Note on the boundary terms in AdS/CFT correspondence for Rarita-Schwinger field

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Abstract

In this letter the boundary problem for massless and massive Rarita-Schwinger field in the AdS/CFT correspondence is considered. The considerations are along the lines of a paper by Henneaux (hep-th/9902137) and are based on the requirement the solutions to be a stationary point for the action functional. It is shown that this requirement, along with a definite asymptotic behavior of the solutions, fixes the boundary term that must be added to the initial Rarita-Schwinger action. It is also shown that the boundary term reproduce the known two point correlation functions of certain local operators in CFT living on the boundary.

1 Introduction

Recently a fascinating conjecture by Maldacena [1] has been proposed. According to [1] the supergravity theory in d+1 dimensional Anti-de Sitter space (AdS) with a compact extra space is related by holographic correspondence principle [2] to a certain conformal field theory (CFT) living on the boundary of AdS space. The underlying principle behind this AdS/CFT correspondence was elaborated in explicit form by Gubser, Klebanov and Polyakov [3] and Witten [4]. According to [3] and [4], the action for the supergravity theory on AdS considered as a functional of the asymptotic values of the fields on the boundary is interpreted as a generating functional for the correlation functions in the conformal field theory living on the boundary. The explicit form of this interpretation is:

\[ \int \mathcal{D}\Phi \exp\{ -S[\Phi] \} = \langle \exp \int_{\partial\text{AdS}} d^d x O \Phi_0 \rangle \]

where \( \Phi_0 \) is the boundary data for AdS theory which couples to a certain conformal operator \( O \) on the boundary. This interpretation has already a large number of examinations by computing various correlation functions of a local operators in CFT induced by AdS scalar fields [4, 6, 7, 8, 9], spinor fields [12, 10], vector fields [4, 8, 10, 11], antisymmetric fields [16, 17, 18] and Rarita-Schwinger fields [19, 20, 21]. All the examples confirmed the validity of the AdS/CFT correspondence principle. The essence of these examinations is in studying of the field behavior near the boundary of AdS space and calculation of the boundary terms which produce the corresponding correlation functions in CFT. While in the case of an action of AdS theory with derivatives of order higher than one the considerations are, more or less, transparent, in the case of Dirac-like actions (Dirac and Rarita-Schwinger) the situation is more subtle. That is because the naive limit to the boundary lead to vanishing of the action (it is zero on shell) which obviously spoil the correspondence principle.

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Recently two different but equivalent ways of treatment of the boundary term problem in the case of spinor field were proposed [25, 24]. In [25] the considerations are based on the Hamiltonian approach treating $x^0$ as an evolution parameter. A half of the components of the boundary spinor turn out to be canonically conjugated to the other half and the boundary term naturally appears to be $p_0q$.

In the paper [24] another approach is used, namely the stationary phase method. It is based on the fact that one can expand the path integral of a given theory with action $S[\Phi]$:

$$Z = \int D\Phi e^{iS[\Phi]}$$

near the stationary point given by the solutions of the classical theory. In the classical limit ($\hbar \to 0$) the leading order is simply:

$$Z \sim \exp \{ \frac{i}{\hbar} S_{\text{class}} \}$$

where $S_{\text{class}}$ is the action functional evaluated on the classical field configurations. The stationary point is determined by the requirement:

$$\frac{\delta S}{\delta \Phi} = 0 \quad (2)$$

which is nothing but the classical equations of motion with given boundary data. The main immediate but important observation is that while the action $S[\Phi]$ may satisfy (2) on the classical field configurations, it is not true in general if there is a surface term $B_\infty$, i.e. there may be $\delta B_\infty \neq 0$ and therefore $\delta (S + B_\infty) \neq 0$. Such a situation appears for instance in the gauge field theory where $B_\infty$ gives the conserved charges of the symmetry currents. It follows that in order the classical configurations to be a true stationary point it is necessary a boundary term to be added ensuring the wanted requirement. This scheme was used in [24] in order to obtain the boundary term in the case of spinor field $\Psi$.

In this letter we would like to study the boundary term in the case of massless and massive Rarita-Schwinger field. In our analysis we will use the receipt described in [24] for derivation of the boundary term in the spinor case.

The paper is organized as follows. In Section 2 the derivation of the boundary term in case of massless Rarita-Schwinger field is given. It is shown that the result reproduce exactly the two-point correlation functions derived in [20]. In Section 3 the same analysis is performed in case of massive Rarita-Schwinger field. The result for the correlation functions is in complete agreement with that given in [21]. In the Conclusions we give some remarks and brief comments.

## 2 Boundary term for massless Rarita-Schwinger field

In this Section we will consider a massles Rarita-Schwinger field on $AdS_{d+1}$ given by the following action [5, 20]:

$$S_{R-S} = \int_{AdS} d^{d+1}x \sqrt{G} \bar{\Psi}_\mu \left( \Gamma^\mu_\nu \Lambda D_\nu - m \Gamma^\mu_\nu \right) \Psi_\Lambda \quad (3)$$

We used the notations:

$$\Gamma^\mu_\nu = \Gamma^{[\mu} \Gamma^{\nu]}, \quad \Gamma^\mu_\nu = \Gamma^{[\mu} \Gamma^{\nu]}$$

$$\bar{\Psi}_\mu = \Psi_{\mu a}$$

where $\Gamma^\mu$ are the gamma matrices in AdS connected to the flat gamma matrices by the relation $\Gamma^\mu = e^\mu_a \gamma^a$. The vielbein is given below and the flat gamma matrices satisfy the usual anticommutation relations $\{ \gamma^a, \gamma^b \} = 2\delta^{ab}$. We choose to work in coordinates $x^a = (x^0, x^i) = (x^0, \mathbf{x}); i = 1, \ldots, d$ defining...
$d + 1$-dimensional Euclidean Anti-de Sitter space as Lobachevski upper half plane $x^0 > 0$ with a metric of the form:

$$ds^2 = \frac{1}{x^0} (dx^0 + dx^2)$$  \hspace{1cm} (5)$$

With this choice the vielbein and the corresponding non-zero components of the spin connection are given by the expressions:

$$e_a^\mu = \frac{\delta_a^\mu}{x^0}; \quad \omega_i^0 = -\omega_i^0 = \frac{\delta_i}{x^0}; \quad a = 0, \ldots, d$$  \hspace{1cm} (6)$$

The boundary of the AdS space consists in a hypersurface $x^0 = 0$ and a single point $x^0 = \infty$.

In this frame the covariant derivative and the Dirac operator reads off:

$$D\nu = \partial\nu + \frac{1}{2x^0} \gamma_0 \gamma_\nu; \quad \Gamma^\mu D\mu = \partial_\nu \psi_0 - \frac{d}{2} \gamma^\nu \partial^\nu$$  \hspace{1cm} (7)$$

where $\gamma = (\gamma^i); \quad \nabla = (\partial_i); \quad i = 1 \ldots d$.

The equation of motion for Rarita Schwinger field following from the action (3):

$$\Gamma^{\mu\nu\lambda} D_\nu \Psi_\lambda = m \Gamma^{\mu\nu} \Psi_\lambda$$  \hspace{1cm} (8)$$

can be rewritten in the form [20]:

$$x_0 \gamma^\nu \partial_\nu \psi_0 - \left( \frac{d}{2} + 1 \right) \gamma_0 \psi_0 + m \psi_a = 0$$  \hspace{1cm} (9)$$

$$x_0 \gamma^\nu \partial_\nu \psi_i - \frac{d}{2} \gamma_0 \psi_i + m \psi_i = \gamma_a \psi_0$$

where $\psi_a = e_a^\mu \Psi_\mu$.

In order to find the boundary contributions we are interested in studying of the behavior of the solutions near the boundary $x^0 = 0$. For this purpose one can use the Frobenius procedure looking for solutions of the form [24]:

$$(x^0)^\rho \sum_{n=0}^{\infty} c_n^a (\vec{x}) (x^0)^n$$

Substitution of these series into the equations of motion determines the values of the parameter $\rho$:

$$\rho = \frac{d}{2} \pm m + \delta_{a0}$$

(3) (the values of $\rho$ turn out to be the same as in the spinor case [24] for $a \neq 0$). Therefore, we are dealing with two types of solutions:

$$\psi_a^- (x_0, \mathbf{x}) = (x_0)^{\frac{d}{2} - m + \delta_{a0}} \varphi_a (\mathbf{x}) + o \left( (x_0)^{\frac{d}{2} - m + \delta_{a0}} \right)$$  \hspace{1cm} (10)$$

$$\psi_a^+ (x_0, \mathbf{x}) = (x_0)^{\frac{d}{2} + m + \delta_{a0}} \chi_a (\mathbf{x}) + o \left( (x_0)^{\frac{d}{2} + m + \delta_{a0}} \right)$$  \hspace{1cm} (11)$$

where:

$$\gamma_0 \varphi_a (\mathbf{x}) = -\varphi (\mathbf{x})$$

$$\gamma_0 \chi_a (\mathbf{x}) = \chi (\mathbf{x})$$  \hspace{1cm} (12)$$

3
It follows that \( \psi_0^\pm \) are eigenvectors of \( 1/2 (I \pm \gamma_0) \) with eigenvalues \( \pm 1 \) respectively. The conjugated 
Rarita-Schwinger field can be treated analogously giving the following result:

\[
\bar{\psi}_a^+ (x) = \bar{\varphi}_a (x) (x_0)^{\frac{d}{2} + m + \delta_0 a} + o \left( (x_0)^{\frac{d}{2} + m + \delta_0 a} \right) \\
\bar{\psi}_a^- (x) = \bar{\chi}_a (x) (x_0)^{\frac{d}{2} - m + \delta_0 a} + o \left( (x_0)^{\frac{d}{2} - m + \delta_0 a} \right)
\]

where the fields \( \bar{\varphi}_a \) and \( \bar{\chi}_a \) are subject to the constraints:

\[
\bar{\varphi}_a \gamma^0 = \bar{\varphi}_a; \quad \bar{\chi}_a \gamma^0 = -\bar{\chi}_a
\]

Note that using the equations of motion one can find the subleading terms recursively but since they
doesn't contribute to the boundary we will skip their explicit form.

Let us consider the general solutions of the equations of motion:

\[
\psi_a (x_0, \mathbf{x}) = \int d^d p e^{ip \cdot x} \left[ F_+ (x_0, \mathbf{p}) \tilde{\varphi}_a^+ (\mathbf{p}) + F_- (x_0, \mathbf{p}) \tilde{\varphi}_a^- (\mathbf{p}) \right]
\]

Since the solutions must be regular in the bulk (up to \( x_0 = \infty \)) the boundary spinor fields \( \tilde{\varphi}_a^\pm \) are not
independent and the solutions can be expressed in terms of \( \tilde{\varphi}_a^- \) only \(^2\):

\[
\psi_0 (x_0, \mathbf{x}) = \int d^d p e^{ip \cdot x} (x_0 p) \left[ i \frac{\hat{p}}{p} K_{m+\frac{1}{2}} (x_0 p) + K_{m-\frac{1}{2}} (x_0 p) \right] \tilde{\varphi}_0^- (\mathbf{p})
\]

\[
\psi_1 (x_0, \mathbf{x}) = \int d^d p (x_0 p) \left[ i \frac{\hat{p}}{p} K_{m+\frac{1}{2}} (x_0 p) + K_{m-\frac{1}{2}} (x_0 p) \right] \tilde{\varphi}_1^- (\mathbf{p})
\]

\[
+ \left[ (2m + 1) \frac{p \hat{p}}{p^2} - i p_0 x_0 + \gamma_i \right] K_{m+\frac{1}{2}} (x_0 p) - \frac{p \hat{p}}{p} x_0 K_{m-\frac{1}{2}} (x_0 p) \right] \tilde{\varphi}_0^- (\mathbf{p}) \}
\]

\((\hat{p} = \gamma^i p_i). In order to see how the fields \( \tilde{\varphi}_a^- \) are related to the fields \( \varphi_a^- \) (which gives the asymptotic of \( \psi_a \) at \( x_0 \to 0 \)) we use the small argument expansion of the modified Bessel function \( K_\nu \):

\[
K_\nu (z) = \frac{1}{2} \left[ \left( \frac{z}{2} \right)^{-\nu} \Gamma (\nu) [1 + \ldots] + \left( \frac{z}{2} \right)^{\nu} \Gamma (-\nu) [1 + \ldots] \right]
\]

where dots stands for positive integer powers of \( z^2 \). Substitution of (17) into (15,16) gives the following
behavior:

\[
\psi_0^+ (x_0, \mathbf{x}) = x_0^{\frac{d}{2} - m + 1} \int d^d p e^{ip \cdot x} \left[ i p \frac{\hat{p} \hat{p}}{p^2} - m + \frac{1}{2} \right] \tilde{\varphi}_0^- (\mathbf{p}) + o \left( (x_0)^{\frac{d}{2} - m + 1} \right)
\]

\[
\psi_0^- (x_0, \mathbf{x}) = x_0^{\frac{d}{2} + m + 1} \int d^d p e^{ip \cdot x} \left[ p \frac{\hat{p} \hat{p}}{p^2} + \frac{1}{2} \right] \tilde{\varphi}_0^- (\mathbf{p}) + o \left( (x_0)^{\frac{d}{2} + m + 1} \right)
\]

\[
\psi_1^- (x_0, \mathbf{x}) = x_0^{\frac{d}{2} - m} \int d^d p e^{ip \cdot x} \left[ i p \frac{\hat{p} \hat{p}}{p^2} - m + \frac{1}{2} \right] \tilde{\varphi}_1^- (\mathbf{p})
\]

\[
+ \left. \left[ p \frac{\hat{p} \hat{p}}{p^2} + \frac{1}{2} \right] \Gamma \left( m + \frac{1}{2} \right) \left( 2m + 1 \right) + \gamma_i \right] \tilde{\varphi}_1^- (\mathbf{p}) + o \left( (x_0)^{\frac{d}{2} - m} \right)
\]

\(^2\)In what follows we will always suppose that the Fourier transform is well defined, i.e. \( \varphi_a (\mathbf{x}) \) and \( \bar{\varphi}_a (\mathbf{x}) \) are of
compact support and vanish at \( x \to \infty \) and therefore can be Fourier transformed.
\[ \psi_i^\pm (x_0, \mathbf{x}) = x_0^{\frac{d}{2} + m} \int d^d \mathbf{p} e^{i \mathbf{p} \cdot \mathbf{x}} \frac{1}{2^{m + \frac{d}{2}} (2m + d - 1) \Gamma \left( \frac{1}{2} + m \right)} \tilde{\psi}_i^\pm (\mathbf{p}) + o \left( (x_0)^{\frac{d}{2} + m} \right) \]  

(20)

The components of the Rarita-Schwinger field are subject to one more constraint:

\[ \gamma^0 \psi_0 + \gamma^i \psi_i = 0 \]

which relate the \( \tilde{\psi}_0^- \) and \( \tilde{\psi}_i^- \) components as follows [20]:

\[ \tilde{\psi}_0^- (\mathbf{p}) = -\frac{2i p \tilde{\psi}_i^- (\mathbf{p})}{(2m + d - 1) p}; \quad \gamma^i \tilde{\psi}_i^- (\mathbf{p}) = 0 \]

(21)

Using the asymptotic expressions for \( \psi_a \) (10, 11) it is easy to find the relation between \( \varphi_a (\mathbf{p}) \) and \( \chi_a (\mathbf{p}) \) and \( \tilde{\psi}_a^- (\mathbf{p}) \):

\[ \varphi_0 (\mathbf{p}) = i p p \tilde{\psi}_\frac{d}{2} - m - \frac{1}{2} \Gamma \left( \frac{1}{2} + m \right) \tilde{\psi}_0^- (\mathbf{p}) \]

\[ \chi_0 (\mathbf{p}) = \frac{p^2 + m + 1}{2m + \frac{d}{2}} \Gamma \left( \frac{1}{2} - m \right) \tilde{\psi}_0^- (\mathbf{p}) \]

(22)

and:

\[ \varphi_i (\mathbf{p}) = p^\frac{d}{2} - m - \frac{1}{2} \Gamma \left( \frac{1}{2} + m \right) \left[ \frac{p^2 \tilde{\psi}_i^- + \left( 2m + 1 \right) p \tilde{\psi}_i^- \tilde{\psi}_i^- + \left( 2m + 1 \right) \gamma_i \right) \tilde{\psi}_i^- \right] \]

\[ \chi_i (\mathbf{p}) = \frac{p^2 + m + 1}{2m + \frac{d}{2}} \frac{1}{2} \Gamma \left( \frac{1}{2} - m \right) \tilde{\psi}_i^- (\mathbf{p}) \]

(23)

From (18, 19, 21) it follows that \( \psi_a^\pm \) can be expressed in terms of \( \varphi_a^+ (\mathbf{p}) \) only and that \( \chi_a \) and \( \varphi_a \) are related "on-shell".

The final lesson from the above considerations is that a half of the boundary data is expressible in terms of the other half but the relations are valid "on-shell". The main conclusion is that the general solutions of the equations of motion in the whole AdS space are determined by the fields annihilated by \( (I + \gamma_0) \), a quite similar result as in case of spinor field [24].

Now we are going to apply the variational principle to the Rarira-Schwinger action (3). Since the Rarira-Schwinger equations of motion are first order differential equations we cannot fix all the components of \( \psi_a \) at the boundary but only a half of them, \( \varphi_a \) or \( \chi_a \). The basic idea is to use AdS correspondence principle which tells us that the fields \( \varphi_a \) serve as a sources for the bulk-boundary Green functions [12, 19, 20]. Thus, it is appropriate to fix \( \varphi_a \) at the boundary and to leave \( \chi_a \) to vary. The variational principle will be applied to all configurations of the form:

\[ \psi_a = \psi_a^- + \psi_a^+ \]

\[ \tilde{\psi}_a = \tilde{\psi}_a^- + \tilde{\psi}_a^+ \]

(24)

The fields \( \psi_a^- \) and \( \psi_a^+ \) have the asymptotic (10, 13) while \( \varphi_a \) and \( \tilde{\psi}_a^- \) are fixed on the boundary. The other part of \( \psi_a^- \) and \( \tilde{\psi}_a^- \) behaves near the boundary as it is described in (11,14), but in this case the values of \( \chi_a \) and \( \tilde{\chi}_a \) on the boundary are free to vary. The relations between \( \varphi_a \) and \( \chi_a \) (22, 23) are only on-shell and will not affect the variational principle (the same is true for \( \tilde{\varphi}_a \) and \( \tilde{\chi}_a \)).

After the above preparations we are ready to vary the action (3) with respect to \( \psi_a \) and \( \tilde{\psi}_a \). As in [24], the variation will be in the class of fields defined in (24) but varying (3) we will take into account the surface terms:

\[ \delta S_{R-S} = B_\infty + \left[ 0 \right]_{\text{on-shell}} \]

(25)
where:

\[ B_\infty = -\frac{1}{2} \int d^d x \left[ \tilde{\varphi}_i (x) g^{ij} \delta \chi_j (x) + \delta \tilde{\varphi}_i (x) g^{ij} \varphi_j (x) + \varphi_i (x) \gamma^i \gamma^j \delta \chi_j (x) 
+ \delta \chi_i (x) \gamma^i \gamma^j \varphi_j (x) \right] \]  

(26)

(we recall that \( \sqrt{G} = (x_0)^{-d-1} \) and \( g^{ij} \) is the induced metric on the boundary).

The term \( B_\infty \) is nothing but the variation of the surface term at infinity [24]:

\[ B_\infty = -\delta C_\infty \]  

(27)

where:

\[ C_\infty = \frac{1}{2} \int d^d x \left[ \tilde{\varphi}_i g^{ij} \chi_j + \chi_i g^{ij} \varphi_j + \varphi_i \gamma^i \gamma^j \chi_j + \chi_i \gamma^i \gamma^j \varphi_j \right] \]  

(28)

Note that since \( \gamma^i \chi_i = 0 \) the last two terms don’t contribute. The requirement for the action \( S_{R-S} \) to be stationary on the solutions of the equations of motion imposes to consider a new, improved action of the form:

\[ S = S_{R-S} + C_\infty \]  

(29)

It is obvious that \( \delta S = 0 \) on-shell and reproduce the correct solutions of the equations of motion.

Of course, the above boundary term is not unique. This can be achieved by imposing three natural conditions, namely:

a) \( C_\infty \) is local
b) \( B_\infty \) is without derivatives
c) \( C_\infty \) preserves the AdS symmetry.

Under the above requirements \( C_\infty \) is unique.

Having the explicit solutions for \( \psi_\alpha \) and its asymptotics, one can rewrite the boundary term (up to irrelevant for our considerations contact terms) as:

\[ C_\infty = \lim_{\varepsilon \to 0} \frac{1}{2} \int_{M_t} d^d \sqrt{g}_c \psi_\alpha g^{ij} \psi_j \]  

(30)

where \( M_t \) is a d-dimensional surface approaching the boundary \( \varepsilon \to 0 \) and \( g_c \) is the induced metric on \( M_t \). The boundary term (30) is in complete agreement with that of [19, 20].

Using the explicit expressions for \( \langle \tilde{\varphi}_i, \chi_i \rangle \) and \( \langle \tilde{\varphi}_i, \chi_i \rangle \) it is straightforward to calculate the correlation functions produced on the boundary. Since the Rarita-Schwinger action is zero on-shell the contributions will come only from the boundary terms (28). According to the AdS/CFT correspondence principle one must replace \( \chi_i \) and \( \tilde{\chi}_i \) in (28) with their on-shell values (23). The substitution gives:

\[ S_{class} = \int \frac{d^d p}{(2\pi)^d} \langle \tilde{\varphi}_i (-p) \chi_i (p) + \tilde{\chi}_i (p) \varphi_i (-p) \rangle \]

\[ = i \frac{\Gamma \left( \frac{d}{2} - m \right)}{2^{2m} \Gamma \left( \frac{d}{2} + m \right)} \int \frac{d^d p}{(2\pi)^d} \tilde{\varphi}_i (p) \left( \delta_{ij} \hat{p}_n - \frac{2 (2m + 1) p_i p_j \hat{p}_n}{d + 2m - 1} \right) \varphi_i (p) \]

\[ = - \frac{\Gamma \left( \frac{2m + d + 1}{2} \right)}{\pi^\frac{d}{2} \Gamma \left( \frac{d}{2} + m \right)} \int d^d x d^d y \tilde{\varphi}_i (x) \frac{(x - y)_i \gamma^i}{|x - y|^{2m + d + 1}} \left[ \delta_{ij} - 2 \frac{(x - y)_i (x - y)_j}{|x - y|^2} \right] \varphi_j (y) \]  

(31)

The above expression coincides with the two point correlation functions found in [19, 20]:

\[ \Omega (x, y) \sim \frac{(x - y)_i \gamma^i}{|x - y|^{2m + d + 1}} \left[ \delta_{ij} - 2 \frac{(x - y)_i (x - y)_j}{|x - y|^2} \right] \]  

(32)

corresponding to conformal operator of dimension \( \Delta = \frac{d}{2} + m \).
3 Massive case

We proceed with the analysis of the boundary term in the case of massive Rarita-Schwinger field. The most general action is given by [5]:

\[
S_{mR} = \int d^d x \sqrt{G} \left[ \bar{\Psi}_\mu \Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - m_1 \bar{\Psi}_\mu \Psi^\nu - m_2 \bar{\Psi}_\mu \Gamma^{\mu\nu} \Psi_\nu \right]
\] (33)

The equations of motion following from (33)

\[
\Gamma^{\mu\nu\rho} D_\nu \Psi_\rho - m_1 \Psi^\mu - m_2 \Gamma^{\mu\nu} \Psi_\nu = 0
\]

can be rewritten in more convenient form [19, 21]:

\[
\Gamma^{\nu} \left( D^{\nu} \Psi_\rho - D_\rho \Psi^{\nu} \right) - m_{-\rho} \gamma^{\nu} \psi_0 = 0
\]

where \( m_\pm = m_1 \pm m_2 \). Applying the standard procedure of passing to flat space equations (\( \psi_a = e^\mu_a \Psi_\mu \)) it is straightforward to obtain the equations for \( \psi_0 \) and \( \psi_i \) [19, 21]:

\[
\psi_0 \sim (x_0)^{d\over 2} \sum_{n=0}^\infty c_n^a (x_0) x_0^n
\]

where \( c_n^a (x_0) \) are \( x_0 \)-independent Rarita-Schwinger fields. If one try to solve the system for fields depending on \( x_0 \) only the leading terms will have rather different values for \( \rho ^3 \) [21]:

\[
\psi_0 \sim x_0^{\frac{d}{2} - 1} \sum_{n=0}^\infty \left[ \begin{array}{c}
C_{+}^a b_0^+ + x_0^{\frac{d}{2} - 1} C_{-}^a b_0^-
\end{array} \right]
\]

where the constant \( C \) is given by:

\[
C = \frac{d(d-1)}{4m_1} + \frac{m_2 + m_1 + dm_2}{m_1 (d-1)}
\]

Similar expressions for the conjugated fields hold. The splitting of \( \phi_a \) into \( \pm \) parts is subject to the conditions:

\[
\gamma_0 b^+_a = \pm \bar{b}^+_a; \quad \bar{b}^+_a = \mp \bar{b}^+_a
\]

As it was noted in [21] simple algebraic operations leads to the following relation between \( \psi_0 \) and \( \eta \) when we consider the dependence on \( x_0 \) only:

\[
(\psi_0 - 2m_0 \gamma_0) \psi_0 = (d-1 + 2m_2 \gamma_0) \eta.
\]

Note that \( b^\pm_a \) are constant spinors.
Let us discuss the general solutions of the system (35, 36). For the $\psi_0$ component it reads off \[21\]:

$$
\psi_0(x_0, \mathbf{p}) = \frac{d^{d+1} \phi_0}{2^{d-C/2} \Gamma \left( \frac{d}{2} + \frac{1}{2} - C \right)} \left[ (d - 3 + 2C_1 + 2i\hat{p}x_0) K_{\frac{1}{2} - C} (x_0 \mathbf{p}) - \frac{4m_1 x_0 p}{d (d - 1 - 2m_2)} \left( \frac{i \gamma_0 \hat{p}}{p} K_{\frac{1}{2} - C} (x_0 \mathbf{p}) - \frac{2p_i \hat{p}}{p} K_{\frac{1}{2} + m - C} (x_0 \mathbf{p}) \right) \right] \tilde{\psi}_0 (\mathbf{p}) - \frac{2x_0^{d+1} p^{\frac{1}{2} - m - C} \Gamma \left( \frac{d}{2} + \frac{3}{2} - m - C \right)}{2^{d - m - C} \Gamma \left( \frac{d}{2} + \frac{1}{2} - C \right)} \left( i \gamma_0 p_i K_{\frac{1}{2} - m - C} (x_0 \mathbf{p}) - \frac{p_i \hat{p}}{p} K_{\frac{1}{2} + m - C} (x_0 \mathbf{p}) \right) \tilde{\psi}_1 (\mathbf{p})
$$

(42)

$$
\psi_1(x_0, \mathbf{p}) = \frac{d^{d+1} \phi_0}{2^{d-C/2} \Gamma \left( \frac{d}{2} + \frac{1}{2} - C \right)} \left[ \frac{2p_i \hat{p}}{p} K_{\frac{1}{2} + C} (x_0 \mathbf{p}) + \frac{2m_1}{d (d - 1 - 2m_2)} \times \left( (d - 1 - 2C_1) \gamma_0 \gamma_i + 2i p_i x_0 K_{\frac{1}{2} - C} (x_0 \mathbf{p}) - \frac{2p_i \hat{p} \gamma_i}{p} \right) K_{\frac{1}{2} + m - C} (x_0 \mathbf{p}) \right] \bar{\psi}_0 (\mathbf{p}) + \frac{2x_0^{d+1} p^{\frac{1}{2} - m - C} \Gamma \left( \frac{d}{2} + \frac{3}{2} - m - C \right)}{2^{d - m - C} \Gamma \left( \frac{d}{2} + \frac{1}{2} - C \right)} \left[ (d - 1 - 2m - C) \gamma_0 \delta_{ij} + 2i \frac{p_j \hat{p} p_i}{p^2} x_0 \right] K_{\frac{1}{2} - m - C} (x_0 \mathbf{p}) - \frac{2p_i p_j}{p} \gamma_0 x_0 \right] K_{\frac{1}{2} + m - C} \left( \tilde{\psi}_j (\mathbf{p}) \right)
$$

(43)

where $\tilde{\psi}_a$ satisfy the relations:

$$
\gamma_0 \tilde{\psi}_a = - \tilde{\psi}_a; \quad \gamma^i \tilde{\psi}_i = 0
$$

and analogous expression for conjugated fields. Since we have two rather different leading terms (of powers $d/2 \pm C$ and $d/2 \pm m_-$), some of the solutions have to be fixed to zero, i.e. we must fix $\tilde{\psi}_0$ or $\tilde{\psi}_i$. A natural criterion for this is the requirement that in the limit $m_2 \to 0$ to reproduce the massless case. This uniquely determine $\tilde{\psi}_0 = 0$. The same arguments as in the massless case for regularity of the solution in the interior of the AdS relate $\tilde{\psi}_i^\dagger$ and $\tilde{\psi}_i$ which reminiscent again the principle that only a half of the components can be fixed on the boundary. \[5\] In order to extract the contribution to the boundary we will use the small argument expansion of the modified Bessel function (17). The

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\[4\] In order to introduce $x$ dependence, in \[21\] $O(d+1,1)$ transformations are used. Such transformations are accompanied by rotation for the spinor fields. One can show that our expressions are equivalent to those of \[21\].

\[5\] This requirement lead to the condition $p_i \hat{p}_j = -i \frac{(d - 1 - 2m_{-}) p_j \hat{p}_i}{p d - 1 - 2m_{-}}$. The relation holds only on-shell. Note that in our expressions the spinors are rotated compared to \[21\] and will be denoted by $\hat{\psi}_i$. 

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result for $\psi_0$ is:

$$
\psi_0 (x_0, p) = x_0^{d/2 + m_+ + 1} \chi_0 (p) + x_0^{2 - m_- + 1} \varphi_0 (p)
$$

(44)

$$
\chi_0 = \frac{\Gamma \left( \frac{d}{2} - m_- \right)}{d - 1 + 2m_-} \left[ \frac{p \hat{p}}{p^2} \right]^{2m_- - \frac{d}{2}} \hat{\varphi}_i
$$

(45)

$$
\varphi_0 = - \frac{\Gamma \left( \frac{1}{2} + m_- \right)}{d - 1 + 2m_-} \left[ ip_i \hat{\varphi}_i \right]
$$

(46)

One can express $\chi_0$ in terms of $\varphi_0$, but since $\psi_0 \sim x_0^{d/2 + m_+ + 1}$ there will be no contribution to the boundary term.

Let us apply the same analysis to the $\psi_i$ components. We split $\psi_i$ into "chiral" and "anti-chiral" parts:

$$
\psi_i (x_0, p) = x_0^{d/2 + m_-} \chi_i (x_0, p) + x_0^{2 - m_-} \varphi_i (x_0, p)
$$

(47)

$$
\gamma_0 \chi_i = \chi_i; \quad \gamma_0 \psi_i = - \psi_i
$$

(48)

The analysis of the $x_0 \to 0$ behavior of the corresponding component gives the following expressions:

$$
\varphi_i (p) = - \frac{\Gamma \left( \frac{1}{2} - m_- \right)}{2^{m_- + \frac{d}{2}}} \left( d - 1 + 2m_- \right) p_i \hat{p} \delta_{ij} + 2i \left( 1 - 2m_- \right) \frac{p_i p_j \hat{p} p_i p_j}{p^4} - 2i \frac{p_i \gamma_i}{p} \right) p^{2m_- - \frac{d}{2}} \hat{\varphi}_j
$$

(49)

$$
\chi_i (p) = - \frac{\Gamma \left( \frac{1}{2} + m_- \right)}{2^{m_- + \frac{d}{2}}} \left( d - 1 + 2m_- \right) \hat{\varphi}_i
$$

(50)

From the above formulae follows that the two expressions (49, 50) are not independent and a half of the boundary data can be expressed (on-shell) in terms of the other half.

We now turn to the variational principle applied to the action for the massive Rarita-Schwinger field (33). Repeating the same considerations as in the massless case we have found the boundary terms in the same form:

$$
C_\infty = \lim_{\epsilon \to 0} \frac{1}{2} \int d^d \epsilon [ \hat{\varphi}_i g^{ij} \chi_j + \hat{\chi}_i g^{ij} \varphi_j + \hat{\varphi}_i \gamma^j \chi_j + \hat{\chi}_i \gamma^j \varphi_j ]
$$

(51)

which again can be written as in the massless case:

$$
C_\infty = \lim_{\epsilon \to 0} \frac{1}{2} \int d^d \sqrt{g} ( \hat{\psi}_i g^{ij} \psi_j + \hat{\psi}_i \gamma^j \psi_j )
$$

(52)

The action ensuring that the classical solutions are true stationary point of the action (33) gets the modification:

$$
S = S_{mR-S} + C_\infty
$$

(53)

which is unique under the requirement of locality, absence of derivatives and presevation of the AdS symmetry.

Now, using (49, 50) it is straightforward to reproduce the correlation functions in the CFT living on the boundary found in [21]:

$$
\Omega (x, y) \sim \frac{(x - y)_i \gamma^i}{|x - y|^{2m_- + d + 1}} \left[ \delta_{ij} - 2 \frac{(x - y)_i (x - y)_j}{|x - y|^2} \right]
$$

(54)

Note that in the massless limit $m_2 \to 0$ the above result coincide with (32).

\(^{6}\)We note that since $\gamma^i \chi_i = 0$ the $3^{rd}$ and $4^{th}$ terms in (51) and the second term in (52) below will not contribute.
Conclusions
In this letter we have analysed the boundary term for Rarita-Schwinger field in the AdS/CFT correspondence. It is shown that, as in the spinor case, one cannot fix simultaneously all the components of the Rarita-Schwinger field on the boundary but only a half of them. Following [24] we used the stationary phase method to determine the surface term. We apply variation of the action over an appropriate class of field configurations. Since we are dealing with a theory with a boundary, it turns out that one half of the components of the spinor field must be kept fixed but the other half are free to vary on the boundary (at $x_0 = 0$). We choose to impose the boundary conditions on the "chiral" part of the Rarita-Schwinger field (annihilated by $1/2(I - \gamma_0)$) which have a minimal value of the leading term in $x_0$. The choice is quite natural thinking of this part as of a source of the bulk-boundary Green function. The "anti-chiral" starts at higher power of $x_0$ but it is shown that these components play an important role since they contribute to the boundary term. Moreover, these components are not off-shell subject to boundary conditions.

It would be interesting to study interacting Rarita-Schwinger–scalars in AdS/CFT correspondence as in [27] and to proceed with an investigation of the $S$-matrix along the lines of [29, 28, 30, 31]. Work on the subject is in progress.

Acknowledgments

I am grateful to N.I.Karchev and M.Stamishkov for comments and critical reading the manuscript.

References


