Loop Soliton Solutions of String Interacting with External Field

Hiroshi Kakuhata and Kimiaki Konno*
Tsuruga Junior College, Tsuruga 914-0814
and

Atomic Energy Research Institute,
Nihon University, Tokyo 101-8308
and

*Department of Physics, College of Science and Technology,
Nihon University, Tokyo 101-8308

Abstract

We give a physical model of the $O(3)$ -invariant coupled integrable dispersionless equations where a string with internal structure is interacting to the external field. We derive loop soliton solutions through the bilinear equations in which solitons can move to positive and negative directions. We study two soliton interactions in detail.

Key Words: string model, dispersionless equations, loop solitons.

SCAN-9904027

CERN LIBRARIES, GENEVA

§1 Introduction

Recently, the present authors and others have been investigating coupled dispersionless systems. Generalized form of the coupled dispersionless system is given by

$$S_{xt} + [S_x, [G, S]] = 0, \quad (1)$$

where the matrix field $S(x, t) = \phi_a(x, t)T^a$ and the constant $G = \kappa_a T^a$ are elements of an arbitrary Lie algebra with the generators T^a of the Lie group satisfying the commutation relation $[T^a, T^b] = if_c^{ab}T^c$. Here f_c^{ab} denotes the structure constant of the Lie algebra. As a special feature, the system has an important conserved quantity

$$P = \frac{1}{2}\text{Tr}(S_x^2), \quad (2)$$

and the inverse scattering scheme^{1, 2)}. For the special cases of the Lie algebra $su(2) \cong o(3)$ and $su(1, 1) \cong o(2, 1) \cong sl(2, R)$, we solved the inverse scattering problem and obtained an infinite number of conserved quantities. Therefore, the system (1) is shown to be integrable^{3, 4, 5, 6)}. We have not their physical applications until now.

In this paper, we give a physical application of the $o(3)$ case which is shown to be equivalent to the Pohlmeyer-Lund-Regge system⁷⁾. The organization of this paper is as follows. As a first step, we give a physical model of the general dispersionless system (1) based on the Lagrangian formalism. Next, we derive loop type of one and two soliton solutions via the bilinear transformation and discuss the model comparing with the loop solitons of elastica⁹⁾. The last section is devoted to concluding remarks.

§2 A Model

Let us consider the Lagrangian

$$\mathcal{L} = \text{Tr} \left(\frac{1}{2} S_x S_t - \frac{1}{3} G[S, [S_x, S]] \right) \quad (3)$$

which give us the general dispersionless equation (1). The canonical conjugate momentum is defined by

$$\Pi \equiv \frac{\partial \mathcal{L}}{\partial S_t} = \frac{1}{2} S_x. \quad (4)$$

Equation(4) represents a constraint equation whose correct treatment is described in ref.2. In terms of the ‘canonical variables’ Π and S , the dispersionless system can be written in the form

$$\Pi_t = [\Pi, [S, G]]. \quad (5)$$

The conserved quantity (2)

$$P = 2\text{Tr}(\Pi^2) \quad (6)$$

is able to be interpreted as momentum conservation. This situation is the same as a charged particle moving in an external magnetic field. Therefore, as we shall discuss below, we may say that the coupled dispersionless system describes equations of motion of the ‘charged’ object in the external field $[S, G]$. The particle dynamics has one independent variable, while, in our system, there are two independent variables x and t . As Lund and Regge derived originally their system from the relativistic string interacting with the totally anti-symmetric tensor field⁸⁾, we suppose that our system describes a string which is an one-dimensionally extended object living in the vector space spanned by elements of the Lie algebra.

Since we restrict ourselves the $o(3)$ case, the physical space is the 3-dimensional Euclidian space $\mathbf{R}^3$. Let us assign the field ϕ_α and the constant κ_α to the position vector $\mathbf{r} = (X, Y, Z)$ of the string and the constant external electric current $\mathbf{J}$, respectively and take the Lie algebra as $[T^a, T^b] = i\varepsilon^{abc}T^c$. Furthermore, we introduce time τ and the arc length σ of the string through the transformation $\tau = x - t$ and $\sigma = x + t$.

Equation(1) becomes

$$\mathbf{r}_{\tau\tau} - \mathbf{r}_{\sigma\sigma} = (\mathbf{r}_\tau + \mathbf{r}_\sigma) \times (\mathbf{J} \times \mathbf{r}). \quad (7)$$

To investigate the physical structure of this equation, at first, let us consider the static case in which eq.(7) is reduced to

$$\mathbf{r}_{\sigma\sigma} = -\mathbf{r}_\sigma \times (\mathbf{J} \times \mathbf{r}). \quad (8)$$

The factor $\mathbf{J} \times \mathbf{r}$ in the right hand side of eq.(8) can be regarded as a magnetic field generated by $\mathbf{J}$ because it satisfies the Maxwell equations

$$\begin{aligned} \text{rot}(\mathbf{J} \times \mathbf{r}) &= 2\mathbf{J}, \\ \text{div}(\mathbf{J} \times \mathbf{r}) &= 0. \end{aligned} \quad (9)$$

If we interpret $\mathbf{r}_\sigma$ as an internal electric current, then right hand side of eq.(8) represents the Lorentz force acting on $\mathbf{r}_\sigma$. The factor $\mathbf{r}_\tau + \mathbf{r}_\sigma$ in the right hand side of eq.(7) is an effective internal current in which $\mathbf{r}_\tau$ is a correction term to $\mathbf{r}_\sigma$, that is, it is induced by the motion of the string. Hence, the right hand side of eq.(7) may be interpreted to represents current-fed string interacting with the external magnetic field $\mathbf{J} \times \mathbf{r}$. Equation(8) shows us that in contrast to the KdV-type equations in which dispersion effect balances the nonlinearity, the nonlinear external force of the dispersionless equations balances to the linear elasticity of the string. With the components of the position vector $\mathbf{r}$, eq.(7) becomes the following coupled nonlinear equations of the Klein-Gordon type

$$\begin{aligned} X_{\tau\tau} - X_{\sigma\sigma} &= -(Z_\tau + Z_\sigma)X, \\ Y_{\tau\tau} - Y_{\sigma\sigma} &= -(Z_\tau + Z_\sigma)Y, \\ Z_{\tau\tau} - Z_{\sigma\sigma} &= (X_\tau + X_\sigma)X + (Y_\tau + Y_\sigma)Y, \end{aligned} \tag{10}$$

with $\mathbf{J} = (0, 0, 1)$.

§3 Loop Solitons

Now, we consider bilinearization of these equations to obtain the soliton solutions under the boundary condition $\mathbf{r} \rightarrow (0, 0, \sigma)$ for $|\sigma| \rightarrow \infty$. The bilinear equations for the dispersionless equations are firstly derived by Alagesan and Porsezian through the Painlevé analysis^{11, 12)}. The other type bilinear equations for the Pohlmeyer-Lund-Regge equations had been derived by Hirota¹³⁾. Unfortunately, the known soliton solutions of eq.(10) do not satisfy the boundary condition. We present the bilinear equations satisfying our physical situation and derive the soliton solutions.

Under the following transformation

$$\begin{aligned} X &= \frac{G}{F}, \\ Y &= \frac{H}{F}, \\ Z &= \sigma + 2(\partial_\tau - \partial_\sigma) \log F, \end{aligned} \tag{11}$$

eqs.(10) become the bilinear equations

$$\begin{aligned}(D_\tau^2 - D_\sigma^2 + 1)F \cdot G &= 0, \\ (D_\tau^2 - D_\sigma^2 + 1)F \cdot H &= 0, \\ (D_\tau - D_\sigma)^2 F \cdot F - \frac{1}{2}(G^2 + H^2) &= 0,\end{aligned}\tag{12}$$

where D s denote the Hirota's derivatives. According to the usual procedure, we expand F , G and H in a formal power series of ε as

$$\begin{aligned}F &= 1 + \varepsilon^2 f_2 + \varepsilon^4 f_4 + \varepsilon^6 f_6 + \cdots, \\ G &= \varepsilon g_1 + \varepsilon^3 g_3 + \varepsilon^5 g_5 + \cdots, \\ H &= \varepsilon h_1 + \varepsilon^3 h_3 + \varepsilon^5 h_5 + \cdots.\end{aligned}\tag{13}$$

We can obtain one soliton solution

$$\begin{aligned}F &= 1 + B e^{2\eta}, \\ G &= A e^\eta, \\ H &= 0,\end{aligned}\tag{14}$$

where the phase is given by $\eta = \omega\tau + p\sigma + \delta$ and the dispersion relation satisfies

$$\omega^2 - p^2 = -1,\tag{15}$$

and the coefficients must be $B = A^2/[16(\omega - p)^2]$. A is a parameter related to amplitude of the loop soliton. Hereafter we take $A = 1$. In the physical space $\mathbf{R}^3$ one soliton solution is given by

$$\begin{aligned}X &= 2\sqrt{\frac{1+v}{1-v}} \operatorname{sech} \frac{\sigma - v\tau}{\sqrt{1-v^2}}, \\ Y &= 0, \\ Z &= Z_0 + \sigma - 2\sqrt{\frac{1+v}{1-v}} \tanh \frac{\sigma - v\tau}{\sqrt{1-v^2}},\end{aligned}\tag{16}$$

under the transformation

$$\begin{aligned}\omega &= -\frac{v}{\sqrt{1-v^2}}, \\ p &= \frac{1}{\sqrt{1-v^2}},\end{aligned}\tag{17}$$

where v is a phase velocity of the soliton and takes $-1 < v < 1$ and Z_0 is a constant. As illustrated in Fig.1, this solution represents a loop shape. It is clear that, as v increases, from (16) the amplitude of the soliton increases and each point of the string stretches according to the equation

$$ds^2 = \mathbf{r}_\sigma^2 d\sigma^2, \quad (18)$$

so that the right moving soliton has large amplitude and the left moving one has small amplitude. The reason is that in more realistic model the large stretching (and shrinking) of the string causes changes of the mass density, equations of motion (7) is valid for sufficiently small v .

For the case of two soliton solution, we study two types of the special cases of the headon collision of the two loop solitons moving with same speed v but opposite directions. One type is that both solitons are in the XZ -plane and solution is given by

$$\begin{aligned} F &= 1 + B_1 e^{2\eta_1} + B_2 e^{2\eta_2} + B_{12} e^{\eta_1 + \eta_2} + E_{12} e^{2(\eta_1 + \eta_2)}, \\ G &= A_1 e^{\eta_1} + A_2 e^{\eta_2} + C_1 e^{2\eta_1 + \eta_2} + C_2 e^{\eta_1 + 2\eta_2}, \\ H &= 0, \end{aligned} \quad (19)$$

where the coefficients are given by

$$\begin{aligned} B_i &= \frac{A_i^2}{16(\omega_i - p_i)^2} \quad (i = 1, 2), \\ B_{12} &= \frac{A_1 A_2}{2(\omega_1 + \omega_2 - p_1 - p_2)^2}, \\ C_1 &= -\frac{(\omega_1 - \omega_2)^2 - (p_1 - p_2)^2}{(\omega_1 + \omega_2)^2 - (p_1 + p_2)^2} A_2 B_1, \\ C_2 &= -\frac{(\omega_1 - \omega_2)^2 - (p_1 - p_2)^2}{(\omega_1 + \omega_2)^2 - (p_1 + p_2)^2} A_1 B_2, \\ E_{12} &= \left[\frac{(\omega_1 - \omega_2)^2 - (p_1 - p_2)^2}{(\omega_1 + \omega_2)^2 - (p_1 + p_2)^2} \right]^2 B_1 B_2, \end{aligned} \quad (20)$$

and the phases by

$$\eta_i = \omega_i \tau + p_i \sigma + \delta_i, \quad (i = 1, 2) \quad (21)$$

with the dispersion relations

$$\omega_i^2 - p_i^2 = -1. \quad (22)$$

According to eq.(17), we have

$$\begin{aligned}\omega_1 &= -\omega_2 = \frac{-v}{\sqrt{1-v^2}}, \\ p_1 &= p_2 = \frac{1}{\sqrt{1-v^2}}.\end{aligned}\tag{23}$$

We take A_i as 1 or -1 corresponding to positive or negative amplitude. When both amplitudes are positive, behaviors of interaction of the loop solitons are given in Figs.2. In the case of small speed ($v = 0.12$) where the amplitudes are nearly same, they exchange their amplitude in such a way that the loops do not overlap and push in each other as shown in Fig.2-a. However they change their shapes in each as a result of nonlinear interaction. For relatively large speed ($v = 0.24$) where two solitons have large amplitude difference, the small loop soliton travels around the large loop as shown in Fig.2-b. In these cases, clearly loop shaped figures are preserved during interaction as the same as the elastica. On the other hand, the headon collision of solitons with positive amplitude and with negative one is completely different from positive-positive amplitude collision and also from the loop-antiloop soliton interaction of the elastic beam. The small loop vanishes during the interacting period and the large one enhances its amplitude such as shown in Fig.2-c. The phase shift of the soliton propagating to the positive direction is given by

$$\begin{aligned}\phi_1 &= \frac{1}{2} \log \frac{E_{12}}{B_1 B_2} - \frac{4(\omega_2 - p_2)}{\sqrt{1-v^2}} \\ &= 2 \log v + \frac{4}{1+v},\end{aligned}\tag{24}$$

and that of the soliton propagating to the negative direction by

$$\begin{aligned}\phi_2 &= -\frac{1}{2} \log \frac{E_{12}}{B_1 B_2} + \frac{4(\omega_1 - p_1)}{\sqrt{1-v^2}} \\ &= -2 \log v - \frac{4}{1-v}.\end{aligned}\tag{25}$$

The first terms of the right hand side of eqs.(24) and (25) show the phase shift given by nonlinear interaction between solitons and the second terms represent the phase shift caused by two loops pushing away in each other or traveling along another loop.

Another type of the solution where asymptotically one soliton is in the XZ -plane and the other in the YZ -plane is given by

$$\begin{aligned} F &= 1 + B_1 e^{2\eta} + B_2 e^{2\eta_2} + E_{12} e^{2(\eta+\eta_2)}, \\ G &= A_1 e^{\eta} + C_2 e^{\eta+2\eta_2}, \\ H &= A_2 e^{\eta_2} + C_1 e^{2\eta+\eta_2}, \end{aligned} \tag{26}$$

where the coefficients, phases and dispersion relations are given by eqs.(20), (21) and (22), respectively. The main differences with the solution (19) are non-vanishing Y -component and the lack of the term $B_{12} e^{\eta+\eta_2}$ in the first equation of eq.(26). We find that the phase shifts of two solitons in the orthogonal planes are also given by the same formula as eqs.(24) and (25).

We take ω 's and p 's as the same form as eq.(23) and choose a parameter such that the left-moving small amplitude soliton is in the YZ -plane and the right-moving large one in XZ -plane. In this case which describes 3-dimensional soliton interaction, the collision behaviors are similar to the positive-positive amplitudes interactions, but, through interaction one of solitons declines to the plane located to another soliton and they are on the same plane when both solitons just collide and return to their own plains after collision as shown in Figs.3-a and 3-b.

§4 Concluding Remarks

We have given physical model of the coupled integrable, dispersionless equations for the $o(3)$ case. The equations represent the current-fed string moving in the 3-dimensional Euclidian space interacting with external magnetic field. We have obtained one and two soliton solutions through the bilinear equations. The solitons have the loop shape. Interaction between solitons has been studied in detail and found the differences from the cases of the elastica. Finally, we would like to point out that the bilinear equations (12) can be easily extended to multi-component equations more than three with other Lie groups. This is useful to derive the soliton solutions for the generalized dispersionless systems given in the Ref.6 which is multi-component version of eq.(7). This type extensions will be report forthcoming paper.

Acknowledgment:

One of the authors (H.K.) thanks Doyo-kai for the valuable discussion.

References

- [1] H.Kakuhata and K.Konno: J. Phys. Soc. Jpn. **65** (1996) 340.
- [2] H.Kakuhata and K.Konno: J. Phys. A **34** (1997) L401.
- [3] K.Konno and H.Oono: J. Phys. Soc. Jpn. **63** (1994) 377.
- [4] K.Konno: Applicable Analysis, **57** (1995) 209.
- [5] K.Konno and H.Kakuhata: J. Phys. Soc. Jpn. **64** (1995) 2707.
- [6] K.Konno and H.Kakuhata: J. Phys. Soc. Jpn. **65** (1996) 713.
- [7] V.P.Kotlyarov: J. Phys. Soc. Jpn. **63** (1994) 3535.
- [8] F.Lund and T.Regge: Phys. Rev. D **14** (1976) 1524.
- [9] K.Konno and A.Jeffrey: “*The loop soliton*” in *Advances in Nonlinear Waves Vol.1*, ed. L.Debnath (Pitman Advanced Publishing Program 1984).
- [10] H.Kakuhata and K.Konno: J. Phys. Soc. Jpn. **65** (1996) 1.
- [11] T.Alagesan and K.Porsezian: Chaos, Solitons & Fractals **7** (1996) 1209.
- [12] T.Alagesan and K.Porsezian: Chaos, Solitons & Fractals **8** (1997) 1645.
- [13] R.Hirota: J. Phys. Soc. Jpn. **51** (1982) 323.

Figure Captions

Fig.1 Shape of one loop soliton.

Fig.2-a The headon collision of the two loop solitons with $v = 0.12$ in ZX -plane where two loops do not overlap but push away. The Z -axis is horizontal and the X -axis vertical.

Fig.2-b The headon collision of the two loop solitons with $v = 0.24$ in ZX -plane. The small loop soliton travels around the large one.

Fig.2-c The headon collision of the two loop solitons with $v = 0.24$ in ZX -plane. The small loop vanishes during the interaction.

Fig.3-a The headon collision of the two loop solitons with $v = 0.12$ in $\mathbf{R}^3$. The small left-moving soliton is in YZ -plane and the large right moving one in YZ -plane. The loops do not overlap, but push away.

Fig.3-b The headon collision of the two loop solitons with $v = 0.24$ in $\mathbf{R}^3$. The small loop travels around the large one.

Fig.1.

Fig.2-a.

Fig.2-b.

Fig.2-c.

Fig.3-a.

Fig.3-b.