Kruskal Coordinates and Mass of Schwarzschild Black Holes

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Schwarzschild coordinates \((r, t)\) fail to describe the region within the event horizon \((\text{EH})\), \((r \leq r_g)\), of a Black Hole \((\text{BH})\) because the metric coefficients exhibit singularity at \(r = r_g\) and the radial geodesic of a particle appears to be null \((ds^2 = 0)\) when actually it must be timelike \((ds^2 > 0)\), if \(r_g > 0\). Thus, both the exterior and the interior regions of BHs are described by singularity free Kruskal coordinates. However, we show that, in this case too, \(ds^2 \to 0\) for \(r \to r_g\). And this result can be physically reconciled only if the EH coincides with the central singularity or if the mass of Schwarzschild black holes \(M \equiv 0\).

The concept of Black Holes (BHs) is one of the most important plinths of modern physics and astrophysics. As is well known, the basic concept of BHs actually arose more than two hundred years ago in the cradle of Newtonian gravitation [1]. In General Theory of Relativity (GTR), the gravitational mass is less than the baryonic mass \((M \leq M_0)\). Further, as the body contracts and emits radiation \(M\) keeps on decreasing progressively alongwith \(r\). Thus, given an initial gravitational mass \(M_i\), one can not predict with certainty the value of \(M_f\) when we would have \(2M_f/r = 1\) \((G = c = 1)\).

Neither are the values of \(M_i\), \(M_f\) and \(M_0\) related by any combination of fundamental constants though, it is generally assumed that \(M_i \approx M_f\). Ideally, one should solve the Einstein equations analytically to fix the value of \(M_f\) for a given initial values of \(M_i\) and \(M_0\) for a realistic equation of state (EOS) and energy transport properties. However even when one does away with the EOS by assuming the matter to behave like a dust, \(p \equiv 0\), one does not obtain any unique solution if the dust is inhomogeneous. Depending on the various initial conditions and assumptions (like self-similarity) employed one may end up finding either a BH or a “naked singularity” [2]. By further assuming the dust to be homogeneous Oppenheimer and Snyder (OS) [3] found asymptotic solution of the problem by approximating Eq.(36) of their paper. The region exterior to the event horizon \((r > r_g = 2M)\) can be described by the Schwarzschild coordinates \(r\) and \(t\) [4,5]:

\[
ds^2 = g_{tt}dt^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2
\]

where \(g_{tt} = (1 - 2M/r), g_{rr} = -(1 - 2M/r)^{-1}, g_{\theta\theta} = -r^2, \) and \(g_{\phi\phi} = -r^2\sin^2\theta\). Here, we are working with a spacetime signature of +1, -1, -1, -1 and \(r\) has a distinct physical significance as the invariant circumference radius. For \(r > r_g = 2M\), the worldline of a free falling radial material particle is indeed timelike \(ds^2 > 0\) and the metric coefficients have the right signature, \(g_{tt} > 0, g_{rr} < 0, g_{\theta\theta} < 0\) and \(g_{\phi\phi} < 0\). But at \(r = 2M\), \(g_{rr}\) blows up and as \(r < 2M\), the \(g_{tt}\) and \(g_{rr}\) suddenly exchange their signatures though the signatures of \(g_{\theta\theta}\) and \(g_{\phi\phi}\) remain unchanged. This is interpreted by saying that, inside the event horizon, \(r\) becomes “time like” and \(t\) becomes “spacelike” [4,5]. However, we see that actually \(r\) continues to retain, atleast partially, its spacelike character by continuing to be “invariant circumference radius”.

Also, note that, if physically measurable quantities like the Riemannian curvature components behaved like \(\sim M/r^3\) outside the EH, they continue to behave in a similar manner, and not like \(\sim M/t^3\) inside the EH. And it should be borne in mind here that by a fresh relabelling or by any other means, the curvature components can not be made to assume the form \(\sim M/t^3\). One particular reason for this is that, we would see later that, inside the EH, we have \(t \to \infty\) while, of course, the value of \(r\) remains finite. Thus it may not actually be justified to conclude that \(r\) becomes the “timelike coordinate” inside the EH even though \(g_{rr}\) changes its sign. So far, it has not been possible to resolve this enigma of the duality in the behaviour of \(r\) for \(r < 2M\), and the present paper intends to attend to this problem. Since \(ds\) is the proper time, we may also write

\[
ds^2 = dt^2 \left(1 - \frac{2M}{r}\right)
\]

Therefore, the radial geodesic of a material particle in the Schwarzschild metric becomes, unphysically null \((ds^2 = 0)\) and then spacelike \((ds^2 < 0)\) as one moves inside the event horizon (EH). In contrast, any physically meaningful coordinate system must be free of such anomalies. Although \(g_{rr}\) blows up at \(r = 2M\), as mentioned before, the curvature components of the Riemannian tensor behave perfectly normally at \(r = r_g\). \(R_{ij}^3 \sim M/r^3\). Further, the determinant of the metric coefficients continues to be negative and finite \(g = r^4\sin^2\theta g_{rr} g_{tt} = r^4\sin^2\theta \leq 0\). Such realizations gave rise to the idea that the Schwarzschild coordinate system suffers from a “coordinate singularity” at
the event horizon and must be replaced some other well behaved coordinate system. It is known that a comoving coordinate system is naturally singularity free and Lemaître suggested that the region inside $r \leq r_g$ may be represented by such a coordinate system [6] whereas the exterior region is still described by the old Schwarzschild coordinates. It is only in 1960 that Kruskal and Szekeres [4–6] discovered a one-piece coordinate system which can describe both the interior and exterior regions of a BH. They achieved this by means of the following coordinate transformation for the exterior region (Sector I):

$$u = f_1(r) \cosh \frac{t}{4M}; \quad v = f_1(r) \sinh \frac{t}{4M}; \quad r \geq 2M$$

(3)

where

$$f_1(r) = \left( \frac{r}{2M} - 1 \right)^{1/2} e^{r/4M}$$

(4)

It would be profitable to note that

$$\frac{df_1}{dr} = \frac{r}{8M^2} \left( \frac{r}{2M} - 1 \right)^{-1/2} e^{r/4M}$$

(5)

And for the region interior to the horizon (Sector II), we have

$$u = f_2(r) \sinh \frac{t}{4M}; \quad v = f_2(r) \cosh \frac{t}{4M}; \quad r \leq 2M$$

(6)

where

$$f_2(r) = \left( 1 - \frac{r}{2M} \right)^{1/2} e^{r/4M}$$

(7)

and

$$\frac{df_2}{dr} = -\frac{r}{8M^2} \left( 1 - \frac{r}{2M} \right)^{-1/2} e^{r/4M}$$

(8)

Given our adopted signature of spacetime ($-2$), in terms of $u$ and $v$, the metric for the entire spacetime is

$$ds^2 = \frac{32M^3}{r} e^{-r/2M} (dv^2 - du^2) - r^2 (d\theta^2 + d\phi^2 \sin^2 \theta)$$

(9)

The metric coefficients are regular everywhere except at the intrinsic singularity $r = 0$, as is expected. Note that, the angular part of the metric remains unchanged by such transformations and $r(u, v)$ continues to signal its intrinsic spacelike nature. In either region we have

$$u^2 - v^2 = \left( \frac{r}{2M} - 1 \right) e^{r/2M}$$

(10)

so that

$$u^2 - v^2 > 1; \quad u/v > \pm 1; \quad r > 2M,$$

(11)

$$u^2 - v^2 \to 0; \quad u = \pm v; \quad r = 2M$$

(12)

and

$$u^2 - v^2 < 0; \quad u/v < \pm 1; \quad r < 2M$$

(13)

So, each of these above three inequalities, and, in particular, the $r = 0$ point corresponds to not one but two conditions!

$$v = \pm (1 + u^2)^{1/2}$$

(14)

Here, one point needs to be hardly overemphasized: astronomical observations and experiments actually conform to the idea that at least far from massive bodies or probable BHs, the spacetime is well described by the $r, t$ coordinate system. In fact, although in the (normal) physical spacetime, in a spherically symmetric spatial geometry (as defined
by the implications of \( r \) as an “invariant circumference radius”), the physical singularity corresponds to a mathematical point, in the Kruskal world view, this central singularity corresponds to a pair of hyperbolas in the \((u-v)\) plane. While the “+ve” sign of equation corresponds to the central BH singularity, the “-ve” sign corresponds to the singularity inside a so-called White Hole which may spew out mass-energy spontaneously in “our universe” [4,5]. The white hole singularity belongs to “other universe” whose presence is suggested by the fact that the Kruskal metric remains unaffected by the following additional transformations:

\[
\begin{align*}
    u &= -f_1(r) \cosh \frac{t}{4M}; \quad v = -f_1(r) \sinh \frac{t}{4M}; r \geq 2M \\
    u &= -f_2(r) \sinh \frac{t}{4M}; \quad v = -f_2(r) \cosh \frac{t}{4M}; r \leq 2M
\end{align*}
\]

defining Sector (III) and

\[
\begin{align*}
    u &= -f_1(r) \cosh \frac{t}{4M}; \quad v = -f_1(r) \sinh \frac{t}{4M}; r \geq 2M \\
    u &= -f_2(r) \sinh \frac{t}{4M}; \quad v = -f_2(r) \cosh \frac{t}{4M}; r \leq 2M
\end{align*}
\]

defining Sector (IV). Thus not only the region interior to the EH corresponds to two different universes, (Sector II and IV) but the structure of the physical spacetime outside the EH, too, effectively corresponds to two universes (Sector I and III). If there exists \( N \) number of BHs, the (normal) physical spacetime may be much more complex. The aim of this paper is to explicitly verify whether the (radial) geodesics of material particles are indeed timelike at the EH which they must be if this idea of a finite mass Schwarzschild BH is physically correct. First we focus attention on the region \( r \geq 2M \) and differentiate Eq.(3) to see

\[
\frac{du}{dr} = \frac{\partial u}{\partial r} + \frac{\partial u}{\partial t} \frac{dt}{dr} = \frac{df}{dr} \cosh \frac{t}{4M} + \frac{f}{4M} \sinh \frac{t}{4M} \frac{dt}{dr}
\]

Now by using Eq. (4-6) in the above equation, we find that

\[
\frac{du}{dr} = \frac{ru}{8M^2} (r/2M - 1)^{-1} + \frac{v}{4M} \frac{dt}{dr}; \quad r \geq 2M
\]

and

\[
\frac{dv}{dr} = \frac{rv}{8M^2} (r/2M - 1)^{-1} + \frac{u}{4M} \frac{dt}{dr}; \quad r \geq 2M
\]

By dividing equation (18) by (19), we obtain

\[
\frac{du}{dv} = \frac{ru}{2M^2} + \frac{v}{2M^2} \left( \frac{r}{r/2M - 1} \right)
\]

Similarly, starting from Eq. (6), we end up obtaining a form of \( du/dv \) for the region \( r < 2M \) which is exactly similar to the foregoing equation. Now, by using Eq.(12) \((u = \pm v)\) in Eq. (20), we promptly find that

\[
\frac{du}{dv} \rightarrow \frac{r}{2M^2} \pm \frac{dt}{dr} \left( \frac{r}{r/2M - 1} \right) \rightarrow \pm 1; \quad r \rightarrow 2M
\]

Thus, we are able to find the precise value of \( du/dv \) at the EH in a most general manner irrespective of the precise relationship between \( t \) and \( r \). Armed with this value of \( du/dv \), we are in a position now to complete our task by rewriting the radial part of the Kruskal metric \((d\theta = d\phi = 0)\) as

\[
ds^2 = \frac{32M^3}{r} e^{-r/2M} \left[ 1 - \left( \frac{du}{dv} \right)^2 \right]
\]

Or,

\[
ds^2 = 16M^2 e^{-1} \left( 1 - \frac{1}{r} \right); \quad r = 2M
\]

We have found that for the Lemaitre coordinate too, \( ds^2 = 0 \) at \( r = 2M \). This implies that although the metric coefficients can be made to appear regular, the radial geodesic of a material particle becomes null at the event horizon of a finite mass BH in contravention of the basic premises of GTR! And since, now, we can not blame the coordinate system to be faulty for this occurrence, the only way we can explain this result is that the Event Horizon itself
corresponds to the physical singularity or, in other words, the mass of the Schwarzschild BH \( M \equiv 0 \). And then, the entire conundrum of “Schwarzschild singularity”, “swapping of spatial and temporal characters by \( r \) and \( t \) inside the event horizon (when the angular part of all metrics suggest that \( r \) has a spacelike character even within the horizon), “White Holes” and “Other Universes” get resolved. Here we recall the conjecture of Rosen \[8\] “so that in this region \( r \) is timelike and \( t \) is spacelike. However, this is an impossible situation, for we have seen that \( r \) defined in terms of the circumference of a circle so that \( r \) is spacelike, and we are therefore faced with a contradiction. We must conclude that the portion of space corresponding to \( r < 2M \) is non-physical. This is a situation which a coordinate transformation even one which removes a singularity can not change. What it means is that the surface \( r = 2M \) represents the boundary of physical space and should be regarded as an impenetrable barrier for particles and light rays.” This idea of Rosen is also in accordance with the idea of Einstein that the Schwarzschild type singularity is unphysical and can not occur for realistic cases \[9\]. And this paper indeed shows in order that the radial worldlines of free falling material particles do not become null at a mere coordinate singularity, Nature (GTR) refuses to have any spacetime within the EH.

Although, having made our basic point, we could have ended this paper at this point, for the sake of further insight, we shall study the behaviour of \( ds^2 \) for the entire spacetime by, again assuming, for a moment, the existence of a finite mass BH. It can be found that in the region \( r > 2M \), one would indeed have \( ds^2 > 0 \) for \( r > 2M \). And to see the behaviour of \( du/dv \) inside the EH, we recall the relationship between \( t \) and \( r \) (see pp. 824 of ref.\[4\] or pp. 343 of ref.\[5\]):

\[
\frac{t}{2M} = \ln \left| \frac{(r_\infty/2M - 1)^{1/2} + \tan(\eta/2)}{(r_\infty/2M - 1)^{1/2} - \tan(\eta/2)} \right| + 2M \left( \frac{r_\infty}{2M} - 1 \right)^{1/2} \left[ \eta + \left( \frac{r_\infty}{4M} \right) (\eta + \sin \eta) \right]
\]  

(24)

where the particle is released with zero velocity from \( r = r_\infty \) at \( t = 0 \) and the “cyclic” coordinate \( \eta \) is defined by

\[
r = \frac{r_\infty}{2} (1 + \cos \eta)
\]  

(25)

Since \( \tan(\eta/2) = (r_\infty/r - 1) \) we find from Eq. (24) that, as \( r \to 2M \), the logarithmic term blows up and \( t \to \infty \), which is a well known result. And since \( t \) continues to increase as the particle enters the EH, we have the general result that \( t = \infty \) for \( r \leq 2M \) In this limit, we have

\[
\cosh \frac{t}{4M} \to \sinh \frac{t}{4M} \to \frac{e^{t/4M}}{2} = \infty
\]  

(26)

Consequently, even though, \( u^2 - v^2 \) continues to be finite we obtain

\[
\frac{u}{v} = \pm 1; \quad r \leq 2M
\]  

(27)

Hence we obtain a more general form of Eq. (21)

\[
\frac{du}{dv} \to \pm 1; \quad r \leq 2M
\]  

(28)

irrespective of the precise form of \( dt/dr \). Then from Eq. (22), we find that the metric would continue to be null for \( r < 2M \):

\[
ds^2 = 0; \quad r \leq 2M
\]  

(29)

And this unphysical happening is of course avoided when we realize that \( M = 0 \) and there is no additional spacetime between the EH and the central singularity. We may mention now that we have recently shown that the OS work too actually suggests that the mass of the resultant BH must be \( M \equiv 0 \) \[10\]. The basic reason for this assertion is extremely simple. The Eq.(36) of OS paper connects  \( t \) and \( r \) through a relationship which, for large values of \( t \) is

\[
t \sim \ln \frac{y^{1/2} + 1}{y^{1/2} - 1}
\]  

(30)

where at the boundary of the fluid

\[
y = \frac{r}{r_0} = \frac{r}{2M}
\]  

(31)
Since the argument of a logarithmic function can not be negative, in order that \( t \) is definable at all we must have

\[
y = \frac{r}{2M} \geq 1; \quad \frac{2M}{r} \leq 1
\]  

(32)

Thus at least for the collapse of a homogeneous dust, “trapped surfaces” do not form and if the collapse continues to the point \( r \to 0 \) we must have \( M_f \to 0 \). This independent finding is in complete agreement with what we have shown in the present paper that Schwarzschild BHs must have \( M = 0 \). Although, there is no modulus here in the argument of the logarithmic of Eq. (30) (unlike Eq. [24]), some readers may wish there were one. Even if one imagined the existence of such a modulus, one would run into contradiction in the following way. Of course we will have \( t \to \infty \) as \( r \to 2M \). But during the collapse if one would enter \( r < 2M \) (if \( M > 0 \)), \( t \) would start decreasing!

However, unlike the case of Newtonian gravity, in GTR, \( M = 0 \) state need not correspond to a configuration with zero baryonic mass. The \( M = 0 \) state is simply one in which the negative gravitational energy exactly offsets the positive energy associated with \( M_0 \) and internal energy, and may indeed represent a physical singularity with infinite energy density and tidal acceleration. For instance, if the collapse process leads to the \( y = 1 \) limit, then the curvature components \( R_{ij} \sim M/r^3 \sim r^{-2} \to \infty \) as \( r \to 0 \). Note also that, the metric coefficients \( g_{uu} \) and \( g_{vv} \) for the zero-mass BH blow up in a similar fashion at the EH. It may be noted that the “naked singularities” too may be characterized by \( M = 0 \) [11]. In the context of the dust collapse, we see that, for, \( M = 0 \), the proper time for the formation of the BH would be infinite

\[
\tau = \pi \left( \frac{r_3}{8M} \right)^{1/2} = \infty
\]  

(33)

Further, we have shown elsewhere that the crucial condition (32), \( y \geq 1 \), is valid not only for the OS problem, but also for any generic spherical gravitational collapse [12]. And similarly, \( \tau \to \infty \) as \( r \to 0 \) not only for dust collapse, but also for the collapse of any physical fluid [12]. Thus at any given finite proper time there would be no BH, and on the other hand there could be dynamically collapsing configurations with arbitrary high surface redshifts. In fact it can be found that the \emph{proper length} of a radial geodesic becomes infinite too [12]. And therefore, even if, such dynamically configurations with large surface red-shifts may be collapsing with relativistic velocities, the collapse process will never terminate in any finite amount of time. This happens because spacetime would get infinitely stretched by infinite curvature near \( r = 0 \). This is a purely general relativistic effect, and is difficult to comprehend by “common astronomical sense”. Observationally, such configurations may be identified as Black Holes. And if some of these configurations are collapsing with nearly free fall speed, accretion onto such configurations would emit little radiation if the accretion flow happens to be advection dominated. To conclude, irrespective of the observational consequences, we have \emph{directly} shown that, if GTR is correct, Schwarzschild BHs must have \( M \equiv 0 \) in order that the radial geodesics of material particles remain timelike at a finite value of \( r \).

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