton equations (87) exemplify this gauge-type condition when \( \mathfrak{s} = J^1 s \) is a solution of the Euler–Lagrange equations. At the same time, the above-mentioned freedom characterizes solutions of the Cartan equations, but not the Euler–Lagrange ones. First of all, this freedom reflects the degeneracy of the Cartan equations (26a). Therefore, in the Hamiltonian gauge theory, the above freedom is not related directly to the familiar gauge invariance. Nevertheless, the Hamilton equations (87) are not gauge invariant, and also can play the role of gauge conditions in gauge theory. Indeed, given a Hamiltonian form \( H \) (85), the corresponding Lagrangian \( L_H \) (8) reads

\[
L_H = \mathcal{R}_\lambda^i (\mathcal{S}_i^\lambda - \Gamma_\lambda^i) + \mathcal{P}_i^\lambda \mathcal{F}_i^\lambda - \frac{1}{2} \sigma_{\lambda\mu}^{ij} \mathcal{P}_i^\lambda \mathcal{P}_j^\mu + c'.
\] (92)

In comparison with the Lagrangian \( L \) (71) and the constrained Lagrangian \( L_H \mid_{J^1 N_L} \), the Lagrangian (92) includes the additional gauge fixing term \( \mathcal{R}_\lambda^i (\mathcal{S}_i^\lambda - \Gamma_\lambda^i) \).

VII. VERTICAL EXTENSION OF POLYSYMPLECTIC FORMALISM

The extension of polysymplectic formalism to the vertical tangent bundle \( VY \) of \( Y \rightarrow X \) is a preliminary step toward its BRST extension. The Legendre bundle (3) over \( VY \rightarrow X \), called the vertical Legendre bundle, is

\[
\Pi_{VY} = V^* VY \wedge_{VY}^{n-1} T^* X.
\]

We will use the compact notation

\[
\dot{\partial}_i = \frac{\partial}{\partial \dot{y}^i}, \quad \dot{\partial}_\lambda^i = \frac{\partial}{\partial \dot{p}_i^\lambda}, \quad \partial_V = \dot{y}^i \partial_i + \dot{p}_i^\lambda \partial_\lambda^i.
\]

Lemma 27: There exists the bundle isomorphism

\[
\Pi_{VY} \cong \Pi_{VII}, \quad \dot{p}_i^\lambda \leftrightarrow \dot{p}_i^\lambda, \quad \dot{q}_i^\lambda \leftrightarrow p_i^\lambda,
\] (93)
written relative to the holonomic coordinates \((x^\lambda, y^i, \dot{y}^i, p^\lambda_i, q^\lambda_i)\) on \(\Pi_{VY}\) and \((x^\lambda, y^i, \dot{y}^i, \dot{p}^\lambda_i)\) on \(\Pi\).

**Proof:** Similar to the well-known isomorphism between the fiber bundles \(TT^*X\) and \(T^*TX^5\), the isomorphism
\[
VV^*Y \cong V^*VY, \quad p_i \longleftrightarrow \dot{v}_i, \quad \dot{p}_i \longleftrightarrow \dot{y}_i,
\]
can be established by inspection of the transformation laws of the holonomic coordinates \((x^\lambda, y^i, p^\lambda_i)\) on \(V^*Y\) and \((x^\lambda, y^i, v^i)\) on \(VY\).

It follows that Hamiltonian formalism on the vertical Legendre bundle \(\Pi_{VY}\) can be developed as the vertical extension onto \(V\Pi\) of Hamiltonian formalism on \(\Pi\), where the canonical conjugate pairs are \((y^i, \dot{p}^\lambda_i)\) and \((\dot{y}^i, p^\lambda_i)\). In particular, due to the isomorphism (93), \(V\Pi\) is endowed with the canonical polysymplectic form (33) which reads
\[
\Omega_{VY} = \left[dp^\lambda_i \wedge dy^i + dp^\lambda_i \wedge d\dot{y}^i \right] \wedge \omega \otimes \partial_\lambda. \tag{94}
\]

Let \(Z_{VY}\) be the homogeneous Legendre bundle (2) over \(VY\) with the corresponding coordinates \((x^\lambda, y^i, \dot{y}^i, p^\lambda_i, q^\lambda_i, p)\). It can be endowed with the canonical form \(\Xi_{VY}\) (28). Sections of the affine bundle
\[
Z_{VY} \rightarrow V\Pi, \tag{95}
\]
by definition, provide Hamiltonian forms on \(V\Pi\). Let us consider the following particular case of these forms which are related to those on the Legendre bundle \(\Pi\). Due to the fiber bundle
\[
\zeta : VZ_Y \rightarrow Z_{VY}, \tag{96}
\]
\[(x^\lambda, y^i, \dot{y}^i, p^\lambda_i, q^\lambda_i, p) \circ \zeta = (x^\lambda, y^i, \dot{y}^i, p^\lambda_i, p),\]
the vertical tangent bundle \(VZ_Y\) of \(Z_Y \rightarrow X\) is provided with the exterior form
\[
\Xi_V = \zeta^*\Xi_{VY} = \dot{p} \omega + (\dot{p}^\lambda_i dy^i + p^\lambda_i d\dot{y}^i) \wedge \omega_\lambda.
\]
Given the affine bundle \(Z_Y \rightarrow \Pi\) (5), we have the fiber bundle
\[
V_{\pi_{Z\Pi}} : VZ_Y \rightarrow V\Pi, \tag{97}
\]
where \(V_{\pi_{Z\Pi}}\) is the vertical tangent map to \(\pi_{Z\Pi}\). The fiber bundles (95), (96) and (97) form the commutative diagram.

Let \(h\) be a section of the affine bundle \(Z_Y \rightarrow \Pi\) and \(H = h^*\Xi\) the corresponding Hamiltonian form (6) on \(\Pi\). Then the section \(Vh\) of the fiber bundle (97) and the corresponding section \(\zeta \circ Vh\) of the affine bundle (95) defines the Hamiltonian form
\[
H_V = (Vh)^*\Xi = (\dot{p}^\lambda_i dy^i + p^\lambda_i d\dot{y}^i) \wedge \omega_\lambda - \mathcal{H}_V \omega, \tag{98}
\]
\[
\mathcal{H}_V = \partial_V \mathcal{H} = (\dot{y}^i \partial_i + \dot{p}^\lambda_i \partial_\lambda) \mathcal{H},
\]
\[24\]
It is called the vertical extension of $H$. In particular, given the splitting (35) of $H$ with respect to a connection $\Gamma$ on $Y \to X$, we have the corresponding splitting
\[ \mathcal{H}_V = \dot{y}^i \Gamma^i_{\lambda} + \dot{y}^j \partial_j \Gamma^i_{\lambda} + \partial_V \mathcal{H}_\Gamma \]
of $H_V$ with respect to the vertical connection $V\Gamma$ (16) on $VY \to X$.

**Proposition 28:** Let $\gamma$ (39) be a Hamiltonian connection on $\Pi$ associated with a Hamiltonian form $H$. Then its vertical prolongation $V\gamma$ (16) on $V\Pi \to X$ is a Hamiltonian connection associated with the vertical Hamiltonian form $H_V$ (98).

**Proof:** The proof follows from a direct computation. We have
\[ V\gamma = \gamma + dx^\mu \otimes [\partial_V \gamma^i_{\lambda} \dot{\gamma}^i_{\lambda} + \partial_V \gamma^\lambda_{\mu i} \dot{\gamma}^i_{\lambda}] \]
Components of this connection obey the Hamilton equations (41) and the equations
\[ \dot{\gamma}^i_{\lambda} = \partial^i_{\mu} \mathcal{H}_V = \partial_V \partial^i_{\mu} \mathcal{H}, \quad \dot{\gamma}^\lambda_{\mu i} = -\partial^i_{\lambda} \mathcal{H}_V = -\partial_V \partial^i_{\lambda} \mathcal{H}. \]

In order to clarify the physical meaning of the Hamilton equations (99), let us suppose that $Y \to X$ is a vector bundle. Given a solution $r$ of the Hamilton equations for $H$, let $r$ be a Jacobi field, i.e., $r + \varepsilon \tau$ is also a solution of the same Hamilton equations modulo terms of order $> 1$ in $\varepsilon$. Then it is readily observed that the Jacobi field $r$ satisfies the Hamilton equations (99). At the same time, the Lagrangian $L_{\mathcal{H}_V}$ (8) on $J^1V\Pi$, defined by the Hamiltonian form $H_V$ (98), takes the form
\[ L_{\mathcal{H}_V} = h_0(H_V) = \dot{p}^\lambda_{\mu}(y^i - \dot{y}^i \partial^i_{\lambda} \mathcal{H}) - \dot{y}^i (p^\lambda_{\mu} + \partial^i_{\lambda} \mathcal{H}) + d_{\lambda}(\dot{p}^\lambda_{\mu} \dot{y}^i), \]
where $\dot{p}^\lambda_{\mu}$, $\dot{y}^i$ play the role of Lagrange multipliers. The corresponding generating functional reduces to Dirac’s $\delta$-functions at classical solutions.

**VIII. BRST-EXTENDED HAMILTONIAN FORMALISL**

The BRST extension of Hamiltonian mechanics\textsuperscript{11,12} shows that: (i) one should consider vector bundles $Y \to X$ in order to introduce generators of BRST and anti-BRST transformations, and (ii) one can narrow the class of superfunctions under consideration because the BRST extension of a Hamiltonian is a polynomial of a finite degree in odd variables. Therefore, we will formulate the BRST extension on the polysymplectic Hamiltonian formalism in the terms of simple graded manifolds.

Recall\textsuperscript{24,25} that by a graded manifold is meant the pair $(Z, \mathcal{A})$ of a smooth manifold $Z$ and a sheaf $\mathcal{A}$ of graded-commutative $\mathbf{R}$-algebras such that
(i) there is the exact sequence of sheaves
\[ 0 \to \mathcal{J} \to \mathcal{A} \to C^\infty(Z) \to 0, \quad \mathcal{J} = \mathcal{A}_1 + (\mathcal{A}_1)^2, \quad (101) \]

(ii) \( \mathcal{J}/\mathcal{J}^2 \) is a locally free \( C^\infty(Z) \)-module of finite rank, and \( \mathcal{A} \) is locally isomorphic to the exterior bundle \( \bigwedge_{C^\infty(Z)} (\mathcal{J}/\mathcal{J}^2) \). The exact sequence (101) admits the canonical splitting \( C^\infty(Z) \to \mathcal{A} \), and the well-known Batchelor’s theorem takes place.

**Theorem 29.**\(^{25,26}\) Let \((Z, \mathcal{A})\) be a graded manifold. There exists a vector bundle \( E \to Z \) with an \( m \)-dimensional typical fiber \( V \) such that \( \mathcal{A} \) is isomorphic to the sheaf \( \mathcal{A}_E \) of sections of the exterior bundle
\[ \bigwedge E^* = \mathbb{R} \oplus (\bigoplus_{k=1}^{m} \bigwedge E^*) \quad (102) \]
whose typical fiber is the finite Grassman algebra \( \bigwedge V^* \).

This isomorphism fails to be canonical, and restricts transformations of a graded manifold to those induced by the bundle automorphisms of \( E \to Z \). Nevertheless, this class of transformations is sufficient for our purposes because we consider the graded extension of Hamiltonian formalism on smooth manifolds when the vector bundle \( E \) (115) below is fixed. We will call \((Z, \mathcal{A}_E)\) the simple graded manifold. This is not the terminology of Ref. [27] where this term is applied to all finite graded manifolds, but in connection with Batchelor’s isomorphism.

Global sections of the exterior bundle (102) are called superfunctions due to the equivalence between the graded manifolds \((Z, \mathcal{A})\) and the De Witt supermanifolds whose body is \( Z^{25,28} \). This isomorphism is important for for functional integration over superfunctions. Superfunctions make up a \( \mathbb{Z}_2 \)-graded ring \( \mathcal{A}_E(X) \). Let \( \{c^a\} \) be the holonomic bases for \( E^* \to Z \) with respect to some bundle atlas with transition functions \( \{\rho^a_b\} \), i.e., \( c^a = \rho^a_b(z)c^b \).

Then superfunctions read
\[ f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \ldots a_k} c^{a_1} \ldots c^{a_k}, \quad (103) \]
where \( f_{a_1 \ldots a_k} \) are local functions on \( Z \), and we omit the symbol of exterior product of elements \( c \). The coordinate transformation law of superfunctions (103) is obvious. We will use the notation \([\cdot] \) of the Grassman parity.

Given a graded manifold \((Z, \mathcal{A})\), the sheaf \( \text{Der} \mathcal{A} \) of graded derivations of \( \mathcal{A} \) is introduced. This is a subsheaf of endomorphisms of \( \mathcal{A} \) whose sections \( u \) on an open subset \( U \subset Z \) are graded derivations of the restriction \( \mathcal{A}|_U \) of the sheaf \( \mathcal{A} \) to \( U \), i.e.,
\[ u(ff') = u(f)f' + (-1)^{|u||f|}fu(f') \]
for the homogeneous elements \( u \in (\text{Der}\mathcal{A})(U) \) and \( f, f' \in \mathcal{A}|_U \). In the case of graded manifolds, derivations of \( \mathcal{A} \) are local operators. It means that \((\text{Der}\mathcal{A})(U) = \text{Der}\mathcal{A}(U)\), i.e., if
$U' \subset U$ are open sets, there is the restriction morphism $\text{Der}_A(U) \to \text{Der}_A(U')$. It follows that the sheaf $\text{Der}_A$ coincides with the sheaf of graded $A$ modules $U \to \text{Der}_A(U)$. Its sections are called supervector fields on a manifold $Z$. The dual of the sheaf $\text{Der}_A$ is the sheaf $\text{Der}^* A$ generated by the $A$-linear morphisms

$$\phi : \text{Der}_A(U) \to A_U. \quad (104)$$

One can think of its sections as being 1-superforms on a manifold $Z$.

In the case of a simple graded manifold $(Z, A_E)$ supervector fields and 1-superforms can be represented by sections of vector bundles as follows. Due to the canonical splitting $VE = E \times E$, the vertical tangent bundle $VE \to E$ can be provided with the fiber bases $\{\partial_a\}$ dual of $\{e^a\}$. These are fiber bases for $\text{pr}_2 VE = E$. Let $(z^A)$ be coordinates on $Z$. Then a supervector field on a trivialization domain $U$ read $u = u^A \partial_A + u^a \partial_a$ where $u^A, u^a$ are local superfunctions. It yields a graded endomorphism of $A_E(U)$ by the rule

$$u(f_{a...b} e^a \cdots c^b) = u^A \partial_A(f_{a...b}) e^a \cdots c^b + u^a f_{a...b} \partial_a (e^a \cdots c^b). \quad (105)$$

This implies the corresponding coordinate transformation law

$$u'^A = u^A, \quad u'^a = \rho^i_a u^i + u^A \partial_A(\rho^i_a) c^i$$

of supervector fields. It follows that supervector fields on $Z$, which we agree to call $E$-determined supervector fields, can be represented by sections of the vector bundle $V_E \to Z$ which is locally isomorphic to the vector bundle

$$V_E \big|_U \approx \wedge E^* \otimes (\text{pr}_2 VE \oplus T_Z) \big|_U,$$

and has the transition functions

$$z_{i_1 \ldots i_k}^A = \rho^{-1}_{i_1} \cdots \rho^{-1}_{i_k} z_{a_1 \ldots a_k}^A,$$

$$v^i_{j_1 \ldots j_k} = \rho^{-1}_{j_1} \cdots \rho^{-1}_{j_k} \left[ \rho^i_{j_1} v^b_{j_1 \ldots j_k} + \frac{k!}{(k-1)!} z^A_{b_1 \ldots b_{k-1}} \partial_A(\rho^i_{b_k}) \right]$$

of the bundle coordinates $(z^A_{a_1 \ldots a_k}, v^i_{b_1 \ldots b_k}), k = 0, \ldots, m$. These transition functions fulfill the cocycle relations. There is the exact sequence over $Z$ of vector bundles

$$0 \to \wedge E^* \otimes \text{pr}_2 VE \to V_E \to \wedge E^* \otimes \text{pr}_2 T_Z \to 0. \quad (106)$$

Due to the above mentioned locality property the sheaf of sections of the vector bundle $V_E \to Z$ is isomorphic to the sheaf $\text{Der}_A E$. Global sections of $V_E \to Z$ constitute the $A_E(Z)$-module of supervector fields on $Z$, which is also a Lie superalgebra with respect to the bracket

$$[u, u'] = uu' + (-1)^{|u||u'|+1} u'u.$$
One can think of a splitting
\[ \tilde{\gamma} : z^A \partial_A \mapsto \tilde{z}^A (\partial_A + \tilde{\gamma}_a^A \partial_a) \] (107)
of the exact sequence (106) as being a graded connection, though this is not a true connection on $\mathcal{V}_E \to Z$. A graded connection can be represented by a section
\[ \tilde{\gamma} = dz^A \otimes (\partial_A + \tilde{\gamma}_a^A \partial_a) \] (108)
of the vector bundle $T^*Z \otimes \mathcal{V}_E \to Z$ such that the composition
\[ Z \xrightarrow{\tilde{\gamma}} T^*Z \otimes \mathcal{V}_E \to T^*Z \otimes (\wedge E^* \otimes T_Z) \to T^*Z \otimes TZ \]
is the canonical form $dz^A \otimes \partial_A$ on $Z$. Such a graded connection $\tilde{\gamma}$ transforms every vector field $\tau$ on $Z$ into a supervector field
\[ \tau = \tau^A \partial_A \mapsto \tilde{\gamma} \tau = \tau^A (\partial_A + \tilde{\gamma}_a^A \partial_a), \]
and provides the corresponding decomposition
\[ u = u^A \partial_A + u^a \partial_a = u^A (\partial_A + \tilde{\gamma}_a^A \partial_a) + (u^a - u^A \tilde{\gamma}_a^A) \partial_a \]
of supervector fields on $Z$. For instance, every linear connection
\[ \gamma = dz^A \otimes (\partial_A + \gamma_a^A b v^b \partial_a) \]
on the vector bundle $E \to Z$ defines the graded connection
\[ \gamma_S = dz^A \otimes (\partial_A + \gamma_a^A b c^b d x^c \partial_a) \] (109)
such that, for any vector field $\tau$ on $Z$ and any superfunction $f$, the graded derivation $\gamma_S \tau(f)$ is exactly the covariant derivative $\tau^A \nabla_A f$ relative to the connection $\gamma$.

**Remark 6:** Let now $Z \to X$ be a fiber bundle, coordinated by $(x^\lambda, z^i)$. Let
\[ \gamma = \Gamma + \gamma_\lambda^a b c^b d x^c \partial_a \]
be a connection on $E \to X$ which is a linear morphism over a connection $\Gamma$ on $Z \to X$. Then we have the bundle monomorphism
\[ \gamma_S : \wedge E^* \otimes T_X \ni u^\lambda \partial_\lambda \mapsto u^\lambda (\partial_\lambda + \Gamma_\lambda^i \partial_i + \gamma_\lambda^a b c^b \partial_a) \in \mathcal{V}_E \]
over $Z$, called a composite graded connection on $Z \to X$. It is represented by a section
\[ \gamma_S = \Gamma + \gamma_\lambda^a b c^b d x^c \otimes \partial_a \] (110)
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of the fiber bundle $T^* X \otimes V_E \to Z$ such that the composition

$$Z \xrightarrow{\gamma} T^* X \otimes V_E \to T^* X \otimes (\wedge^E \otimes TZ) \to T^* X \otimes TX$$

is the pull-back onto $Z$ of the canonical form $dx^\lambda \otimes \partial_\lambda$ on $X$.

The $\wedge^E$-dual $V_E^*$ of $V_E$ is a vector bundle over $Z$ which is locally isomorphic to the vector bundle

$$V_E^* \mid_U \approx \wedge^E \otimes Z (\text{pr}_2 V^* \oplus \wedge^E \otimes Z) \mid_U,$$

and has the transition functions

$$v'_{j_1 \ldots j_k} = \rho_{j_1}^{-a_{j_1}} \ldots \rho_{j_k}^{-a_{j_k}} v_{a_1 \ldots a_k},$$

$$z'_{i_1 \ldots i_k} = \rho_{b_{i_1}}^{-b_{i_1}} \ldots \rho_{b_{i_k}}^{-b_{i_k}} \left[ z_{b_1 \ldots b_k} + \frac{k!}{(k-1)!} v_{b_1 \ldots b_k} \partial_A (\rho_{b_k}) \right]$$

of the bundle coordinates $(z_{a_1 \ldots a_k}, v_{b_1 \ldots b_k})$, $k = 0, \ldots, m$, with respect to the dual bases $\{dz^A\}$ for $T^* Z$ and $\{dc^b\}$ for $\text{pr}_2 V^* = E^*$. There is the exact sequence

$$0 \to \wedge^E \otimes T^* Z \to V_E^* \to \wedge^E \otimes \text{pr}_2 V^* \to 0. \quad (111)$$

The sheaf of sections of $V_E^* \to Z$ is isomorphic to the sheaf $\text{Der}^* A_E$. Global sections of the vector bundle $V^* \to Z$ constitute the $A_E(Z)$-module of $E$-determined exterior 1-superforms $\phi = \phi_A dz^A + \phi_a dc^a$ on $Z$ with the coordinate transformation law

$$\phi'_a = \rho^{-b_a} \phi_b, \quad \phi'_A = \phi_A + \rho^{-b_a} \partial_A (\rho^j_b) \phi_b c^j.$$

Then the morphism (104) can be seen as the interior product

$$u \cdot \phi = u^A \phi_A + (-1)^{|\phi_A|} a^a \phi_a. \quad (112)$$

Any graded connection $\tilde{\gamma}$ (108) also yields the splitting of the exact sequence (111), and defines the corresponding decomposition of 1-superforms

$$\phi = \phi_A dz^A + \phi_a dc^a = (\phi_A + \phi_a \tilde{\gamma}_A) dz^A + \phi_a (dc^a - \tilde{\gamma}_A dz^A).$$

Accordingly, $k$-superforms $\phi$ are sections of the graded exterior bundle $\wedge^k_Z V_E^*$ such that

$$\phi \wedge \phi = (-1)^{|\phi| + |\phi|} \sigma \wedge \phi.$$

The interior product (112) is extended to higher degree superforms by the rule

$$u \cdot (\phi \wedge \phi) = (u \cdot \phi) \wedge \phi + (-1)^{|\phi| + |\phi|} \phi \wedge \phi (u \cdot \phi).$$
Recall that the graded exterior differential $d$ of superfunctions is introduced in accordance with the condition $u \downarrow df = u(f)$ for an arbitrary supervector field $u$, and is extended uniquely to higher degree superforms by the rules
\[ d(\phi \wedge \sigma) = (d\phi) \wedge \sigma + (-1)^{|\phi|} \phi \wedge (d\sigma), \quad d \circ d = 0. \]
It takes the coordinate form
\[ d\phi = dz^A \partial_A(\phi) + dc^a \partial_a(\phi), \]
where the left derivatives $\partial_A, \partial_a$ act on the coefficients of superforms by the rule (105), and they are graded commutative with the forms $dz^A, dc^a$. The Lie derivative of a superform $\phi$ along a supervector field $u$ is given by the familiar formula
\[ L_u \phi = u \downarrow d\phi + d(u \downarrow \phi). \quad (113) \]

Remark 7: Given a vector bundle $E \rightarrow Z$, let us consider the jet manifold $J^1 E$, coordinated by $(z^A, v^a, v^A_a)$. This is also a vector bundle over $Z$. Then one can construct the corresponding fiber bundles $V_{J^1 E}$ and $V^*_{J^1 E}$. Due to the monomorphism $E^* \rightarrow (J^1 E)^*$, there is the monomorphism $V_E^* \rightarrow V_{J^1 E}^*$, i.e., every $E$-determined superform on $Z$ can be also seen as a $J^1 E$-determined superform. In particular, the horizontal projection $h_0$ (14) gives rise to the 0-graded homomorphism
\[ h_0 : dc^a \rightarrow c^a_A dz^A \quad (114) \]
which sends $E$-determined superforms onto horizontal $J^1 E$-determined superforms.

Turn now to the BRST extension of covariant Hamiltonian formalism on the Legendre bundle $\Pi (3)$ when $Y \rightarrow X$ is a vector bundle. Let us apply the above construction of simple graded manifolds to the case of the vertical tangent bundle
\[ E = VV\Pi = \Pi \oplus \Pi \overset{pr_1}{\rightarrow} \Pi \]
over the vertical Legendre bundle $Z = \Pi \rightarrow X$. Let $(x^\lambda, y^i, p^i_\lambda, \dot{y}^i, \dot{p}^i_\lambda)$ be the holonomic coordinates on $V\Pi$. Then the dual $E^*$ of $E$ can be endowed with the associated fiber bases $\{c^i, c^A_\lambda, c^a, c^A_{a_\lambda}\}$ such that $c^i$ and $c^a$ have the same linear coordinate transformation law as the coordinates $y^i$ and $\dot{y}^i$, while $c^A_\lambda$ and $c^A_{a_\lambda}$ have those of the coordinates $p^i_\lambda$ and $\dot{p}^i_\lambda$. The corresponding supervector fields and superforms are introduced on $V\Pi$ as sections of the vector bundles $V_{VV\Pi}$ and $V^*_{VV\Pi}$, respectively. Let us complexify these bundles as $C \otimes_{\bar{X}} V_{VV\Pi}$ and $C \otimes_{\bar{X}} V^*_{VV\Pi}$. 

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As in mechanics, the main criterion of the BRST extension of covariant Hamiltonian formalism is its invariance under BRST and anti-BRST transformations whose generators are the supervector fields

$$\vartheta_Q = \partial_c + iy^i \frac{\partial}{\partial c^i} + i\dot{p}_\lambda^i \frac{\partial}{\partial c^i}_\lambda, \quad \vartheta_{\overline{c}} = \partial_{\overline{c}} - iy^i \frac{\partial}{\partial \overline{c}^i} - i\dot{p}_\lambda^i \frac{\partial}{\partial \overline{c}^i}_\lambda, \quad (116)$$

on \( V \). They fulfill the nilpotency rules

$$\vartheta_Q \vartheta_Q = 0, \quad \vartheta_{\overline{c}} \vartheta_{\overline{c}} = 0, \quad \vartheta_{\overline{c}} \vartheta_Q + \vartheta_Q \vartheta_{\overline{c}} = 0.$$  

The BRST- and anti-BRST-invariant extension of the polysymplectic form \( \Omega_{vy} \) (94) on \( V \) is the \( TX \)-valued superform

$$\Omega_S = [d\dot{p}_\lambda^i \wedge dy^i + dp_\lambda^i \wedge d\dot{y}^i + i(d\overline{c}_\lambda^i \wedge dc^i - dc^i \wedge d\overline{c}_\lambda^i)] \wedge \omega \otimes \partial_{\lambda}, \quad (117)$$

on \( V \), where \((c^i, -\overline{c}_\lambda^i)\) and \((\overline{c}_\lambda^i, ic^i)\) are the conjugate pairs. Let \( \gamma \) be a Hamiltonian connection for a Hamiltonian form \( H \). The double vertical connection \( VV\gamma \) on \( VV \rightarrow X \) is a linear morphism over the vertical connection \( V\gamma \) on \( V \rightarrow X \), and so defines the composite graded connection

$$(VV\gamma)_S = V\gamma + dx^\mu \otimes [g^i_\mu \frac{\partial}{\partial c^i} + g^\lambda_\mu \frac{\partial}{\partial \overline{c}^i}_\lambda + g^i_\mu \frac{\partial}{\partial c^i} + g^\lambda_\mu \frac{\partial}{\partial \overline{c}^i}_\lambda], \quad (110)$$

on \( VV \rightarrow X \), whose components \( g \) and \( \overline{g} \) are given by the expressions

$$g^i_\lambda = \partial_c \partial^i_\lambda H, \quad g^\lambda_i = -\partial_{\overline{c}} \partial_i H, \quad g^i_\lambda = \partial_c \partial^i_\lambda H, \quad g^\lambda_i = -\partial_{\overline{c}} \partial_i H.$$  

This composite graded connection satisfies the relation

$$(VV\gamma)_S \Omega_S = -dH_S,$$

and so is a Hamiltonian graded connection for the Hamiltonian superform

$$H_S = [\dot{p}_\lambda^i dy^i + p_\lambda^i d\dot{y}^i + i(\overline{c}_\lambda^i dc^i + dc^i \overline{c}_\lambda^i)]\omega_\lambda - (\partial_V + i\partial_{\overline{c}}\partial_c)H\omega \quad (117)$$

on \( V \). This superform is BRST- and anti-BRST-invariant, i.e., \( L_\theta H_S = 0 \). Thus, it is the desired BRST extension of the Hamiltonian form \( H \).

The Hamiltonian superform \( H_S \) (117) defines the corresponding BRST-extension of the Lagrangian \( L_H \) (8). Following Remark 7, let us consider the vector bundle

$$J^1(VV) = VJ^1(V) \rightarrow J^1(VV) = VJ^1V$$
and the corresponding fiber bundle $\mathcal{V}_{V^J(VII)} \rightarrow VJ^1\Pi$. It is readily observed that $VV^{\Pi}$-determined superforms on $V^{\Pi}$ can be seen as particular $VJ^1(VII)$-determined superforms on $VJ^1\Pi$. Moreover, combining the horizontal projections $h_0$ (14) and (114) for exterior forms and superforms, we obtain the 0-graded homomorphism $h_0$ which sends $VV^{\Pi}$-determined superforms on $V^{\Pi}$ onto the horizontal $VJ^1(VII)$-determined superforms on $VJ^1\Pi$. Then the horizontal superdensity

$$L_{SH} = h_0(H_S) = L_{VH} + i[(\overline{c}_i^\lambda \partial_i^\lambda + c_i^\lambda \overline{c}_i^\lambda) - \partial_\tau \partial_\lambda \mathcal{H}]\omega = L_{VH} +$$

$$+ i[\overline{c}_i^\lambda (c_j^\lambda - \partial_\lambda \partial_j^\lambda \mathcal{H}) + (c_i^\lambda - \partial_\lambda \partial_i^\lambda \mathcal{H}) c^\lambda_j + \overline{c}_i^\lambda \overline{c}_j^\lambda \partial_\lambda \partial_\mu \mathcal{H} - \overline{c}_i^\lambda \partial_\lambda \partial_\mu \mathcal{H}]$$

on $VJ^1\Pi \rightarrow X$ can be treated as the desired BRST extension of the Lagrangian $L_H$ (8). Note that, in comparison with the Lagrangian $L_{VH}$ (100), the generating functional determined by the BRST-extended Lagrangian (118) is not reduced to $\delta$-functions.

The BRST-extended Lagrangian $L_{SH}$ (118) is also invariant under the jet prolongations $J^1\vartheta = \vartheta^a \partial_a + d_\lambda \vartheta^a \partial^\lambda_a$ of the BRST and anti-BRST transformations (116). Moreover, it is easily verified that both $L_{HS}$ and $H_S$ are invariant under transformations whose generators are the supervector fields $\vartheta_Q = c_i^\lambda \partial_i^\lambda + c_j^\lambda \overline{c}_j^\lambda$, $\vartheta_{\overline{Q}} = \overline{c}_i^\lambda \partial_i^\lambda + c^j_i \overline{c}_j^\lambda$, $\vartheta_K = c_i^\lambda \partial_i^\lambda$, $\vartheta_{\overline{K}} = \overline{c}_i^\lambda \partial_i^\lambda$, $\vartheta_C = c_i^\lambda \partial_i^\lambda - c_i^\lambda \partial_\lambda \partial_i^\lambda - c^j_i \partial_i^j \overline{c}_j^\lambda - \overline{c}_i^\lambda \partial_\lambda \partial_i^\lambda$.

The supervector fields (116) and (119) constitute the Lie superalgebra of the well-known group $\text{ISp}(2)$:

$$[Q, Q] = [\overline{Q}, Q] = [\overline{Q}, \overline{Q}] = [K, Q] = [\overline{K}, Q] = 0,$$

$$[K, Q] = Q, \quad [K, \overline{Q}] = \overline{Q}, \quad [K, \overline{K}] = C, \quad [C, K] = 2K, \quad [C, \overline{K}] = -2\overline{K}.$$  

Similarly to the lift $\tilde{u}$ (67) onto $\Pi$ of a vector field $u$ on $Y$, the supervector fields $\vartheta$ (116) and (119) can be represented as the corresponding graded lift

$$\vartheta = \tilde{u} = u^a \partial_a - (-1)^{[u^a][p^b] + [u^b]} \partial_a u^b \partial_{p^a}$$

of some $VV^Y$-determined supervector fields $u$ on $VY$ which are sections of the fiber bundle $\mathcal{V}_{V^VY \rightarrow VY}$. These supervector fields read

$$u_Q = c_i^\lambda \partial_i^\lambda, \quad u_{\overline{Q}} = \overline{c}_i^\lambda \partial_i^\lambda, \quad u_K = c_i^\lambda \partial_i^\lambda, \quad u_{\overline{K}} = \overline{c}_i^\lambda \partial_i^\lambda, \quad u_C = c^j_i \partial_i^j - \overline{c}_i^\lambda \partial_\lambda \partial_i^\lambda.$$
They also constitute the Lie superalgebra (120). Then by analogy with (70), we obtain the corresponding supercurrents $J_\vartheta = \vartheta \bar{c}_i^\lambda \bar{\epsilon} \omega_\lambda$, $K = -ic_i^\lambda \epsilon \omega_\lambda$, $C = i(\tau^\lambda e^i - \epsilon^i c_i) \omega_\lambda$

on VII. They form the Lie superalgebra (120) with respect to the product (11). It should be emphasized that the Lie superalgebra (120) provides the canonical symmetries of any BRST-extended Hamiltonian system.

The following construction is similar to that is met in supersymmetric mechanics and BRST mechanics. Given a function $F$ on the Legendre bundle $\Pi$, let us consider the operators

$$F_\beta = e^{\beta\epsilon} \circ \vartheta \circ e^{-\beta\epsilon} = \vartheta - \beta \partial_\kappa F, \quad \overline{F}_\beta = e^{-\beta\epsilon} \circ \vartheta \circ e^{\beta\epsilon} = \vartheta + \beta \partial_\kappa F, \quad \beta > 0,$$

(122)

called the BRST and anti-BRST charges, which act on superfunctions on VII. These operators are nilpotent, i.e.,

$$F_\beta \circ F_\beta = 0, \quad \overline{F}_\beta \circ \overline{F}_\beta = 0.$$

(123)

By the BRST- and anti-BRST-invariant extension of $F$ is meant the superfunction

$$F_S = -\frac{i}{\beta}(\overline{F}_\beta \circ F_\beta + F_\beta \circ \overline{F}_\beta).$$

(124)

We have the relations

$$F_\beta \circ F_S - F_S \circ F_\beta = 0, \quad \overline{F}_\beta \circ F_S - F_S \circ \overline{F}_\beta = 0.$$

These relations together with the relations (123) provide the operators $F_\beta$, $\overline{F}_\beta$, and $F_S$ with the structure of the Lie superalgebra $\text{sl}(1/1)^\perp$.

Let now $\Gamma^\lambda_i = \Gamma^i_{\lambda} y^j$ be a linear connection on $Y \rightarrow X$ and $\tilde{\Gamma}$ some Hamiltonian connection on $\Pi \rightarrow X$ for the Hamiltonian form $H_\Gamma$ (34). Given the splitting (35) of the Hamiltonian form $H$ with respect to the connection $\Gamma$, there is the corresponding splitting of the BRST-extended Hamiltonian form

$$\mathcal{H}_S = \mathcal{H}_{FS} + \mathcal{H}_{ST} = p_i^\lambda \Gamma^i_{\lambda j} y^j + \dot{y}^i p_i^\lambda \Gamma^i_{\lambda j} + i(\tau^\lambda e^i \epsilon + \epsilon^i c_i^\lambda) + (\partial \nu + i \partial_\kappa \partial_\lambda) \tilde{H}_\Gamma$$

with respect to the composite graded connection $(VV\tilde{\Gamma})_S$ (110) on the fiber bundle $V\Pi \rightarrow X$.

Let $dV$ be a volume element on $X$ and $\tilde{H}_{\Gamma \omega} = F dV$, where $F$ is a function on $\Pi$. Then

$$\tilde{H}_{ST \omega} = F_S dV = -i(\overline{F}_1 \circ F_1 + F_1 \circ \overline{F}_1) dV,$$

where $F_1$, $\overline{F}_1$ and $F_S$ are the BRST and anti-BRST charges (122) and (124). The similar splitting of a super-Hamiltonian is the corner stone of supersymmetric mechanics.


