Newtonian limit of conformal gravity

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We study the weak-field limit of the static spherically symmetric solution of the locally conformally invariant theory advocated in the recent past by Mannheim and Kazanas as an alternative to Einstein’s General Relativity. In contrast with the previous works, we consider the physically relevant case where the scalar field that breaks conformal symmetry and generates fermion masses is nonzero. In the physical gauge, in which this scalar field is constant in space-time, the solution reproduces the weak-field limit of the Schwarzschild–(anti) De Sitter solution modified by an additional term that, depending on the sign of the Weyl term in the action, is either oscillatory or exponential as a function of the radial distance. Such behavior reflects the presence of, correspondingly, either a tachyon or a massive ghost in the spectrum, which is a serious drawback of the theory under discussion.

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In a series of papers (see [1–8] and references therein), Mannheim and Kazanas explored the possibility that the gravity is described by the conformally invariant theory with the key ingredient in the action being the Weyl term

\[ I_W = -\alpha \int d^4 x \sqrt{-g} C_{\lambda\mu\nu\kappa} C^{\lambda\mu\nu\kappa}, \]

where \( C_{\lambda\mu\nu\kappa} \) is the conformal Weyl tensor and \( \alpha \) is a purely dimensionless gravitational coupling constant (we use the system of units in which \( \bar{\hbar} = 1 \) and \( \bar{c} = 1 \)). In particular, they obtained the complete \textit{conformally} static spherically symmetric solution [1,6] of the theory Eq. (1) with the line element given by

\[ ds^2 = C^2(x) ds_0^2, \quad ds_0^2 = -G(r) dt^2 + dr^2/G(r) + r^2 d\Omega, \]

where \( C(x) \) is an arbitrary nonzero function of the spacetime coordinates \( x \), and \( G(r) \) is given by

\[ G(r) = 1 - \beta(2 - 3\beta\gamma)/r - 3\beta\gamma + \gamma r - \kappa r^2. \]

Here, \( \beta, \gamma, \) and \( \kappa \) are integration constants. Having tacitly assumed that test bodies move along the geodesics of the metric with the line element \( ds_0^2 \) of equation (2), Mannheim and

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Kazanas then claimed to recover the Newtonian term ($\propto 1/r$) in the potential of solution (3) of the conformal gravity theory and also suggested [1,5,7] that the additional linear term $\gamma r$ in Eq. (3) might account for the flat galactic rotation curves without having to invoke dark matter.

However, solution (2), (3) of the purely gravitational conformal theory defined by Eq. (1) is not quite relevant to the observations, since it is obtained without regard of the matter part of the theory that includes the mass generation mechanism for the elementary particles and thereby for test bodies such as stars and planets. Such a feature of this solution is reflected in the unrestricted freedom of choosing the conformal factor $C(x)$ in Eq. (2) which clearly affects the timelike geodesics of the metric, but which is totally undetermined thus far. Moreover, the electrovac generalization of solution (2), (3) was previously obtained by Riegert [9], who also asserted that one of the integration constants can be eliminated by further coordinate and conformal transformations. This property of the solution, with $\gamma$ being such constant, was noted also in [10,11] and very recently explicitly demonstrated in [12]. All this makes very problematic the use of the metric given by the second expression in Eq. (2) and by Eq. (3) as an observable one.

In the present paper, we consider this problem taking the matter to be represented by the following generic conformally invariant action [3,4,6]

\[
I_M = -\int d^4x \sqrt{-g} \left[ \partial^\mu S \partial_{\mu} S/2 + \lambda S^4 - S^2R/12 + i\bar{\psi}\gamma^\mu(x)\nabla_\mu \psi - \zeta S\bar{\psi}\psi \right],
\]

where $\psi$ is the fermion field, $S$ is the scalar field, $R$ is the curvature scalar of the metric, and $\lambda$ and $\zeta$ are dimensionless coupling constants. In the theory defined by Eqs. (1), (4), once the scalar field $S$ is everywhere nonzero it can be gauged to an identical constant $S_0$ by a conformal transformation. In this gauge, the fermion part of the action acquires the standard form with constant mass, hence all physical effects receive standard description; in particular, massive particles and test bodies move along the timelike geodesics of the metric as in General Relativity. It is clear that since conformal symmetry is broken and there are massive particles in the real world, one should take solutions with $S$ being nonzero. The physical vacuum is then regarded as the state without excitations of the rest of the matter fields, in our case, the field $\psi$.

We consider solutions outside a compact source formed by the matter fields (represented in our model by the single field $\psi$). The equations of the theory, first introduced by Bach [13] for the case of generic matter, have the form

\[
4\alpha W_{\mu\nu} = T_{\mu\nu},
\]

where the two sides stem, respectively, from the variation of the action (1) and the action (4) with respect to the metric, and the expression of the stress-energy tensor $T_{\mu\nu}$ in the gauge $S \equiv S_0$ and with the $\psi$ field being zero is given by [3,6]

\[
T_{\mu\nu} = -S_0^2 (R_{\mu\nu} - g_{\mu\nu}R/2)/6 - \lambda S_0^4 g_{\mu\nu}.
\]

Equations (5) with the right-hand side given by (6) are nothing but the Bach–Einstein equations with the cosmological constant term — the last term in Eq. (6) (see, e.g., [14]).
Note that the left-hand side of Eq. (5) is identically traceless, and it is convenient to rewrite system (5) as

\[ 4\alpha W_{\mu\nu} = T_{\mu\nu}, \quad R = 24\lambda S_0^2, \quad (7) \]

where \( T_{\mu\nu} \equiv -S_0^2 \left( R_{\mu\nu} - g_{\mu\nu}R/4 \right) /6 \) is the traceless part of the stress-energy tensor \( T_{\mu\nu} \), and the second equation of system (7) is the trace of equation (5).

We restrict ourselves to the static spherically symmetric case. As we explained above, we are interested in the situation where \( T_{\mu\nu} \) is given by Eq. (6) with constant nonzero \( S_0 \). In this gauge, a static spherically symmetric metric can be put in the form

\[ ds^2 = -B(r)dt^2 + A(r)dr^2 + r^2d\Omega, \quad (8) \]

Performing transformation of the radial coordinate as in [1], it is convenient to rewrite this metric in the form

\[ ds^2 = C_2(\rho) \left[ -D(\rho)dt^2 + d\rho^2 / D(\rho) + \rho^2 d\Omega \right], \quad (9) \]

where \( r_2(\rho) = \rho^2 C_2(\rho) \), and the functions \( C(\rho) \) and \( D(\rho) \) are simply related to \( A(r) \) and \( B(r) \) (see [1]).

Due to spherical symmetry and stationarity, the first set of equations (7) will give two independent equations which are conveniently chosen to be

\[ 4\alpha \left( W_0^0 - W_1^1 \right) = T_0^0 - T_1^1, \quad 4\alpha W_1^1 = T_{11}, \quad (10) \]

where index "0" labels the time \( t \) and index "1" labels the radial coordinate \( \rho \). The expressions for the quantities in the left-hand sides of equations (10) for the metric of Eq. (9) were obtained by Mannheim and Kazanas [1,6]:

\[ W_0^0 - W_1^1 = \frac{D(\rho D)^{(4)}}{3\rho C^4}, \quad (11) \]

\[ W_{11} = \frac{1}{3C^2D} \left( \frac{D''D}{2} - \frac{D'^2}{4} \right) \left( \frac{DD'}{\rho} - \frac{2DD'' + D'^2}{\rho^2} + \frac{2DD'}{\rho^3} - \frac{D^2}{\rho^4} + \frac{1}{\rho^4} \right), \quad (12) \]

where the superscript "\( (n) \)" indicates the \( n \)th derivative with respect to the radial coordinate. Note that the function \( C(\rho) \) has been factored out in equations (11) and (12) due to the conformal symmetry of the theory. The expressions for the right-hand sides of equations (10) are

\[ T_0^0 - T_1^1 = \frac{S_0^2 D}{3C} \left( \frac{C'}{C^2} \right)', \quad (13) \]

\[ T_{11} = \frac{S_0^2}{12} \left( \frac{D'}{2D} + \frac{1 - D}{\rho^2 D} + 3F'' + \frac{D'F'}{D} - \frac{2F}{\rho} - 3F' \right), \quad (14) \]
where \( F = \log C \). The scalar curvature is given by
\[
R = \frac{6(\rho^2DC)'' - C(\rho^2(1-D))''}{\rho^2C^3}.
\] (15)

In a general case, it appears to be difficult to obtain an exact solution for \( C(\rho) \) and \( D(\rho) \). However, it is possible to obtain the weak-field limit of the solution. Let the physical metric of Eq. (8) in the spatial region of interest be sufficiently close to the flat one, so that
\[
A(r) = 1 + \epsilon a(r), \quad B(r) = 1 - \epsilon b(r),
\] (16)

where \( \epsilon \) is an auxiliary small parameter to be set equal to unity in the end. Then, in the same region, the functions \( C(\rho) \) and \( D(\rho) \) of the metric of Eq. (9) and the radial coordinate transformation \( r(\rho) \) to the first order in \( \epsilon \) are given by
\[
C(\rho) = 1 + \epsilon f(\rho), \quad D(\rho) = 1 - \epsilon h(\rho), \quad r(\rho) = \rho[1 + \epsilon f(\rho)],
\] (17)

with the functional relation
\[
a(r) = h(r) - 2r f'(r), \quad b(r) = h(r) - 2f(r).
\] (18)

To obtain the system of equations for the functions \( h(\rho) \) and \( f(\rho) \), we linearize the equations of system (7) for the metric of Eqs. (9), (17) in the small parameter \( \epsilon \). First, we note that the scalar curvature \( R \) of this metric is of order \( \epsilon \). Hence, the second equation of system (7) implies that the dimensionless value of \( \lambda S_0^2r^2 \) should also be at least of order \( \epsilon \) in the spatial region under consideration. On the observational grounds, this restriction on the value of \( \lambda S_0^2r^2 \) is quite natural since this value represents the effect of the cosmological constant which is believed to be small on the galactic and stellar spatial scales. However, from the theoretical viewpoint, such a restriction constitutes the fine-tuning problem of the cosmological constant. The solution of this long-standing problem is absent, so we formally replace \( \lambda \) by \( \epsilon \lambda \), thus taking into account the smallness of the corresponding parameter.

It remains to linearize equations (10) and the last equation of system (7) in the small parameter \( \epsilon \). Omitting the simple but cumbersome calculations, we present the corresponding result in the form of the following system for the functions \( h(\rho) \) and \( f(\rho) \):
\[
-(\rho h)^{(4)} = 6p pf'',
\] (19)
\[
\frac{1}{3p} \left( h^{(3)} + \frac{h''}{\rho} - \frac{2h'}{\rho^2} + \frac{2h}{\rho^3} \right) = \frac{p}{2} \left( \frac{h''}{2} - \frac{h}{\rho^2} - 3f'' + \frac{2f'}{\rho} \right),
\] (20)
\[
6 \left( \rho^2f' \right)' = \left( \rho^2h \right)'' + 24q \rho^2,
\] (21)

where we made the notation
\[ p = \frac{S_0^2}{24 \alpha}, \quad q = \lambda S_0^2. \]  \hspace{1cm} (22)

We proceed to the solution of these equations. First, it is convenient to set
\[ h(\rho) = 2m/\rho - 2q\rho^2 + v(\rho), \]  \hspace{1cm} (23)
where \( v(\rho) \) is the new unknown function and \( m \) is a constant. Equations (19) and (21) can be integrated once. The integration constant that appears after the integration of Eq. (19) can be set to any value by rescaling the time and length which shifts \( f(\rho) \) by a constant. We use this property to eliminate the parameter \( q \) from the equations for \( v(\rho) \) and \( f(\rho) \). The integration constant that appears after the integration of Eq. (21) can be eliminated by the redefinition of the constant \( m \) in Eq. (23). As a result, we obtain the following system of equations:
\[ \rho v^{(3)} + 3v'' + 6p(\rho f' - f) = 0, \]  \hspace{1cm} (24)
\[ v^{(3)} + \frac{v''}{\rho} - \frac{3p}{4} \rho v'' - \frac{2v'}{\rho^2} + \frac{(4 + 3p\rho^2)}{2\rho^3} v + \frac{9p}{2} \rho f'' - 3pf' = 0, \]  \hspace{1cm} (25)
\[ v' + 2v/\rho - 6f' = 0. \]  \hspace{1cm} (26)

Now it is convenient to proceed to the new unknown functions \( \tilde{a}(\rho) \) and \( \tilde{b}(\rho) \) related to \( v(\rho) \) and \( f(\rho) \) as follows [cf. Eq. (18)]:
\[ \tilde{a}(\rho) = v(\rho) - 2\rho f'(\rho), \quad \tilde{b}(\rho) = v(\rho) - 2f(\rho). \]  \hspace{1cm} (27)
From system (24)–(26), one easily obtains the following system of equations for \( \tilde{a}(\rho) \) and \( \tilde{b}(\rho) \):
\[ \tilde{b}'' + \frac{2\tilde{b}'}{\rho} + p \tilde{b} = 0, \quad \rho \tilde{b}' + 2\tilde{a} = 0. \]  \hspace{1cm} (28)
Its solution is straightforward and depends on the sign of the constant \( p \) that coincides with the sign of \( \alpha \) [see Eq. (22)]. First, we consider the case where \( p > 0 \). We obtain
\[ \tilde{a}(\rho) = n \left[ \frac{\sin (k\rho + \phi)}{\rho} - k \cos (k\rho + \phi) \right], \]  \hspace{1cm} (29)
\[ \tilde{b}(\rho) = \frac{2n \sin (k\rho + \phi)}{\rho}, \]  \hspace{1cm} (30)
where \( k = \sqrt{p} \), and \( n \) and \( \phi \) are integration constants. Combining Eqs. (18), (23), (27), and (29), (30), we eventually obtain the following solution of our problem:
\begin{align*}
a(r) & = \frac{2m}{r} - 2qr^2 + n \left[ \frac{\sin (kr + \phi)}{r} - k \cos (kr + \phi) \right], \quad (31) \\
b(r) & = \frac{2m + 2n \sin (kr + \phi)}{r} - 2qr^2. \quad (32)
\end{align*}

We see that in the Newtonian limit, apart from the universal term $qr^2$, there arises the additional gravitational potential
\[ V(r) = - \frac{m + n \sin (kr + \phi)}{r}, \quad (33) \]

in which the constants $m$, $n$, and $\phi$ are to be related to the source. The constants $k = \sqrt{p}$ and $q$ are universal and are given by Eq. (22).

We note that the linearized static spherically symmetric solutions in a generic (not conformally invariant) second-order gravitational theory without the cosmological constant were obtained in [15]. Their structure is similar to that of (31), (32) and to the solutions (42), (43) below. However, it is not possible to pass to a direct limit of conformal invariance in the solutions of [15], because the case of conformal invariance is characterized by a nontrivial degeneracy, in particular, the massive scalar degree of freedom which is present in a generic case is missing here (see also [14] in this respect).

Now suppose that a static compact source is composed of identical “atoms” (these may be real atoms or elementary particles) and that each of these atoms produces static gravitational potential as given by Eq. (33) with identical constants $m$, $n$, and $\phi$. In view of the weakness of the potential, we also assume the validity of the superposition principle. Then, if $\mu(r)$ is the spatial distribution of the “atoms” in the source, the total potential is given by the expression
\[ \Phi(r) = \int V(|r - r'|) \mu(r') d^3r'. \quad (34) \]

This potential is the sum of two terms: $\Phi(r) = \Phi_m(r) + \Phi_n(r)$. They satisfy the equations
\begin{align*}
\Delta \Phi_m(r) & = 4\pi m \mu(r), \\
\Delta \Phi_n(r) + p \Phi_n(r) & = 4\pi n \sin \phi \mu(r), \quad (35)
\end{align*}

that, in the theory under investigation, correspond to the unique Poisson equation of the linearized General Relativity.

For a spherically symmetric compact distribution $\mu(r)$, the potential given by Eq. (34) with the kernel given by Eq. (33) is easily calculated:
\[ \Phi(r) = - \int_r^{\infty} \frac{M(r')}{r'^2} dr' - \frac{N \sin (kr + \phi)}{kr} - \frac{4\pi n \sin \phi}{kr} \int_r^{\infty} \mu(r') \sin [k (r - r')] r' dr', \quad (36) \]

where
\[ M(r) = 4\pi m \int_0^r \mu(r') r'^2 dr', \quad N = \frac{4\pi n}{k} \int_0^\infty \mu(r') r' dr'. \] (37)

Thus, outside the source, the potential of the form (33) is reproduced with the same phase \( \phi \), but with different coefficients \( m \) and \( n \). Moreover, while the coefficient \( m \) is additive (it plays the role of the gravitational mass of the source), the coefficient \( n \) is not: its new value \( N \) is given by the second expression in Eq. (37). However, the coefficient \( n \) becomes approximately additive for a distribution whose spatial size is significantly less than \( 1/k \).

If the product \( kr < 1 \) in the region of interest (say, on galactic scales), one can expand the oscillatory part of Eq. (33) in powers of \( kr \) to obtain

\[ V(r) = V_0 - \frac{M_0}{r} + \frac{\Gamma r}{2} + Qr^2 + \mathcal{O}\left((kr)^3\right), \] (38)

where \( V_0 = -nk \cos \phi \), \( M_0 = m + n \sin \phi \), \( \Gamma = nk^2 \sin \phi \), and \( Q = q + nk^3 \cos \phi / 6 \). We thus recover the linear term in the potential of Eq. (38), similar to that which was used by Mannheim and Kazanas [1,5,7] to account for the flat galactic rotation curves. However, there exists an important observational bound that rules out the possibility for the linear term in the expansion (38) to play a significant role on galactic scales. Note that the coefficients \(-g_{00}(r)\) and \(g_{rr}(r)\) of the metric of our solution are not mutually inverse, which is reflected in the fact that the functions \(a(r)\) and \(b(r)\), given, respectively, by Eqs. (31) and (32), are not equal to each other. At small enough distances, both functions reproduce Newtonian potentials with the masses, respectively, \( m_0 = m + n \sin \phi \) and \( m_1 = m + (n \sin \phi)/2 \), the difference between them being \( \Delta m = (n \sin \phi)/2 \). At the same time, the *Viking* spacecraft observations in the vicinity of the Sun indicate that the ratio \( \Delta m/m \lesssim 2 \times 10^{-3} \) [16] (see also [17]). This implies the following observational bound for the Sun:

\[ \frac{n \sin \phi}{m} \lesssim 4 \times 10^{-3}. \] (39)

Since we assume that the parameter \( k \) is sufficiently small so that the expansion (38) is legitimate on galactic scales, the values of both \( m \) and \( n \) are additive on such scales and the estimate (39) is valid on galactic scales as well. Now, the linear term in Eq. (38) formally becomes comparable in magnitude to the Newtonian one only at the distance \( r \sim \sqrt{M/\Gamma} \approx \sqrt{m/(nk^2 \sin \phi)} \). But, for such distances, we would have \( kr \sim \sqrt{m/(n \sin \phi)} \gtrsim 10 \) because of estimate (39), which contradicts the original assumption \( kr < 1 \). Thus, the linear term in the expansion (38) cannot play a significant role on galactic scales, and one should rather try the whole potential in the form (36) for a spherically symmetric source with the bound (39) to account for the galactic rotation curves. Such a possibility still remains to be investigated.

It is instructive to estimate the realistic value of the constant \( \alpha \) in Eq. (1) for which the value of \( kr \) is of order unity on a typical galactic scale of 10 kpc, thus making the potential of the form (33) in principle relevant to the galactic rotation curves. Whatever scalar fields are present in the theory, they all contribute to the value of \( p \) given by Eq. (22). Thus, at least the scalar Higgs field of the Standard Model of strong and electroweak interactions should be taken into account. The mean value of this field is known to be \( \eta \approx 246 \text{ Gev} \),
this value will contribute to $S_0$ in Eq. (22) and, in order that $k \times (10 \text{ kpc}) \lesssim 1$ be valid, we must have

$$\alpha \gtrsim 10^{74};$$

(40)

which, of course, is a severe restriction. It is possible to conceive models in which this restriction is weakened; for instance, one can introduce another scalar field with the “wrong” overall sign in the action, so that its contribution to the value of the parameter $p$ will be of the opposite sign. Its vacuum expectation value has to be fine-tuned to counterbalance the contribution of $\eta$. Another possibility is to try to construct particle theory without fundamental scalar fields, although it may turn out to be very difficult to do this in a conformally invariant manner.

On the other hand, if we take $\alpha \sim 1$, then the expectation value $\eta \simeq 246 \text{ GeV}$ of the Standard Model Higgs field leads to the spatial scale

$$1/k \sim 10^{-16} \text{ cm}$$

(41)

on which the potential (33) oscillates. Its significance might only be manifest on the spatial scales of elementary particles, where, of course, the whole theory must be quantized.

In the case of $p < 0$, which corresponds to $\alpha < 0$, the solution for $a(r)$ and $b(r)$ has the form

$$a(r) = \frac{2m}{r} - 2qr^2 + n_1 (1 + kr) \frac{e^{-kr}}{r} + n_2 (1 - kr) \frac{e^{kr}}{r},$$

(42)

$$b(r) = \frac{2m}{r} - 2qr^2 + 2n_1 \frac{e^{-kr}}{r} + 2n_2 \frac{e^{kr}}{r},$$

(43)

where now $k = \sqrt{-p}$, and $n_1$ and $n_2$ are integration constants. Similar solutions in a generic second-order gravitational theory (not conformally invariant) without the cosmological constant were obtained in [15]. Solutions in the conformally invariant second-order theory with the Einstein term but without the cosmological-constant term were also obtained in [14]. The physically meaningful solution is selected by imposing boundary conditions at infinity, what leads to the condition $n_2 = 0$. For sufficiently small values of $k$, the observational bound similar to (39) implies

$$\frac{n_1}{m} \lesssim 4 \times 10^{-3},$$

(44)

and makes the extra exponential potential in (43) uninteresting.

Finally, we note that in the case of $p < 0$, which corresponds to $\alpha < 0$, one can also obtain solutions by formally replacing the trigonometric functions in Eqs. (31), (32), and (33) by their hyperbolic counterparts and taking $k = \sqrt{-p}$. Equations (35) will then remain valid in this case as well, with the replacement of $\sin \phi$ by $\sinh \phi$. The structure of the left-hand sides of equations (35) reflects, besides the presence of the massless graviton, also the well-known
presence of a spin-two tachion (in the case of $\alpha > 0$) or a spin-two massive ghost (in the case of $\alpha < 0$) on the background with $S \neq 0$ of the theory described by Eqs. (1), (4) (see [15,18,9] and references therein). The presence of a tachion in the case of $\alpha > 0$ indicates instability of a large class of classical solutions, including the flat space-time solution in the case of $\lambda = 0$; and the presence of a ghost in the case of $\alpha < 0$ implies possible absence of perturbative unitarity in the corresponding quantum theory. This appears to be the main drawback of the conformal theory under discussion.

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[15] K. S. Stelle, Gen. Rel. and Grav. 9, 353 (1978). There appears to be an error in Eq. (3.6b) of this reference: the terms $\propto r^{-1} \exp(\pm m_2 r)$ should be multiplied by 1/2; see the second reference in [14].