Curvature conditions for the occurrence of a class of spacetime singularities

Wiesław Rudnicki* and Paweł Ziuba†

Institute of Physics, University of Rzeszów,
ul. Rejtana 16A, PL 35-310 Rzeszów, Poland

It has previously been shown [W. Rudnicki, Phys. Lett. A 224, 45 (1996)] that a generic gravitational collapse cannot result in a naked singularity accompanied by closed timelike curves. An important role in this result plays the so-called inextendibility condition, which is required to hold for certain incomplete null geodesics. In this paper, a theorem is proved that establishes some relations between the inextendibility condition and the rate of growth of the Ricci curvature along incomplete null geodesics. This theorem shows that the inextendibility condition may hold for a much more general class of singularities than only those of the strong curvature type. It is also argued that some earlier cosmic censorship results obtained for strong curvature singularities can be extended to singularities corresponding to the inextendibility condition.

I. INTRODUCTION

Recently, one of us [1] has shown that, under certain physically reasonable conditions, a generic gravitational collapse developing from a regular initial state cannot lead to the formation of a final state resembling the Kerr solution with \( a^2 > m^2 \)--i.e., of a naked singularity accompanied by closed timelike curves. This result supports the validity of Penrose’s cosmic censorship hypothesis [2] and suggests that there may exist some deeper connection between cosmic censorship and the chronology protection conjecture put forward by Hawking [3]. An important role in this result plays the so-called inextendibility condition (see Sec. II), which is assumed to be satisfied for certain incomplete null geodesics. This condition enables one to rule out artificial naked singularities that could easily be created by simply removing points from otherwise well-behaved spacetimes. The inextendibility condition is based on the idea that physically essential singularities should always be associated with large curvature strengths, which are in turn usually associated with the focusing of Jacobi fields along null geodesics.

It is easily seen that the inextendibility condition will always hold for null geodesics terminating at the so-called strong curvature singularities defined by Tipler [4] (see below). Singularities of this type are sometimes considered to be the only physically reasonable singularities (cf., e.g., [5,6]). However, strong curvature singularities can exist only if the curvature in their neighborhood diverges strong enough [7], while it is not unlikely that some singularities occurring in generic collapse situations will involve a weaker divergence of the curvature. In fact, one cannot a priori exclude the existence of some “real” singularities near which the curvature would remain even bounded (such singularities occur, for example, in Taub-NUT space). Accordingly, since we still have no fully accepted necessary condition on the behavior of the curvature near generic singularities, one should try to prove any cosmic censorship result under as weak a curvature condition as possible. It would be therefore of interest, in view of the mentioned censorship result [1], to know what are curvature conditions for the occurrence of singularities corresponding to the inextendibility condition. Furthermore, the inextendibility condition has also been used in proving some other recent results [8,9] that restrict a class of possible causality violations in classical general relativity.

In this paper, we formulate and prove a theorem that establishes some relations between the inextendibility condition and the rate of growth of the Ricci curvature along incomplete null geodesics. This theorem shows that the inextendibility condition may hold for a much more general class of possible singularities than only those of the strong curvature type. Our theorem will be stated in Sec. II of the paper. In Sec. III we present a proof of the theorem; our main mathematical tool in this proof is a Sturm-type comparison lemma for nonoscillatory solutions of second-order differential equations. In Sec. IV we give a few concluding remarks; in particular, we argue that some earlier cosmic censorship results obtained for strong curvature singularities can be extended to singularities corresponding to the inextendibility condition.

*E-mail: rudnicki@atena.univ.rzeszow.pl
†E-mail: pzieba@atena.univ.rzeszow.pl
II. THE THEOREM

To begin with, we clearly need to recall the precise formulation of the inextendibility condition. Let \( \eta(t) \) be an affinely parametrized null geodesic, and let \( Z_1 \) and \( Z_2 \) be two linearly independent spacelike vorticity-free Jacobi fields along \( \eta(t) \). The exterior product of these Jacobi fields defines a spacelike area element, whose magnitude at affine parameter value \( t \) we denote by \( A(t) \). If we now introduce the function \( z(t) \) defined by \( A(t) \equiv z^2(t) \), then one can show [4] that \( z(t) \) satisfies the following equation:

\[
\frac{d^2 z}{dt^2} + \frac{1}{2}(R_{ab}K^aK^b + 2\sigma^2)z = 0,
\]

where \( K^a \) is the tangent vector to \( \eta(t) \) and \( \sigma^2 \) is a non-negative function of \( t \) defined as follows: \( 2\sigma^2 \equiv \sigma_{mn}\sigma^{mn} \) \((m, n = 1, 2)\). Here \( \sigma_{mn} \) is the shear tensor (see [10], p. 88) that satisfies the equation [4]:

\[
\frac{d}{dt}\sigma_{mn} = -C_{mamb}K^aK^b - \frac{2}{z}\frac{dz}{dt}\sigma_{mn}.
\]

In the following, by \( M \) we shall denote a spacetime, i.e., a smooth, boundaryless, connected, four-dimensional Hausdorff manifold with a globally defined \( C^2 \) Lorentz metric.

**Definition:** Let \( \eta : (0, a] \rightarrow M \) be an affinely parametrized, incomplete null geodesic. Assume also that \( \eta(t) \) generates an achronal set, i.e., a set such that no two points of it can be joined by a timelike curve. Then \( \eta(t) \) is said to satisfy the **inextendibility condition** if for some affine parameter value \( t_1 \in (0, a) \) there exists a solution \( z(t) \) of Eq. (1) along \( \eta(t) \) such that \( z(t_1) = 0 \), \( dz/dt|_{t_1} \neq 0 \) and \( \lim_{t 
rightarrow 0} z(t) = 0 \).

The key idea behind the inextendibility condition is based on the fact that any two zeros of any solution of Eq. (1), which is not identically zero along a given null geodesic, correspond to a pair of conjugate points along the geodesic (see [4]). From Proposition 4.5.12 of Ref. [10] it follows that incomplete null geodesics generating achronal sets cannot contain any pairs of conjugate points. One can thus easily show [8] that if a geodesic \( \eta : (0, a] \rightarrow M \) satisfies the inextendibility condition, then there is no extension of the spacetime \( M \), preserving all the above mentioned properties of \( M \), in which \( \eta(t) \) could be extended beyond a point \( \eta(0) \). This means, according to the standard interpretation, that \( \eta(t) \) should then approach a genuine singularity of the spacetime \( M \) at affine parameter value \( 0 \). Formally, this singularity has the same status as those predicted by the familiar singularity theorems [10], because these theorems predict in fact the existence of incomplete causal (usually null) geodesics in maximally extended spacetimes satisfying just the same topological and smoothness conditions as those imposed on \( M \).

Let us now compare the inextendibility condition with the concept of a strong curvature singularity [4]. Consider a null geodesic \( \lambda : (0, a] \rightarrow M \) that terminates in a strong curvature singularity at affine parameter value \( 0 \). This means that every solution \( z(t) \) of Eq. (1) along \( \lambda(t) \), which vanishes for at most finitely many points in \( (0, a] \), satisfies \( \lim_{t \rightarrow 0} z(t) = 0 \) (cf. Ref. [5], p. 160). Suppose now that \( \lambda(t) \) generates an achronal set; then any solution of Eq. (1), which is not identically zero along \( \lambda(t) \), cannot vanish for any two points in \( (0, a] \) by the argument with conjugate points mentioned above. Thus, for all \( t_1 \in (0, a] \) and for all solutions \( z(t) \) of Eq. (1) along \( \lambda(t) \) with initial conditions \( z(t_1) = 0 \) we will have \( \lim_{t \rightarrow 0} z(t) = 0 \). It is thus clear that any null geodesic terminating in Tipler’s strong curvature singularity and generating an achronal set must always satisfy the inextendibility condition. Notice also that the terms “all” emphasized above imply, via Eqs. (1) and (2), that \( \lambda(t) \) can terminate in the strong curvature singularity only if the curvature diverges strong enough along \( \lambda(t) \) as \( t \rightarrow 0 \), while the inextendibility condition could actually be satisfied for \( \lambda(t) \) even if the curvature along it would remain bounded. Indeed, the theorem stated below makes it clear [see condition (ii)] that the curvature need not necessarily diverge along geodesics satisfying the inextendibility condition.

**Theorem:** Let \( \eta : (0, a] \rightarrow M \) be an affinely parametrized, incomplete null geodesic generating an achronal set. Suppose that the Ricci tensor term \( r(t) \equiv R_{ab}K^aK^b \) along \( \eta(t) \), where \( t \) is the affine parameter and \( K^a \) is the tangent vector to \( \eta(t) \), obeys at least one of the following conditions.

(i) There exists an affine parameter value \( b \in (0, a) \) such that \( \inf\{r(t) | 0 < t < b\} \geq (\pi/b)^2 \).

(ii) There exist an affine parameter value \( c \in (0, a) \) and a constant \( \mu \in (0, 2) \) such that \( r(t) \geq \kappa t^{-\mu} \) for all \( t \in (0, c] \), where \( \kappa = (2/3)(33 - 26\mu + 5\mu^2)^{\mu^{-2}} \).

Then \( \eta(t) \) satisfies the inextendibility condition.

**Remark 1:** From the proof of this theorem, which is given below, it may be seen that the parameter values \( b \) and \( c \) mentioned above in conditions (i) and (ii) correspond to the parameter value \( t_1 \) occurring in the definition of the inextendibility condition.
Remark 2: Since in the theorem \( \eta(t) \) is assumed to be a generator of an achronal set, \( \eta(t) \) cannot contain any pair of conjugate points, and so one can expect that there should exist an upper limit on the rate of growth of the curvature along \( \eta(t) \). Indeed, from Theorems (3) and (4) of Ref. [11] it follows immediately that the Ricci tensor term \( r(t) \) along \( \eta(t) \) must satisfy the following two conditions: (1) there is no affine parameter value \( b' \in (0, a] \) such that \( \inf \{ r(t) | 0 < t < b' \} > 8(\pi/b')^2 \); and (2) if \( r(t) \geq 0 \) on \( \eta(t) \), then \( \lim_{t \to 0} t^2 r(t) \leq 1/2 \). Similar restrictions on the growth of the Weyl part of the curvature along \( \eta(t) \) can be obtained from Proposition 2.2 of Ref. [12].

In the context of our theorem, it is worth recalling the analogous results obtained by Clarke and Królak [7] for singularities of the strong curvature type. They have been obtained for two definitions of a strong curvature singularity: the original one formulated by Tipler [4] and its modification proposed by Królak [6]. According to these results, if a null geodesic \( \eta : [0, a] \to M \) terminates at affine parameter value 0 in a strong curvature singularity defined by Tipler (resp., by Królak), then there must exist some affine parameter value \( c \in (0, a] \) such that \( R_{ab} K^a K^b > A t^{-2} \) (resp., \( R_{ab} K^a K^b > A t^{-1} \)) on \( (0, c] \), where \( K^a \) is the tangent vector to \( \eta(t) \), \( t \) is the affine parameter, and \( A \) is some fixed positive constant. [Or very similar conditions on the rate of growth of the Weyl part of the curvature along \( \eta(t) \) must be satisfied; see Corollary 2 of Ref. 7.] Comparing these results with condition (ii) of our theorem we see that singularities of the strong curvature type involve a considerably stronger divergence of the Ricci tensor term \( R_{ab} K^a K^b \) than singularities corresponding to the inextendibility condition. There may thus exist a large class of curvature singularities that are not strong in the sense of the definition of Tipler or Królak, but they may still satisfy the inextendibility condition. Note also that the above conditions for strong curvature singularities are the necessary ones, whereas conditions (i) and (ii) of our theorem are only sufficient to ensure that the inextendibility condition does hold for a given geodesic. This implies that the inextendibility condition might be satisfied in more general situations than only those characterized by conditions (i) and (ii).

III. PROOF OF THE THEOREM

Now we shall prove the theorem; our main tool in this proof will be the following comparison lemma.

**Lemma (The comparison lemma):** Suppose that \( u(s) \) is a solution of the equation

\[
\frac{d^2 u}{ds^2} + F(s)u(s) = 0
\]

on an interval \((a, b]\) with initial conditions: \( u(b) = 0 \) and \( du/ds|_b \neq 0 \). Let \( v(s) \) be a solution of

\[
\frac{d^2 v}{ds^2} + G(s)v(s) = 0
\]

on \((a, b]\) such that \( v(b) = 0 \), \( dv/ds|_b = du/ds|_b \), and \( v(s) > 0 \) on \((a, b]\). Assume also that \( F(s) \) and \( G(s) \) are piecewise continuous on \((a, b]\), and let \( G(s) \geq F(s) \) on \((a, b]\). Then \( u(s) \geq v(s) \) on \((a, b]\).

**Proof:** The proof of this lemma is based essentially on Theorem 1.2 of Ref. [13], p. 210. To apply this theorem in its original form, it is convenient to reparametrize both of the equations in the lemma introducing the parameter \( t = -s \) instead of \( s \). Note that this reparametrization does not change the form of the equations. Clearly, we shall now have established the lemma if we show that for any \( c \in (a, b] \), \( u(t) \geq v(t) \) on \([-b, -c]\).

Consider the ratio \( u(t)/v(t) \). Since \( v(t) > 0 \) on \((-b, -a)\), it is well defined on \((-b, -c]\). Using l’Hospital’s rule, we get

\[
\lim_{t \to -b} \frac{u(t)}{v(t)} = 1.
\]

Therefore, as \( v(t) > 0 \) on \((-b, -c]\), to show that \( u(t) \geq v(t) \) on \([-b, -c] \), it suffices to show that

\[
\frac{d}{dt} \left[ \frac{u(t)}{v(t)} \right] \geq 0
\]

on \((-b, -c]\). It is easy to see that this inequality holds if

\[
\frac{v(t)}{v(t)} \geq \frac{u(t)}{u(t)} \quad \text{or} \quad \frac{v(t)}{u(t)} \leq \frac{u(t)}{v(t)}
\]

(3)
on \([-b, -c]\), where the overdot denotes the first derivative with respect to \(t\). Since \(F(t)\) and \(G(t)\) are piecewise continuous on \([-b, -c]\), by Theorem 1.2 of Ref. 13, p. 210, we have

\[
\tan^{-1}\left[\frac{v(t)}{v(t)}\right] \geq \tan^{-1}\left[\frac{u(t)}{u(t)}\right]
\]

for all \(t \in [-b, -c]\). Thus, as \(\tan^{-1}\) is an increasing function, the inequality (3) does hold as it is desirable.

\(\square\)

**Proof of the theorem:**

(Part I) Suppose the condition (i) is satisfied. Let \(z_0(t)\) be a solution of Eq. (1) along \(\eta(t)\) such that \(z_0(t)\) is not identically zero on \((0, b)\) and \(z_0(b) = 0\), where \(b\) is the parameter value mentioned in condition (i). Clearly, such a solution will always exist. Since \(\eta(t)\) generates an achronal set, \(z_0(t)\) can vanish nowhere in \((0, b)\); otherwise \(\eta(t)\) would have a pair of conjugate points in \([0, b]\) [see Ref. [4]], which would contradict, by Proposition 4.5.12 of Ref. [10], the achronality of \(\eta(t)\). Notice also that Eq. (1) is linear, and so the function \(-z_0(t)\) will be a solution of Eq. (1) as well. Thus, as \(z_0(t) \neq 0\) on \((0, b)\), without loss of generality we can assume that \(z_0(t) > 0\) on \((0, b)\). This implies, as \(z_0(b) = 0\), that \(dz_0/dt|_b \leq 0\). Since \(z_0(t) > 0\) on \((0, b)\), and condition (i) holds, from Eq. (1) we see at once that \(z_0(t)\) must be a concave function on \((0, b)\). This makes it obvious that \(dz_0/dt|_b \neq 0\), and so we must have \(dz_0/dt|_b = \alpha < 0\).

Let us now define the function \(z_1(t) = -1/(\alpha)z_0(t)\). As Eq. (1) is linear, it is clear that \(z_1(t)\) will be a solution of Eq. (1) along \(\eta(t)\); notice also that \(z_1(t) > 0\) on \((0, b)\), \(z_1(b) = 0\) and \(dz_1/dt|_b = -1\).

Consider now the equation

\[
\frac{d^2x}{dt^2} + \omega x(t) = 0, \quad (4)
\]

where \(\omega = \frac{1}{2}\inf\{r(t)|0 < t < b\}\) and \(r(t)\) is the function defined in the theorem. Notice that \(\omega > 0\) by condition (i). Let \(x_1(t)\) be a solution of Eq. (4) on \((0, b)\) with initial conditions: \(x_1(b) = 0\) and \(dx_1/dt|_b = -1\). It is a simple matter to see that \(x_1(t) = \omega^{-1/2}\sin[\omega^{1/2}(b - t)]\). Let us now apply the comparison lemma to the equations (1) and (4) and their solutions \(z_1(t)\) and \(x_1(t)\). Since \(\omega = \frac{1}{2}\pi\) on \((0, b)\), by the comparison lemma we must have \(x_1(t) \geq z_1(t)\) on \((0, b)\). Consequently, as \(z_1(t) > 0\) on \((0, b)\), we obtain \(x_1(t) > 0\) on \((0, b)\). This implies, by the above form of \(x_1(t)\), that \(\omega \leq (\pi/2)^2\). But \(\omega \geq (\pi/2)^2\) by condition (i). We must thus have \(\omega = (\pi/2)^2\), which gives \(lim_{t \to 0} x_1(t) = 0\). Therefore \(lim_{t \to 0} z_1(t) = 0\) since \(x_1(t) \geq z_1(t) > 0\) on \((0, b)\). This means that \(\eta(t)\) does satisfy the inextendibility condition.

(Part II) The task is now to prove the theorem in the case when condition (ii) holds. For this purpose, let us consider the following equation

\[
\frac{d^2y}{dt^2} + Bt^{-\mu}y(t) = 0, \quad (5)
\]

on \((0, c)\), where \(B = \kappa/2\), and \(\kappa, \mu\) and \(c\) are some fixed constants mentioned in the condition (ii). Let \(y_1(t)\) be a solution of this equation with initial conditions: \(y_1(c) = 0\) and \(dy_1/dt|_c = -1\). Let \(z_2(t)\) be a solution of Eq. (1) along \(\eta(t)\) such that \(z_2(c) = 0\) and \(dz_2/dt|_c = -1\). [There is no loss of generality in assuming \(z_2(t)\) to exist; the existence of \(z_2(t)\) can be established in the same manner as the existence of the analogous solution \(z_1(t)\) considered in the first part of the proof.] Clearly, the solution \(z_2(t)\), just as \(z_1(t)\), can vanish nowhere in \((0, c)\) by the argument with conjugate points. Therefore, as \(dz_2/dt|_c = -1\), we must have \(z_2(t) > 0\) on \((0, c)\). Let us now apply the comparison lemma to the equations (1) and (5) and their solutions \(z_2(t)\) and \(y_1(t)\). By condition (ii) we have \(r(t) \geq \kappa t^{-\mu}\) on \((0, c)\). Thus by the comparison lemma, we must have \(y_1(t) \geq z_2(t)\) on \((0, c)\). Of course, in order to prove the theorem, it suffices to show that \(lim_{t \to 0} z_2(t) = 0\). Thus, as \(y_1(t) \geq z_2(t) > 0\) on \((0, c)\), to complete the proof it suffices to show that \(lim_{t \to 0} y_1(t) = 0\). We shall show below that \(y_1(t)\) does possess this property.

To this end, let us first find the general solution of Eq. (5). It is easy to check that if one puts \(x = t, \alpha = 1/2, \beta = 2\sqrt{B}(2 - \mu)^{-1}, \gamma = (2 - \mu)/2\) and \(n = (2 - \mu)^{-1}\) into the equation (4.1) of Ref. [14], p. 138, then this equation reduces to our equation (5). Thus, according to the solution (4.3) of Eq. (4.1) of Ref. [14], our equation (5) has the following general solution

\[
y(t) = t^{1/2}[C_1J_n(\beta t^n) + C_2Y_n(\beta t^n)], \quad (6)
\]

where \(C_1\) and \(C_2\) are arbitrary constants of integration, and \(J_n(\beta t^n)\) and \(Y_n(\beta t^n)\) are the Bessel functions of order \(n\), of the first and second kind, respectively. Since \(\mu \in (0, 2)\), from the above relations it follows that 1/2 < \(n < \infty, \sqrt{B} < \beta < \infty\) and 0 < \(\gamma < 1\).

Let us recall that any Bessel function of the first kind has infinitely many positive zeros (cf., e.g., [15], p. 29). Let \(j_{n,1}\) be the first positive zero of the function \(J_n(\beta t^n)\), i.e., \(J_n(j_{n,1}) = 0\) and \(J_n(\beta j_{n,1}^n) \neq 0\) as long as \(0 < \beta t^n < j_{n,1}\). Since \(n > 1/2\), \(j_{n,1}\) must satisfy the following relation (see Eq. (2) of Ref. [15], p. 29):
For $J_n(\beta \gamma')$ we now define $L$ to be the number such that $J_{n,1} = L \beta c'$. Putting this into (7), and taking into account the fact that $\beta = (2\kappa)^{1/2}(2 - \mu)^{-1}$, $\kappa = 3^{3/2}(66 - 52\mu + 10\mu^2)\kappa^{n_2}$, $\gamma = (2 - \mu)/2$ and $n = (2 - \mu)^{-1}$, we readily find that $L^2 < 1$.

Consider now equation (5) with $B$ replaced by $B' = L^2 B$. Let $y_2(t)$ be a solution of this equation on $(0, \xi]$ with initial conditions: $y_2(c) = 0$ and $dy_2/ct|_c = -1$. The general form of this solution is given by (6), where $\beta$ should be replaced by $\beta' = 2\sqrt{B'}(2 - \mu)^{-1}$ (notice that $\beta' = L\beta$). Let us now insert the initial conditions for $y_2(t)$ into this general solution in order to determine for $y_2(t)$ the constants $C_1$ and $C_2$ occurring in (6). To find the first derivative of the general solution (6), we use the following recurrence formula

$$
\frac{dJ_n(x)}{dx} = -J_{n+1}(x) + \frac{n}{x} J_n(x),
$$

which is also valid for $Y_n(x)$ (see [15], p. 197). We can now easily calculate the constants $C_1$ and $C_2$; the result is as follows

$$
C_1 = \frac{Y_n(\beta' c')}{\beta' \gamma c'_{-1/2} [Y_n(\beta' c')J_{n+1}(\beta' c') - Y_{n+1}(\beta' c')J_n(\beta' c')]} \quad (8)
$$

and

$$
C_2 = \frac{-J_n(\beta' c')}{\beta' \gamma c'_{-1/2} [Y_n(\beta' c')J_{n+1}(\beta' c') - Y_{n+1}(\beta' c')J_n(\beta' c')]} \quad (9)
$$

As $\beta' = L\beta$, from the above definition of $L$ it is clear that $\beta' c' = j_{n,1}$. Thus $J_n(\beta' c') = 0$ and the numerator in (9) must vanish. As $J_n(\beta' c') = 0$, the denominator in (9) can vanish only if $Y_n(\beta' c')J_{n+1}(\beta' c') = 0$. But the Bessel functions $J_{n+1}$ and $Y_n$ cannot have any common zeros with the Bessel function $J_n$ (see [15], pp. 29-32), and so the denominator in (9) cannot vanish. We thus have $C_2 = 0$ and, by (6) and (8), the solution $y_2(t)$ can be written as follows

$$
y_2(t) = C_1 t^{1/2} J_n(\beta' c'), \quad (10)
$$

where $C_1 = [\beta' \gamma c'_{-1/2} J_{n+1}(\beta' c')]^{-1}$.

Let us now compare the solutions $y_1(t)$ and $y_2(t)$ by means of the comparison lemma. Recall that $y_1(t)$ is a solution of equation (5) with $B = \kappa/2$, while $y_2(t)$ is a solution of the same equation with $B$ replaced by $B' = L^2 \kappa/2$. Since $L^2 < 1$, by the comparison lemma we must have $y_2(t) \geq y_1(t)$ for all $t \in (0, \xi]$. We recall that any Bessel function $J_k(x)$ of the first kind with real $x$ and $k > 0$ is continuous at $x = 0$ (cf. [15], p. 182). Thus, as $n > 1/2$ and $0 < \gamma < 1$, from (10) it follows immediately that $\lim_{t \to 0} y_2(t) = 0$. Therefore, as $y_2(t) \geq y_1(t) > 0$ on $(0, \xi)$, we obtain $\lim_{t \to 0} y_1(t) = 0$, which completes the proof.

IV. CONCLUDING REMARKS

We have been concerned in this paper with the problem of determining what are curvature conditions for the occurrence of singularities corresponding to the inextendibility condition. We have found two such sufficient conditions concerning the behavior of the Ricci tensor term $R_{ab}K^a K^b$ along incomplete null geodesics—these are conditions (i) and (ii) of the theorem stated in Sec. II. This theorem shows that the inextendibility condition may hold for a considerably larger class of possible singularities than only those of the strong curvature type. In particular, condition (i) of the theorem shows that the inextendibility condition may hold even if the curvature along incomplete geodesics would remain bounded. In this context, it is worth recalling that singularities predicted by the famous singularity theorems [10] can be interpreted as regions of the universe at which the normal classical spacetime picture and/or certain energy conditions break down, and this may occur in regions where the curvature, though extremely large, still remains finite. Accordingly, if one attempts to establish, for example, whether or not these singular regions will conform to any cosmic censorship principle, it would be well to try to characterize, if necessary, incomplete geodesics terminating in these regions by a condition that may hold even if the curvature along the geodesics would remain bounded. One possible candidate for such a condition may thus be the inextendibility condition.

It should also be stressed here that some earlier cosmic censorship theorems [6,16,17] proved for strong curvature singularities can be extended to singularities corresponding to the inextendibility condition. To see this, let us first
recall that these theorems show, briefly, that under certain restrictions imposed on the causal structure, strong curvature singularities are censored (see Refs. [6,16,17] for details). Proofs of these theorems are, in essence, alike. In a brief outline, they run as follows. First, one shows that if the theorem under consideration were false, then there would have to exist a sequence \( \{\mu_i\} \) of future endless, future complete null geodesics converging to a null geodesic \( \mu \) that terminates in the future at a strong curvature singularity. One also shows that \( \mu \) and all the \( \mu_i \) must be generators of achronal sets. As all \( \mu_i \) are achronal, none of them can have a pair of conjugate points, and so any irrotational congruence of Jacobi fields along any \( \mu_i \) cannot be refocused. As \( \{\mu_i\} \) converges to \( \mu \), this must then imply, by continuity, that any irrotational congruence of Jacobi fields along \( \mu \) cannot be refocused as well. However, as \( \mu \) terminates in a strong curvature singularity, all irrotational congruences of Jacobi fields along \( \mu \) should be refocused. This gives the required contradiction. It is not difficult to see, however, that this contradiction can equally well be obtained if \( \mu \) would be assumed to satisfy the inextendibility condition, for this condition holds if at least one irrotational congruence of Jacobi fields along a given geodesic is refocused. It is thus clear that the censorship theorems given in Refs. [6,16,17] are unnecessarily restricted to strong curvature singularities and they can be extended to singularities corresponding to the inextendibility condition.

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