Fractional Supersymmetry and $F^{th}$--roots of Representations

M. Rausch de Traubenberg$^1$

Laboratoire de Physique Théorique, Université Louis Pasteur
3-5 rue de l’université, 67084 Strasbourg Cedex, France

and

Laboratoire de Physique Mathématique et Théorique, Université de Montpellier 2
place Eugène Bataillon, case 70, 34095 Montpellier Cedex 5, France

M. J. Slupinski$^2$

Institut de Recherches en Mathématique Avancée
Université Louis-Pasteur, and CNRS
7 rue R. Descartes, 67084 Strasbourg Cedex, France

Abstract

A generalization of super-Lie algebras is presented. It is then shown that all known examples of fractional supersymmetry can be understood in this formulation. However, the incorporation of three dimensional fractional supersymmetry in this framework needs some care. The proposed solutions lead naturally to a formulation of a fractional supersymmetry starting from any representation $\mathcal{D}$ of any Lie algebra $g$. This involves taking the $F^{th}$--roots of $\mathcal{D}$ in an appropriate sense. A fractional supersymmetry in any space-time dimension is then possible. This formalism finally leads to an infinite dimensional extension of $g$, reducing to the centerless Virasoro algebra when $g = \mathfrak{sl}(2,\mathbb{R})$.

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$^1$rausch@lpt1.u-strasbg.fr, rausch@lpt1.u-strasbg.fr

$^2$slupinski@math.u-strasbg.fr
1 Introduction

Describing the laws of physics in terms of underlying symmetries has always been a powerful tool. In this respect, it is interesting to study the kind of symmetries which are allowed in space-time. Within the framework of Quantum Field Theory (unitarity of the $S$ matrix etc) it is generally admitted that we cannot go beyond supersymmetry (SUSY). However, the no-go theorem stating that supersymmetry is the only non-trivial extension beyond the Poincaré algebra is valid only if one considers Lie or Super-Lie algebras. Indeed, if one considers Lie algebras, the Coleman and Mandula theorem [1] allows only trivial extensions of the Poincaré symmetry, i.e. extra symmetries must commute with the Poincaré generators. In contrast, if we consider superalgebras, the theorem of Haag, Lopuszanski and Sohnius [2] shows that we can construct a unique (up to the number of supercharges) superalgebra extending the Poincaré Lie algebra non-trivially. It may seem that these two theorems encompass all possible symmetries of space-time. But, if one examines the hypotheses of the above theorems, one sees that it is possible to imagine symmetries which go beyond supersymmetry. Several possibilities have been considered in the literature [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13], the intuitive idea being that the generators of the Poincaré algebra are obtained as an appropriate product of more fundamental additional symmetries. These new generators are in a representation of the Lorentz group which can be neither bosonic nor fermionic (bosonic charges close under commutators and generate a Lie algebra, whilst fermionic charges close under anticommutators and induce super-Lie algebras). In this paper we propose an algebraic structure, called an $F$–Lie algebra, which makes this idea precise in the context of fractional supersymmetry (FSUSY) of order $F$. Of course, when $F = 1$ this is a Lie algebra, and when $F = 2$ this is a Super-Lie algebra. We show that all examples of FSUSY considered in the literature can be described within this framework.

FSUSY ($F > 2$) has been investigated in dimensions one, two and three. In 1D the algebraic structure is relatively simple [6, 7, 8] (one just adds a new supercharge $Q$ such that $Q^2 = \partial_t$). In two dimensions, one can add either two or an infinite number of additional generators [9, 10, 11]. In three dimensions the situation is more complicated. We showed [12] that it is possible to inject equivariantly the vector representation of $so(1,2)$ in a quotient of the $F$–th symmetric product of an appropriate representation $D_{1/F}$ of $so(1,2)$. In other words, we were able to express the generators of space-time translations as symmetric $F$–order polynomials in more fundamental generators but with the new supercharges satisfying extra constraints. We also constructed explicitly in [12] unitary representations of the corresponding algebraic structure which can be understood as relativistic anyons [14, 15, 16, 17]. However, it was not possible to consider translations as $F$–order symmetric products of the new supercharges without imposing extra constraints. In contrast, this problem exists.
neither in dimensions one and two \([6, 7, 8, 9, 10]\), nor in any dimension when \(F = 2\) (SUSY). To understand the results of our paper \([12]\) in terms of \(F\)-Lie algebras, we propose two solutions: (i) extending the vectorial representation or, (ii) extending the Poincaré algebra \(B = t \oplus so(1, 2)\) to \(\hat{B} = \hat{t} \oplus Vir\), where \(so(1, 2) \subset Vir\) is the Virasoro (without central charge) algebra and \(\hat{t}\) a representation of Vir which extends the vectorial representation of \(so(1, 2)\). Correspondingly we also have to extend the \(D_{1/F}\) representation of \(so(1, 2)\) to \(\hat{D}_{1/F}\).

The problem encountered in 3D FSUSY and especially the solution we propose to solve it, enables us to define a general method of associating an FSUSY to any representation \(D\) of any Lie algebra \(g\). This algebraic structure is in general associated to a non unitary infinite dimensional representation of \(g\). Furthermore, as for \(so(1, 2)\), one can define an infinite dimensional Lie algebra \(V(g)\) having \(g\) as a sub-algebra and leading to an \(F\)-lie algebra.

The content of this paper is as follows. In section two, we give a precise mathematical definition of the algebraic structure which underlies FSUSY. Several simple examples are then given. In section 3, we show how one can incorporate 3D FSUSY into this general mathematical description by extending the vectorial representation to an appropriate reducible (but indecomposable) representation. We then construct FSUSY starting from any semi-simple Lie algebra \(g\) (playing the role of \(so(1, 2)\)) and any representation \(D\) (playing the role of the vector representation). This construction involves taking the \(F^{th}\)-root of \(D\) in some sense. In particular this means that one can construct FSUSY in all space-time dimensions. In section 4, we study an \(F\)-Lie algebra associated to an infinite dimensional algebra \(V(g)\) having \(g\) as a sub-algebra. For \(g = so(1, 2)\), \(V(g)\) reduces to the centerless Virasoro algebra.

## 2 Algebraic Structure of Fractional Supersymmetry

In this section, we give the abstract mathematical structure which underlies this paper and which generalizes the theory of Lie super-algebras and their (unitary) representations. Let \(F\) be a positive integer and \(q = \exp(\frac{2\pi i}{F})\). We consider a complex vector space \(S\) together with a linear map \(\varepsilon\) from \(S\) into itself satisfying \(\varepsilon^F = 1\). We set \(A_k = S_{q^k}\) and \(B = S_1\) (where \(S_\lambda\) is the eigenspace corresponding to the eigenvalue \(\lambda\) of \(\varepsilon\)) so that \(S = B \oplus_{k=1}^{F-1} A_k\). The map \(\varepsilon\) is called the grading. If \(S\) is endowed with the following structures we will say that \(S\) is a fractional super Lie algebra (\(F\)-Lie algebra for short):

1. \(B\) is a Lie algebra and \(A_k\) is a representation of \(B\). We write these structures as a bracket \([b, X]\) with the understanding that \([b, X] = -[X, b]\) if \(X \in A_k, b \in B\). It is clear that \([\varepsilon(X), \varepsilon(Y)] = \varepsilon([X, Y])\).

2. There are multilinear, \(B\)-equivariant (\(i.e.\) which respect the action of \(B\)) maps
\( \{ \varepsilon(a_1), \ldots, \varepsilon(a_F) \} = \varepsilon(\{a_1, \ldots, a_F\}), \forall a_i \in A_k. \) (2.1)

3. For \( b_i \in B \) and \( a_j \in A_k \) the following “Jacobi identities” hold:

\[
[[b_1, b_2], b_3] + [[b_2, b_3], b_1] + [[b_3, b_1], b_2] = 0 \\
[[b_1, b_2], a_3] + [[b_2, a_3], b_1] + [[a_3, b_1], b_2] = 0 \\
[b, \{a_1, \ldots, a_F\}] = \{[b, a_1], \ldots, a_F\} + \ldots + \{a_1, \ldots, [b, a_F]\} \\
\sum_{i=1}^{F+1} [a_i, \{a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{F+1}\}] = 0. 
\] (2.2)

The first identity is the usual Jacobi identity for Lie algebras, the second says that the \( A_k \) are representation spaces of \( B \) and the third is just the Leibniz rule (or the equivariance of \( \{ \cdot, \cdot, \cdot \} \)). The fourth identity is the analogue of the graded Leibniz rule of Super-Lie algebras for \( F \)-Lie algebras.

If we want to be able to talk about unitarity, we also require the following additional structure and in this case, \( S \) is called an \( F \)-Lie algebra with adjoint.

4. A conjugate linear map \( \dagger \) from \( S \) into itself such that:

\[
\begin{align*}
a) \quad (s\dagger)^\dagger &= s, \forall s \in S \\
b) \quad [a, b]^\dagger &= [b^\dagger, a^\dagger] \\
c) \quad \varepsilon(s^\dagger) &= \varepsilon(s)^\dagger \\
d) \quad \{a_1, \ldots, a_F\}^\dagger &= \{(a_1)^\dagger, \ldots, (a_F)^\dagger\}, \forall a \in A_k. \\
\end{align*}
\] (2.3)

From a) and c) we see that for \( X \in B \) we have \( X^\dagger \in B \), and that for \( X \in A_k \), we have \( X^\dagger \in A_{F-k} \).

A unitary representation of an \( F \)-Lie algebra with adjoint \( S \) is a linear map \( \rho : S \to \text{End}(H) \), (where \( H \) is a Hilbert space and \( \text{End}(H) \) the space of linear operators acting on \( H \)) and a unitary endomorphism \( \hat{\varepsilon} \) such that \( \hat{\varepsilon}^F = 1 \) which satisfy
a) $\rho([x,y]) = \rho(x)\rho(y) - \rho(y)\rho(x)$

b) $\rho\{a_1, \cdots, a_F\} = \frac{1}{F!} \sum_{\sigma \in S_F} \rho\left(a_{\sigma(1)}\right) \cdots \rho\left(a_{\sigma(F)}\right)$

c) $\rho(s)^* = \rho(s)$

d) $\hat{\varepsilon}\rho(s)\hat{\varepsilon}^{-1} = \rho(\varepsilon(s))$

(S_F being the group of permutations of F elements). Note that with the normalisation of b), when $F = 2$, one has $\rho(\{a_1, a_2\}) = 1/2(a_1a_2 + a_2a_1)$ instead of the usual $\rho(\{a_1, a_2\}) = (a_1a_2 + a_2a_1)$. As a consequence of these properties, since the eigenvalues of $\hat{\varepsilon}$ are $F^{th}$ roots of unity, we have the following decomposition of the Hilbert space

$$H = \bigoplus_{k=0}^{F-1} H_k,$$

where $H_k = \{ |h\rangle \in H : \hat{\varepsilon} |h\rangle = q^k |h\rangle \}$. The operator $N \in \text{End}(H)$ (the set of linear operators acting on $H$) defined by $N |h\rangle = k |h\rangle$ if $|h\rangle \in H_k$ is the “number operator” (obviously $q^N = \hat{\varepsilon}$). Since $\hat{\varepsilon}\rho(b) = \rho(b)\hat{\varepsilon}, \forall b \in B$ each $H_k$ provides a representation of the Lie algebra $B$. Furthermore, for $a \in A_\ell$, $\hat{\varepsilon}\rho(a) = q^\ell\rho(a)\hat{\varepsilon}$ and so we have $\rho(a).H_k \subseteq H_{k+\ell (\text{mod } F)}$

**Remark 1:**
For all $k = 1, \cdots, F-1$ it is clear that the subspace $B \oplus A_k$ of $S$ satisfies (2.1-2.2) and the subspace $B \oplus A_k \oplus A_{-k}$ satisfies (2.1-2.3) (when $S$ has an adjoint).

**Remark 2:**
It is important to notice that bracket $\{ \cdots \}$ is a priori not defined for elements in different gradings.

**Remark 3:**
If we set

$$B_R = \{b \in B : b^\dagger = -b\}$$

$$A_R = \{a \in A : a^\dagger = a\},$$

then $S_R = B_R \oplus A_R$ is stable by $\varepsilon$ and satisfies (2.1-2.2). Here we use the normalizations conventionally used in mathematical literature (no $i$ factor in the structure constants of the algebra). For physicists, notice that if $b^\dagger = -b$ then $(ib)$ is hermitian.

**Example 1:**

Obviously, a 1–Lie algebra is just a Lie algebra. A 2–Lie algebra is just a Lie super-algebra: $S = B \oplus A_1$, with even part $B$ and odd part $A_1$. In supersymmetry, because of the spin
statistics theorem $A_1$ is a fermionic representation of $B$. Note that the Jacobi identities (2.2) above reduce to the standard Jacobi identities of a Super-Lie algebra. If we consider unitarity in SUSY, property (2.3), (2.4) above have also to be considered but for the super-Poincaré algebra, the nature of $\dagger$ depends very much on the dimension of the space-time and the signature of the metric (Majorana, Weyl, Majorana-Weyl, $SU(2)$—Majorana and $SU(2)$—Majorana-Weyl conditions).

Example 2:

Let $V$ be finite dimensional complex vector space and let $\varepsilon : V \to V$ be a linear operator satisfying $\varepsilon^F = 1$. Then

$$V = \bigoplus_{k=0}^{F-1} V_k,$$

where $V_k = \{ |v\rangle \in V : \varepsilon |v\rangle = q^k |v\rangle \}$. We define

$$A_k = \left\{ f \in \text{End}(V) : \varepsilon \circ f \circ \varepsilon^{-1} = q^k f \right\}$$

and $S = B \bigoplus_{k=1}^{F-1} A_k$ (with $B = A_0$). Since $A_kA_\ell \subset A_{k+\ell (\text{mod } F)}$ one has

$$[A_0, A_k] \subset A_k$$

and

$$A_kA_k \cdots A_k \subset A_0.$$

The bracket $[,]$ of $S$ is defined by (2.6) and $\{ \cdots \} : S^F(A_k) \to B$ by

$$\{a_1 \cdots a_F\} = 1/F! \sum_{\sigma \in S_F} a_{\sigma(1)} \cdots a_{\sigma(F)}.$$

The first three Jacobi (2.2) identities are clearly satisfied, and calculation shows that the last Jacobi identity also holds. Thus $S$ is an $F-$Lie algebra. If $V$ is endowed with a hermitian metric and $\varepsilon$ is a unitary operator then adjunction defines an adjoint on the $F-$Lie algebra $S$.

Example 3:

In $1D$ [6, 7, 8] the simplest $F-$Lie algebra is two dimensional, and is generated by the operators $\partial_t, Q$ with the relation $Q^F = \partial_t$. We take $B = \langle \partial_t \rangle$, the translation in time, and $A_1 = \langle Q \rangle$. We obviously have $\varepsilon(\partial_t) = \partial_t$ and $\varepsilon(Q) = qQ$. An explicit representation in terms of generalized grassmann variables [18, 19, 20, 21, 22] can be constructed [6, 7, 8]. It is possible to extend this $F-$Lie algebra to a $F-$Lie algebra with adjoint by the addition of one more generator, $Q^\dagger$, such that $\varepsilon(Q^\dagger) = q^{-1}Q^\dagger$, $(Q^\dagger)^F = (\partial_t)^\dagger = -\partial_t$. Let us recall once again that there are no algebraic relations between $Q$ and $Q^\dagger$. 5
Example 4:

In $2D$ there are several possible algebras. The simplest one is obtained by considering the 3 generators $\partial_z, \partial_{\bar{z}}$ and $Q_z$. We set $B = \langle \partial_z, \partial_{\bar{z}} \rangle$ and $A_1 = \langle Q_z \rangle$, the relations are $(Q_z)^F = \partial_z$ and $[\partial_z, Q_z] = [\partial_{\bar{z}}, Q_z] = 0$.

This $F$–Lie algebra can be extended to a 4–dimensional $F$–Lie algebra with adjoint by adding one more generator $Q_{\bar{z}} \in A_{-1}$ such that $(\partial_z)^\dagger = \partial_{\bar{z}}$, $(Q_z)^F = \partial_{\bar{z}}$ and $(Q_z)^\dagger = Q_{\bar{z}}$ [9, 10].

There is also a more complicated algebraic extension, involving an infinite number of generators which corresponds to an extension of the Virasoro algebra without central charge. In addition to the Virasoro generators $L_n, \overline{L}_n, n \in \mathbb{Z}$ we add the generators $G_r, r \in \mathbb{Z} + 1/F$, which correspond to the modes of a field of conformal weight $1+1/F$ and satisfy the following relations [11]

\[
\begin{align*}
[L_n, L_m] &= (n-m)L_{m+n} \\
[\overline{L}_n, \overline{L}_m] &= (n-m)\overline{L}_{m+n} \\
[L_n, \overline{L}_m] &= 0 \\
[L_n, G_r] &= \left( \frac{n}{F} - r \right)G_{n+r} \\
[\overline{L}_n, G_r] &= 0 \\
\{G_{r_1}, \cdots, G_{r_F}\} &= L_{r_1+\cdots+r_F},
\end{align*}
\]

Here we take $B = \text{Vir} \oplus \overline{\text{Vir}}$ and $A_1 = \langle G_r, r \in \mathbb{Z} + 1/F \rangle$. In this extension, we have $L_1 \equiv \partial_z, \overline{L}_1 \equiv \partial_{\bar{z}}$ and $G_{\frac{1}{F}} \equiv Q_z$. We can also include an adjoint by adding $A_{-1} = \overline{A}_1$. As in $1D$, it is possible to construct an explicit realisation of the above algebras using generalised grassmann variables [11, 9, 10].

In all the given examples, appropriate representations have been obtained in terms of adapted superfields [6, 7, 8, 9, 10]. To our knowledge, unitarity remains an unsolved problem.

In three dimensions, the situation is much more complicated and we will study this in the next section.

Example 5

Let $g$ be a complex Lie algebra and let $r, r'$ be representations of $g$ such that there is a $g$–equivariant map $\mu : S^F(g) \to r'$. We set

$$S = B \oplus A_1 = (g \oplus r') \oplus r.$$
\[ B = g \oplus r' \] is a Lie algebra as the semi-direct product of \( g \) and \( r' \) (the latter with the trivial bracket). We can extend the action of \( g \) on \( r \) to an action of \( B \) on \( r \) by letting \( r' \) act trivially on \( r \). This defines the bracket \([ , ]\) on \( S \). For the map \{\cdots\} we take \( \mu \). The first three Jacobi identities (2.2) are clearly satisfied, and the fourth is also satisfied as each term in the expression of the L.H.S vanishes.

For example, if

\[ S^F(r) = \oplus_k r_k, \]

is a decomposition into irreducible summands, then for a given \( k \)

\[ S_k = (g \oplus r_k) \oplus r, \]

is an \( F \)-Lie algebra.

As an illustration, if \( g = \text{so}(1,2) \) and \( r = 2 \oplus 1 \) (the spin representation plus the trivial representation), then \( S^3(2 \oplus 1) = S^3(2) \oplus S^2(2) \oplus 2 \oplus 1 = 4 \oplus 3 \oplus 2 \oplus 1 \) and it is possible to obtain the spinorial or the vectorial representations of \( \text{so}(1,2) \) from a symmetric product of order 3. This can be compared with the result of R. Kerner [3] where a cubic root of the Dirac equation is obtained. More generally, for any \( F \):

\[ S^F(2 \oplus 1) = S^F(2) \oplus S^{F-1}(2) \oplus \cdots \oplus 1 = (F + 1) \oplus F \oplus \cdots \oplus 1. \]

3 Fractional Supersymmetry and finite dimensional Lie algebras

3.1 Fractional Supersymmetry in three dimensions

In [12] we considered FSUSY in three dimensions. In order to understand our results in term of \( F \)-Lie algebras let us introduce a realisation of \( \text{so}(1,2) \) which is convenient for explicit calculations. Let \( \mathcal{F} \) be the vector space of functions on \( \mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 : x, y > 0\} \).

Consider the linear operators acting on \( \mathcal{F} \) given by

\[
\begin{align*}
J_- &= x\partial_y \\
J_0 &= \frac{1}{2}(y\partial_y - x\partial_x) \\
J_+ &= y\partial_x.
\end{align*}
\]

These operators satisfy the commutation relations
\[
\begin{align*}
[J_-, J_+] &= -2J_0 \\
[J_0, J_+] &= J_+ \\
[J_0, J_-] &= -J_-,
\end{align*}
\]

and thus generate the Lie algebra \(\text{so}(1, 2)\). It is easy to check that the following subspaces of \(\mathcal{F}\) are representations of \(\text{so}(1, 2)\):

\[
\begin{align*}
\mathcal{D}_- &= \langle x^{2n} x^{2n-1} y, \ldots, x y^{2n-1}, y^{2n} \rangle, \quad (n \in \mathbb{N}/2) \\
\mathcal{D}^+ &= \langle x^{2\lambda} \left(\frac{y}{x}\right)^m, m \in \mathbb{N} \rangle, \quad (\lambda \in \mathbb{R} \setminus \mathbb{N}/2) \\
\mathcal{D}^- &= \langle y^{2\lambda} \left(\frac{x}{y}\right)^m, m \in \mathbb{N} \rangle, \quad (\lambda \in \mathbb{R} \setminus \mathbb{N}/2).
\end{align*}
\]

Of course other representations can also be obtained (for instance unbounded from below and above) but they are not useful for our purpose.

The representation \(\mathcal{D}_-\) is the \((2n+1)\)-dimensional irreducible representation and the representations \(\mathcal{D}^\pm\) are infinite dimensional representations, bounded from below and above respectively. It is important to emphasize that the representations given in (3.3) do not have the normalizations conventionally used in the literature and the basis is not orthonormal, but those normalizations are convenient for further developments. For a general classification of the representations of three-parameter Lie algebras, see e.g. [23] where analogous monomials (of the form \(x^\alpha y^\beta \left(\frac{z}{x}\right)^m\), with \(\alpha, \beta \in \mathbb{C}, m \in \mathbb{Z}\)) are considered.

In the paper [12] we introduced four representations, \(\mathcal{D}^\pm_{-1/F, \pm}\). These are are related to the above representations by the following isomorphisms

\[
\begin{align*}
\mathcal{D}^+_{-1/F, +} &\simeq \mathcal{D}^+_{-1/F, -} \simeq \mathcal{D}^+_{-1/F} \\
\mathcal{D}^-_{-1/F, +} &\simeq \mathcal{D}^-_{-1/F, -} \simeq \mathcal{D}^-_{-1/F}.
\end{align*}
\]

In this article, for practical reasons we work only with the representations (3.3).

The multiplication map \(m_n : \mathcal{F} \times \cdots \times \mathcal{F} \to \mathcal{F}\) given by

\[
m_n(f_1, \cdots, f_n) = f_1 \cdots f_n
\]

is multilinear and totally symmetric and hence induces a map \(\mu_F\) from \(\mathcal{S}^F(\mathcal{F})\) into \(\mathcal{F}\). Restricting to \(\mathcal{S}^F \left(\mathcal{D}^\pm_{-1/F}\right)\) one sees that
\begin{equation}
S^F(D_{-1/F})_{\text{red}} \overset{\text{def}}{=} \mu_F(S^F(D_{-1/F}^\pm)) = \left\langle x^2 \left( \frac{y}{x} \right)^m, \ m \in \mathbb{N} \right\rangle \supset \mathcal{D}_{-1} \tag{3.5}
\end{equation}

\begin{equation}
S^F(D_{-1/F}^\pm) \overset{\text{def}}{=} \mu_F(S^F(D_{-1/F}^\pm)) = \left\langle y^2 \left( \frac{x}{y} \right)^m, \ m \in \mathbb{N} \right\rangle \supset \mathcal{D}_{-1}
\end{equation}

The simple observation of (3.5) together with example 5 in section 2 naturally lead to the \( F \)-Lie algebra

\begin{equation}
\left( \mathfrak{so}(1, 2) \oplus S^F(D_{-1/F}^\pm) \right) \oplus \mathcal{D}_{-1/F}^\pm. \tag{3.6}
\end{equation}

In [12], by considering an adapted conjugations \( \dagger : \mathcal{D}_{-1/F}^\pm = \left( \mathcal{D}_{-1/F} \right)^\dagger \) and from the Wigner induced representation we proved that the representations of 3d-FSUSY are unitary and induce a symmetry between relativistic anyons.

Looking at the representations defined in (3.5) i.e. \( S^F(D_{-1/F}^\pm) \), one sees that, even though \( \mathcal{D}_{-1} \) is a subspace stable under \( \mathfrak{so}(1, 2) \) there is no complement stable under \( \mathfrak{so}(1, 2) \) [12]. Indeed, these representations cannot be built from a primitive vector. This is due to the fact that \( J_3^3(x_2^3) = 0 \) and consequently we cannot reach \( x^{-1}y^3 \) from \( x^2 \) but conversely \( J_3^3 \left( x^{-1}y^3 \right) = 6x^2 \) (such reducible but indecomposable representations also appear in [23]). This is the reason why there is no \( F \)-Lie algebra structure on \( \mathfrak{so}(1, 2) \oplus \mathcal{D}_{-1} \).

### 3.2 Extension to any Lie algebra

We consider now \( g \) a complex semi-simple Lie algebra of rank \( r \) and \( \mathcal{D} \) an arbitrary representation. The purpose of this section, is to construct an \( F \)-Lie algebra \( S = B \oplus A_1 \) such that the Lie algebra \( B \) contains the semi-direct product \( g \oplus \mathcal{D} \). If \( g = \mathfrak{so}(1, 2) \) and \( \mathcal{D} \) is the vector representation, this construction leads to the \( F \)-Lie algebra (3.6) above.

Let \( h \) be a Cartan sub-algebra of \( g \), let \( \Phi \subset h^* \) (the dual of \( h \)) be the corresponding set of roots and let \( f_\alpha \) be the one dimensional root space associated to \( \alpha \in \Phi \). We choose a basis \( \{ H_i, i = 1, \ldots, r \} \) of \( h \) and elements \( E^\alpha \in f_\alpha \) such that the commutation relations become

\begin{align}
[H_i, H_j] &= 0 \\
[H_i, E^\alpha] &= \alpha^i E^\alpha \\
[E^\alpha, E^\beta] &= \begin{cases} 
\epsilon \{ \alpha, \beta \} E^{\alpha+\beta} & \text{if } \alpha + \beta \in \Phi \\
\frac{2\alpha \cdot H}{\alpha \cdot \alpha} & \text{if } \alpha + \beta = 0 \\
0 & \text{otherwise}
\end{cases}
\end{align}

\( 9 \)
Recall that the real Lie algebra spanned by the $H_i$ and the $E^\alpha$ is the split real form of $g$, and that the real Lie algebra spanned by $iH_j$, $E^\alpha - E^{-\alpha}$ and $i(E^\alpha + E^{-\alpha})$ is the compact real form of $g$.

We now introduce \{\alpha_{(1)}, \ldots, \alpha_{(r)}\} (the positive roots) a basis of simple roots. The weight lattice $\Lambda_W(\mathfrak{g}) \subset \mathfrak{h}^*$ is the set of vectors $\mu$ such that $2\frac{\alpha_{(i)}}{\alpha_{(i)}, \alpha_{(j)}} \in \mathbb{Z}$ and, as is well known, there is a basis of the weight lattice consisting of the fundamental weights $\{\mu_{(1)}, \ldots, \mu_{(r)}\}$ defined by $2\frac{\mu_{(i)}}{\alpha_{(j)}, \alpha_{(i)}} = \delta_{ij}$. A weight $\mu = \sum n_i \mu_{(i)}$ is called dominant if all the $n_i \geq 0$ and it is well known that the set of dominant weights is in one to one correspondence with the set of (equivalence classes of) irreducible finite dimensional representations of $g$.

Recall briefly how one can associate a representation of $g$ to $\mu \in \mathfrak{h}^*$. In $\mathcal{U}(\mathfrak{g})$, the universal enveloping algebra of $g$, let $I_\mu$ be the left-ideal generated by the elements
\[
\{ E^\alpha (\alpha > 0), h_i - \mu(h_i), I(h_i) \mid h_i \in \mathfrak{h} \},
\]
where $I$ is the identity of $\mathcal{U}(\mathfrak{g})$ and $h_i = 2\frac{\alpha_{(i)}, h_i}{\alpha_{(i)} \alpha_{(i)}}$.

The Lie algebra $\mathfrak{g}$ acts on $\mathcal{U}(\mathfrak{g})$ by left multiplication, $I_\mu$ is stable under this action and therefore the quotient $\mathcal{V}_\mu = \mathcal{U}(\mathfrak{g})/I_\mu$ is a representation space of $\mathfrak{g}$: $\mathcal{V}_\mu$ is a highest weight representation and is called the Verma module associated to $\mu$ [24]. If $\mu$ is dominant, then $\mathcal{V}_\mu$ has a unique maximal proper sub-representation $M_\mu$ and the quotient $\mathcal{D}_\mu = \mathcal{V}_\mu/M_\mu$ is an irreducible finite dimensional representation of $g$.

To come back to our original problem, consider a finite dimensional irreducible representation $\mathcal{D}_\mu$ of $g$. If $\mathcal{V}_{\mu/F}$ is the Verma module associated to $\mu/F$, there is a $g$–equivariant map
\[
i : \mathcal{V}_\mu \to \mathcal{S}^F \left( \mathcal{V}_{\mu/F} \right), \tag{3.8}
\]
because $I \otimes \cdots \otimes I \in \mathcal{S}^F \left( \mathcal{V}_{\mu/F} \right)$ is a highest weight vector of weight $\mu$. Taking the quotient by $M_\mu$ one obtains a $g$–equivariant inclusion
\[
\mathcal{D}_\mu \hookrightarrow \mathcal{S}^F \left( \mathcal{V}_{\mu/F} \right)/i(M_\mu), \tag{3.9}
\]
since $\mathcal{D}_\mu = \mathcal{V}_\mu/M_\mu$. Denoting $\mathcal{S}^F \left( \mathcal{V}_{\mu/F} \right)/i(M_\mu)$ by $\mathcal{S}^F \left( \mathcal{D}_{\mu/F} \right)$, then, as in example 5,
\[
S = (g \oplus \mathcal{S}^F \left( \mathcal{D}_{\mu/F} \right) \text{red}) \oplus \mathcal{V}_{\mu/F} \tag{3.10}
\]
is naturally an $F$–Lie algebra.

We can reformulate this construction in less abstract terms. If $|\mu\rangle$ is the primitive vector associated to a dominant weight $\mu$ (i.e. $\langle E^\alpha | \mu \rangle = 0$, $\alpha > 0$ and $h_i |\mu\rangle = n_i |\mu\rangle = \mu(h_i) |\mu\rangle$), the representation $\mathcal{D}_\mu$ is generated by the action of the $E^\alpha$, $\alpha < 0$ on $|\mu\rangle$. Because the representation is finite dimensional, corresponding to the highest weight state $|\mu\rangle$ we have a lowest state $|\mu'\rangle$, $\mu' = \sum n_i' \mu_{(i)}$ ($\langle E^\alpha | \mu' \rangle = 0$, $\alpha < 0$). Of course the two representations built with $\mu$ or $\mu'$ are the same. But, if the weight is not dominant the situation is more
involved. For our purposes, to the representation $\mathcal{D}_\mu$ we associate two infinite dimensional representations: one associated to the weight $\frac{\mu}{F}$ and one to $\frac{\mu'}{F}$, noted $\mathcal{D}^\pm_{\frac{\mu}{F}}$ respectively (the first is bounded from below and the second from above). These two inequivalent representations are characterized by primitive vectors:

$$\mathcal{D}^+_{\frac{\mu}{F}}: \left| \frac{\mu}{F} \right\rangle \langle \frac{\mu}{F} \right| \quad h_i \left| \frac{\mu}{F} \right\rangle = \frac{n_i}{F} \left| \frac{\mu}{F} \right\rangle, E^\alpha \left| \frac{\mu}{F} \right\rangle = 0, \alpha > 0 \quad (3.11)$$

$$\mathcal{D}^-_{\frac{\mu}{F}}: \left| \frac{\mu'}{F} \right\rangle \langle \frac{\mu'}{F} \right| \quad h_i \left| \frac{\mu'}{F} \right\rangle = \frac{n_i}{F} \left| \frac{\mu'}{F} \right\rangle, E^\alpha \left| \frac{\mu'}{F} \right\rangle = 0, \alpha < 0 \quad (3.11)$$

$\mathcal{D}^+_{\frac{\mu}{F}}$ is the Verma module $\mathcal{V}_{\frac{\mu}{F}}$ abstractly defined above: In these terms the projections $\mu_F$ from $S^F(\mathcal{D}^+_{\frac{\mu}{F}}) \rightarrow S^F(\mathcal{D}^+_{\frac{\mu}{F}})_{\text{red}}$ is given by $\mu_F(S^F(\{|h_1\rangle, \cdots, |h_F\rangle\}) = |h_1 + \cdots + \cdots h_F\rangle$.

We observe that that $\mathcal{D}_\mu \subset S^F(\mathcal{D}^+_{\frac{\mu}{F}})_{\text{red}}$ because $\mu_F(S^F(\{|\frac{\mu}{F}\rangle, \cdots, |\frac{\mu}{F}\rangle\})) = |\mu\rangle$ (or $\mu_F(S^F(\{|\frac{\mu'}{F}\rangle, \cdots, |\frac{\mu'}{F}\rangle\})) = |\mu'\rangle$) is the primitive vector of $\mathcal{D}_\mu$.

4 Fractional supersymmetry and infinite dimensional algebras

In the previous section we constructed a canonical $F$–Lie algebra

$$S = (g \oplus S^F(\mathcal{D}_{\mu/F}))_{\text{red}} \oplus \mathcal{D}_{\mu/F} \quad (4.12)$$

associated to a finite dimensional Lie algebra $g$ and an irreducible finite dimensional representation $\mathcal{D}_\mu$. In this section we will show that one can extend the representation $g$ in $\mathcal{D}_\mu$ to a representation of an infinite dimensional Lie algebra $V(g)$ in $\mathcal{D}_\mu$, and construct an $F$–Lie algebra containing (4.12) as a sub-algebra.

4.1 $so(1,2)$ and the Virasoro algebra

It is well known that the Virasoro algebra admits $so(1,2)$ as a sub-algebra. The action (3.1) of $so(1,2)$ on $\mathcal{F}$ extends to an action of the Virasoro algebra (without central extension) on $\mathcal{F}$ by setting

$$L_n = \frac{n+1}{2} \left( \frac{y}{x} \right)^n x \partial_x + \frac{n-1}{2} \left( \frac{y}{x} \right)^n y \partial_y, n \in \mathbb{Z}. \quad (4.13)$$

One can verify the commutation relations

$$[L_n, L_m] = (n - m)L_{n+m} \quad (4.14)$$
and that $J_- = -L_{-1}, J_0 = -L_0, J_+ = L_1$.

In analogy with the representations (3.3) of $so(1, 2)$ we define representations of Vir as follows:

\[
\hat{D}_{-n} = \left\langle f_{m-n}^{(-n)} = x^{2m} \left( \frac{y}{x} \right)^m, m \in \mathbb{Z} \right\rangle, \quad (n \in \mathbb{N}/2)
\]

\[
\hat{D}^{+}_{\lambda} = \left\langle f_{m-\lambda}^{(+,-\lambda)} = x^{2\lambda} \left( \frac{y}{x} \right)^m, m \in \mathbb{Z} \right\rangle, \quad (\lambda \in \mathbb{R} \setminus \mathbb{N}/2)
\]

\[
\hat{D}^{-}_{\lambda} = \left\langle f_{\lambda-m}^{(-,\lambda)} = y^{2\lambda} \left( \frac{x}{y} \right)^m, m \in \mathbb{Z} \right\rangle, \quad (\lambda \in \mathbb{R} \setminus \mathbb{N}/2).
\]

Then, one can check explicitly that the action of Vir on (4.15) is given by

\[
L_k \left(f_p^{(-n)}\right) = (kn - p)f_{k+p}^{(-n)},
\]

\[
L_k \left(f_p^{(+,-\lambda)}\right) = (k\lambda - p)f_{k+p}^{(+,-\lambda)},
\]

\[
L_k \left(f_p^{(-,\lambda)}\right) = (k\lambda - p)f_{k+p}^{(-,\lambda)},
\]

where the indices in (4.15) are chosen in such a way that they correspond to the eigenvalues of $-L_0$, i.e., the helicity. In the language of conformal field theory, $f_p^{(-n)}, f_p^{(+,-\lambda)}$ and $f_p^{(-,\lambda)}$ correspond to the modes of conformal fields of conformal weight $n + 1$ and $\lambda + 1$ respectively.

Finally, we observe that the representations (3.3) are included in the corresponding representations (4.15) and in each case that the action of Vir extends the action of $so(1, 2)$. Let us remark that these representations are all unbounded from below and above (i.e., they cannot be obtained from primitive vectors or a highest/lowest weight state).

The fundamental property of these representations is that there is a Vir-equivariant map from $S^F(\hat{D}^{+}_{-1/F})$ and $S^F(\hat{D}^{-}_{-1/F})$ to $\hat{D}_{-1}$. This is just the multiplication map $\mu_F : \mathcal{F} \rightarrow \mathcal{F}$ (see (3.4)) which is obviously Vir-equivariant. In fact, a direct calculation shows that

\[
S^F(\hat{D}^{\pm}_{-1/F})_{\text{red}} \overset{\text{def}}{=} \mu_F(S^F(\hat{D}^{\pm}_{-1/F})) \cong \hat{D}_{-1},
\]

and hence that $S^F(\hat{D}^{\pm}_{-1/F})_{\text{red}}$ and $\hat{D}_{-1}$ are isomorphic.

By the method explained in example 5,

\[
S = \left(\text{Vir} \oplus \hat{D}_{-1}\right) \oplus \hat{D}^{+}_{-1/F} \oplus \hat{D}^{-}_{-1/F},
\]

is an $F$-Lie algebra.

Denoting $\hat{D}_{-1} = \left\langle P_{m-1} = f_{m-1}^{(-1)}, m \in \mathbb{Z} \right\rangle, \hat{D}^{\pm}_{-1/F} = \left\langle Q_{\pm(m-1/F)} = f_{\mp(m-1/F)}^{(\pm,1/F)}, m \in \mathbb{Z} \right\rangle$, the brackets in $S$ are given explicitly by the following formulae:
\[
\begin{align*}
\left[ L_n, L_m \right] &= (n - m)L_{n+m} \\
\left[ L_n, P_m \right] &= (n - m)P_{n+m} \\
\left[ L_n, Q^\pm_r \right] &= \left( \frac{n}{F} - r \right)Q^\pm_{n+r} \\
\left[ P_n, P_m \right] &= 0 \\
\left[ P_n, Q^\pm_r \right] &= 0 \\
\{ Q^\pm_{\pi_1}, \ldots, Q^\pm_{\pi_F} \} &= P_{\pi_1 + \cdots + \pi_F}.
\end{align*}
\] (4.19)

Remark Any weight $\mu$ of the Lie algebra $so(1, 2)$ can be considered as a weight $\hat{\mu}$ of Vir if we set $\hat{\mu}(L_0) = -\mu(J_0)$ [24]. As in section 3.1, we can construct the associated Verma module and define the $F-$Lie algebra

\[
S = (\text{Vir} \oplus S^F \left( \mathcal{V}_{\mu/F} \right)_{\text{red}}) \oplus \mathcal{V}_{\mu/F}
\] (4.20)

with obvious notation (see (3.10). Of course this construction is different from the previous one (4.18) since $\mathcal{V}_{\mu/F}$ is a highest weight representation of Vir.

### 4.2 The construction of $V(g)$

In this section, we will construct an infinite dimensional Lie algebra $V(g)$ which contains the Lie algebra $g$ in the same way as Vir contains $sl(2)$.

Let $g$ be a semi-simple complex Lie algebra, let $h$ be a Cartan sub-algebra, let $\Phi_+$ be the positive roots and let $(\alpha_{(1)}, \cdots, \alpha_{(r)})$ be the positive simple roots.

We consider the vector space $V$ generated by

\[
V = \left\langle L_0^{\alpha_i}, L_n^{\alpha_i} : \ i = 1, \cdots, r, \ \alpha \in \Phi_+, \ n \in \mathbb{Z}^* \right\rangle
\] (4.21)

satisfying the commutation relations:

1. (a) for positive simple roots $\alpha_{(1)}, \cdots, \alpha_{(r)}$,

\[
\left[ L_0^{\alpha_i}, L_0^{\alpha_j} \right] = 0.
\] (4.22)

(b) for $n > 0$,

\[
\left[ L_0^{\alpha_i}, L_n^{\pm \alpha} \right] = \pm \frac{\alpha_{(i)} \cdot \alpha}{\alpha_{(i)}^2} L_n^{\pm \alpha}, \ i = 1, \cdots, r, \ \alpha \in \Phi_+.
\]
\[
\begin{align*}
\left[ \frac{L^\alpha_n}{n}, \frac{L^\beta_n}{n} \right] &= \begin{cases} 
\epsilon\{\alpha, \beta\} \frac{L^{\alpha + \beta}_n}{n} & \text{if } \alpha + \beta \in \Phi_+ \\
0 & \text{otherwise.}
\end{cases} \\
\left[ \frac{L^\alpha_n}{n}, \frac{L^\beta_{-n}}{-n} \right] &= \begin{cases} 
\epsilon(\alpha, -\beta) \frac{L^{\alpha - \beta}_n}{n} & \text{if } \alpha - \beta \in \Phi_+ \\
\epsilon(\alpha, -\beta) \frac{L^{\alpha + \beta}_{-n}}{-n} & \text{if } -\alpha + \beta \in \Phi_+ \\
2L^\alpha_{-n} & \text{if } \alpha = \beta.
\end{cases} \\
\left[ \frac{L^\alpha_{-n}}{-n}, \frac{L^\beta_{-n}}{-n} \right] &= \begin{cases} 
\epsilon(-\alpha, -\beta) \frac{L^{\alpha + \beta}_{-n}}{-n} & \text{if } \alpha + \beta \in \Phi_+ \\
0 & \text{otherwise.}
\end{cases}
\end{align*}
\]

Thus \( g_n = \langle L^\alpha_0, L^\alpha_{\pm n} : \alpha \in \Phi_+, i = 1, \cdots, r \rangle \) is a Lie algebra isomorphic to \( g \), with Cartan sub-algebra \( \langle L^\alpha_0, i = 1, \cdots, r \rangle \), roots \( \Phi \), simple positive roots \( \alpha^{(1)}, \cdots, \alpha^{(r)} \) and root spaces \( \langle L^\alpha_{\pm n} \rangle \). The isomorphism with (3.7) is:

\[
\begin{align*}
\frac{L^\alpha_i}{-n} &\Leftrightarrow \frac{\alpha(i) \cdot H}{\alpha^2(i)} \\
\frac{L^\alpha_n}{n} &\Leftrightarrow E^\alpha \\
\frac{L^\alpha_{-n}}{-n} &\Leftrightarrow E^{-\alpha}.
\end{align*}
\]

It is important to emphasize that the Cartan sub-algebra of each \( g_n \) is independent of \( n \). Consequently the relation \( [L^\alpha_1, L^{-\alpha}_{-1}] = 2L^\alpha_0 \) defines with no ambiguity \( L^\alpha_0 \) when \( \alpha \) is not a simple root (it is just a linear combination of the \( L^\alpha_0 \) where \( \alpha^{(i)}, i = 1, \cdots, r \) are the simple roots).

2. With the notation \( L^\alpha_0 = \frac{1}{2n}[L^\alpha_n, L^{-\alpha}_{-n}] \) (this is independent of \( n > 0 \) by (4.23)) for \( \alpha \in \Phi_+ \):

\[
[L^\alpha_n, L^\alpha_m] = (n - m)L^\alpha_{n+m}.
\]

The relations (4.22), (4.23) and (4.25) do not specify all commutators \( [L^\alpha_n, L^\beta_m] \) (e.g. \( [L^\alpha_1, L^\beta_2] \) if \( \alpha \neq \beta \)). In order to obtain a Lie algebra from (4.22), (4.23) and (4.25) we define

\[
V(g) = T(V)/\mathcal{I},
\]

where \( T(V) \) is the tensorial algebra on \( V \) and \( \mathcal{I} \) is the two-sided ideal generated by the relations (4.22), (4.23) and (4.25). \( V(g) \) is an associative algebra but we will consider it as a
Lie algebra for the induced Lie bracket (i.e. commutator). The universal property of $V(g)$ is the following: any linear map $f : V \to \text{End}(H)$ such that the $f(L^\alpha_n)$ satisfy the relations (4.22), (4.23) and (4.25) extends to a unique Lie algebra homomorphism $\tilde{f} : V(g) \to \text{End}(H)$. The relations (4.22), (4.23) and (4.25) can be arranged in the following diagram (for $g = su(3)$)

![Diagram for the primitive generators for the $V(su(3))$ algebra. $\alpha, \beta$ are the primitive roots and $\gamma$ is the third positive root.](image)

Figure 1: Diagram for the primitive generators for the $V(su(3))$ algebra. $\alpha, \beta$ are the primitive roots and $\gamma$ is the third positive root.

Of course similar diagrams can be constructed for higher rank Lie algebra, and the results can be easily extend to all Lie algebras $g$. This diagram has a concentric and radial structure. The concentric symmetry is just the manifestation that to any positive $n$ can be associated an algebra isomorphic to $g$ (see (4.23)). The radial symmetries extend the $sl(2)_\alpha$ (generated by $(L^\alpha_0, L^\pm_\alpha)$) algebra to Vir$_\alpha$ a Virasoro algebra (see (4.25)). Finally, composing concentric and radial symmetries through Jacobi identities generates extra symmetries i.e. the secondary, ternary ... generators. The generators $L^\alpha_n$ with $n > 0$ and $\alpha$ a simple root are called the fundamental generators (or primary) of $V(g)$ in the sense that all the others can be obtained from them by taking commutators. The primary generator $L^\alpha_0$ is associated to the root $n\alpha$. Calculating secondary, ternary, ... generators through Jacobi identities induces generators associated to a root $\sum n\alpha_i$. For instance if $\alpha + \beta \in \Phi$, the root associated to $[L^\alpha_n, L^\beta_m]$ is $n\alpha + m\beta$. The Lie algebra $V(g)$ is clearly not a Kac-Moody algebra, but according to
Kac classification of Lie algebras [25], $V(g)$ is an infinite dimensional Lie algebra with a non-polynomial degeneracy of the roots. Let us stress again that, in such diagrams, the Cartan sub-algebra is at the origine of the diagram, and the bracket between two generators is defined only between generators on the same circle, or the same radius.

The decomposition

$$V = \bigoplus \langle L_0^\alpha : i = 1, \cdots, r \rangle \oplus \langle L_\mu^\alpha : \alpha \in \Phi_+, n > 0 \rangle \oplus \langle L_\mu^- : \alpha \in \Phi_+, n > 0 \rangle \quad (4.27)$$

should be thought of as a decomposition of $V$ into a “Cartan” sub-algebra and positive and negative “root” spaces. From this point of view, one expects to be able to construct representations of $V(g)$ from “weights”, i.e. from linear forms on the Cartan sub-algebra $\langle L_0^\alpha : i = 1, \cdots, r \rangle$. Indeed, if $\mu \in \langle L_0^\alpha : i = 1, \cdots, r \rangle^*$ (the dual space of $\langle L_0^\alpha : i = 1, \cdots, r \rangle$) is a weight then $U(V(g))/I_{\mu}$ is a representation of $V(g)$ where $I$ is the two-sided ideal in $U(V(g))$ generated by the $L_0^\alpha - \mu(L_0^\alpha)1$ ($i = 1, \cdots, r$) and the $L_n^\alpha$ ($n > 0, \alpha \in \Phi_+$) (see section 3.2). As in section 3.2

$$S = (V(g) \oplus S^F(V_{\mu/F})_{\text{red}}) \oplus V_{\mu/F} \quad (4.28)$$

is an $F$–Lie algebra (with the obvious notation).

### 4.3 Examples for $so(6)$

The Verma module construction in the previous sub-section gives many examples of $F$–Lie algebras. However, the representations of $V(g)$ obtained are all highest weight representations. In this section, in the spirit of section 4.1, we will construct explicitly non-highest weight representations of $V(so(6))$ and the corresponding $F$–Lie algebras.

First of all, we introduce $H_1, H_2, H_3$ the generators of the Cartan sub-algebra. We denote $\pm e^i$ the eigenvalues of $H_i$, and $\Phi = \{\alpha_1 = e^1 - e^2, \alpha_2 = e^2 - e^3, \alpha_3 = e^2 + e^3, \beta_1 = e^1 + e^2, \beta_2 = e^1 - e^3, \beta_3 = e^1 + e^3\}$ the positive roots, where $\alpha_1, \alpha_2$ and $\alpha_3$ are the simple roots. Then we introduce the six $sl(2) : \{E^{\pm_\phi}, \frac{\partial H}{\partial \phi}, \phi \in \Phi\}$. The three $sl(2)$ associated to the primitive roots are then generated by

$$\begin{align*}
sl(2)_1 : & \quad \{E^{\pm_{\alpha_1}}, \frac{1}{2} (H_1 - H_2)\} \\
sl(2)_1 : & \quad \{E^{\pm_{\alpha_2}}, \frac{1}{2} (H_2 - H_3)\} \\
sl(2)_3 : & \quad \{E^{\pm_{\alpha_3}}, \frac{1}{2} (H_2 + H_3)\}
\end{align*} \quad (4.29)$$

Secondly, we need to understand, using the results of section 4.1, how one can extend the representations (3.3) of $so(1, 2)$ to representations (4.15) of Vir. For that purpose, we introduce commuting variables belonging to the $so(1, 2)$ representation:
• if the representation is a spin-$s$ representation we define $\{x_{-s}, \ldots, x_s\} = D_{-s}$;

• for infinite dimensional representation bounded from below/above we consider $D^\pm_\lambda = \{x_p, p \in \pm(\mathbb{N} - \lambda)\}$.

Next, using the results of section 4.1, we can extend any representations $D$ of $so(1, 2)$ to a representation $\hat{D}$ of the Virasoro algebra. Explicitly:

• For $\hat{D}_{-s}$, one can easily observe that $x_{-ns}$, for any positive $n$ when $s$ integer and for odd positive $n$ when $s$ half-integer, is a primitive vector for the $sl(2)$-sub-algebra spanned by $\{L_{\pm n}/n, L_0/n\}$ so $D^n_{-s} = \{x_{-ns}, x_{-(n-1)s}, \ldots, x_{ns}\}$ is isomorphic to a $D_{-s}$ representation;

• in the case of infinite dimensional representations we have similar results if $\lambda$ is a rational number. In the case of interest $\lambda = 1/F$, we have primitive vector only when $n = \pm(pF + 1)$.

To conclude with the construction of the representation of Vir, we just have to introduce $p_\ell$ the conjugate momentum of $x_k$, $[p_{-\ell}, x_k] = \delta_{k\ell}$

• For $\hat{D}_{-s} = \{x_n, n \in \mathbb{Z} + s\}$ the generators of the Virasoro algebra are $L_n = \sum_{k \in \mathbb{Z}} (ns - m)x_{n+m}p_{-m}$;

• and for $\hat{D}^\pm_{\lambda} = \{x_p, p \in \mathbb{Z} \mp \lambda\}$: $L_n = \sum_{k \in \mathbb{Z} \mp \lambda} (n\lambda - k)x_{n+k}p_{-k}$.

In conformal field theory these expressions for the generators of the Virasoro algebra are known [26]. The $L_n$ are the modes of the stress-energy tensor (for the conformal or superconformal ghosts) associated to conjugate conformal fields of conformal weight $1 + \lambda$ (the $x$'s) and $-\lambda$ (the $p$'s) But it is important to emphasize that in contrast with the case of conformal field theory, here we do not have any normal ordering prescription and consequently no central charge.

The spin representation of $so(6)$

The spinorial representation of chirality $+$ is obtained from the dominant weight $\mu_+ = \mu_{(3)} = \frac{1}{2}(e^1 + e^2 + e^3)$ and the highest weight state is $\left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle$. Then by the action of $E^{-\alpha_i}$, we get the whole representation

$$D_{\mu_+} = \left\{ \left| \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\rangle, \left| \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\rangle, \left| -\frac{1}{2}, 1/2, -1/2 \right\rangle, \left| -\frac{1}{2}, -1/2, 1/2 \right\rangle \right\}.$$ (In $|a_1, a_2, a_3\rangle$, $a_i$ is the eigenvalue of $H_i$).
Now, it is easy to see that with respect to the three $sl(2)$ (noted $sl(2)_i$) we have three spinorial representations (for each $sl(2)_i$, we have $(\mathcal{D}_{-1/2})_i \subset D_{\mu_+}$)

\[
\begin{align*}
sl(2)_1 & : (D_{-1/2})_1 = \left\{ x_{1/2}^{(1)} = \left\{ \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2} \right\}, x_{-1/2}^{(1)} = \left\{ -\frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\} \right. \\
sl(2)_2 & : (D_{-1/2})_2 = \left\{ x_{1/2}^{(2)} = \left\{ -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2} \right\}, x_{-1/2}^{(2)} = \left\{ -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2} \right\} \right. \\
sl(2)_3 & : (D_{-1/2})_3 = \left\{ x_{1/2}^{(3)} = \left\{ \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right\}, x_{-1/2}^{(3)} = \left\{ 1, -\frac{1}{2}, -\frac{1}{2} \right\} \right. 
\end{align*}
\]

(4.30)

where in $x_m^{(i)}$, $i$ indicates to which $sl(2)$ the states belong and $m$ is the eigenvalue of $\frac{1}{2} \alpha_i. H$.

Now, having introduced these variables, we can straightforwardly write the generators associated to the three $sl(2)$. For $sl(2)_i$, we have (at this point we already take the normalizations appropriate for the $V(so(6))$ generalization)

\[
\begin{align*}
E^{-\alpha_i} &= J^{(i)}_- = -x_{-1/2}^{(i)} p_{-1/2}^{(i)} \\
\frac{1}{2} \alpha_i. H &= J^{(i)}_0 = \frac{1}{2} \left( x_{-1/2}^{(i)} p_{1/2}^{(i)} - x_{1/2}^{(i)} p_{-1/2}^{(i)} \right) \\
E^{\alpha_i} &= J^{(i)}_+ = x_{1/2}^{(i)} p_{1/2}^{(i)}
\end{align*}
\]

with $p$ the conjugate momentum of $x$ ($[p_-, x_k] = \delta_{ek}$).

Of course all the $x$ variables are not independent and the following identifications have to be made

\[
\begin{align*}
x_{1/2}^{(1)} &= x_{1/2}^{(2)} \quad p_{1/2}^{(1)} = p_{1/2}^{(2)} \\
x_{1/2}^{(3)} &= x_{-1/2}^{(3)} \quad p_{1/2}^{(3)} = p_{1/2}^{(3)}
\end{align*}
\]

(4.32)

leading to dependent $sl(2)$.

Now, the next point is to introduce new variables which allow us to define the $\hat{\mathcal{D}}_{\mu_+}$ representation of $V(so(6))$. In other words, we need to extend the $J^{(i)}_\pm, J^{(i)}_0$ operators to $L_n^{(i)}, n \in \mathbb{Z}$. This can be done, by considering the three $sl(2)_{2p+1}$ span by $L_{2p+1}^{(i)}/n, L_{2p+1}^{(i)}/n$ with $n = 2p+1, p \in \mathbb{N}$. We extend the $(D_{-1/2})_i$ rep. of $sl(2)_i$ into a rep. of Vir, For that we introduce a highest weight state $|\mu_p \rangle = \left[ \frac{2p+1}{2}, \frac{2p+1}{2}, \frac{2p+1}{2} \right]$ (corresponding to a primitive vector of the $sl(2)_{2p+1}$), and construct explicitly the induced spinorial representation for $so(6)$:

\[
\mathcal{D}_{\mu_+}^p = \left\{ \left[ \frac{2p+1}{2}, \frac{2p+1}{2}, \frac{2p+1}{2} \right], \left[ \frac{2p+1}{2}, \frac{2p+1}{2}, \frac{2p+1}{2} \right] \right. \\
\left[ -\frac{2p+1}{2}, \frac{2p+1}{2}, \frac{2p+1}{2} \right], \left[ -\frac{2p+1}{2}, \frac{2p+1}{2}, \frac{2p+1}{2} \right] \right. 
\]

(4.33)

As previously, we interpret the various states as spinorial multiplets of the three $sl(2)$, and we define 6 variables $x_{\pm(2p+1)}^{(i)}$. Now arguing that the $x_{r}^{(i)}, r \in \mathbb{Z} + 1/2$ span a $(\hat{\mathcal{D}}_{-1/2})_i$ representation of the Virasoro algebra, we set
\begin{equation}
L_n^{(i)} = \sum_{r \in \mathbb{Z} + 1/2} \left( \frac{n}{2} - r \right) x_{n+r}^{(i)} P_{n+r}^{(i)}.
\end{equation}

To conclude the construction of the $\hat{D}_{\mu^+} = \oplus_p \mathcal{D}_{\mu^+}^p$ representation, we make the same identifications as in (4.32)

\begin{equation}
\begin{aligned}
x_{-(2p+1)/2}^{(1)} &= x_{(2p+1)/2}^{(2)}, & p_{(2p+1)/2}^{(1)} &= p_{-(2p+1)/2}^{(2)} \\
x_{(2p+1)/2}^{(1)} &= x_{-(2p+1)/2}^{(2)}, & p_{-(2p+1)/2}^{(1)} &= p_{(2p+1)/2}^{(2)}
\end{aligned}
\end{equation}

This specified the $\hat{D}_{\mu^+}$ representation. The remaining commutators of $V(so(6))$ can then be calculated explicitly.

**The Vector representation**

For the vector representation the dominant weight is $\mu_v = \mu_{(1)} = e^1$. To construct $\hat{D}_{\mu_v}$ from $D_{\mu_v}$, we proceed along the same lines as for the spin representation, so we will be less explicit in our construction. Constructing $\hat{D}_{\mu_v}$, we have for each $sl(2)_i$ the following decomposition $\mathcal{D}_{\mu_v} \supset (\mathcal{D}^{-1/2}) \oplus (\mathcal{D}^{-1/2})$. We define highest weights associated to the three $sl(2)$ generated by $(L_i^{(i)}/n, L_0^{(i)}/n)$ with $n = 2p + 1$. This induce a vector representation $\mathcal{D}_{\mu_v}^p$:

\begin{equation}
\mathcal{D}_{\mu_v}^p = \{ |\pm(2p+1), 0, 0 \rangle, |0, \pm(2p+1), 0 \rangle, |0, 0, \pm(2p+1) \rangle \}.
\end{equation}

We then identify six spinorial representations of the primitive $sl(2)$:

\begin{align*}
sl(2)_1 : & \quad \mathcal{D}_{-1/2}^{1/2} \equiv \{ x_{(2p+1)/2}^{(1)} \equiv |2p+1, 0, 0 \rangle, x_{-(2p+1)/2}^{(1)} \equiv |0, 2p+1, 0 \rangle \} \\
& \quad \mathcal{D}_{-1/2}^{-1/2} \equiv \{ x_{(2p+1)/2}^{(2)} \equiv |0, -2p+1, 0 \rangle, x_{-(2p+1)/2}^{(2)} \equiv |-2p+1, 0, 0 \rangle \}
\end{align*}

\begin{align*}
sl(2)_2 : & \quad \mathcal{D}_{-1/2}^{1/2} \equiv \{ x_{(2p+1)/2}^{(1)} \equiv |0, 2p+1, 0 \rangle, x_{-(2p+1)/2}^{(1)} \equiv |0, 0, 2p+1 \rangle \} \\
& \quad \mathcal{D}_{-1/2}^{-1/2} \equiv \{ x_{(2p+1)/2}^{(2)} \equiv |0, 0, -2p+1 \rangle, x_{-(2p+1)/2}^{(2)} \equiv |0, -2p+1, 0 \rangle \}
\end{align*}

\begin{align*}
sl(2)_3 : & \quad \mathcal{D}_{-1/2}^{1/2} \equiv \{ x_{(2p+1)/2}^{(1)} \equiv |0, 0, 2p+1 \rangle, x_{-(2p+1)/2}^{(1)} \equiv |0, 0, -2p+1 \rangle \} \\
& \quad \mathcal{D}_{-1/2}^{-1/2} \equiv \{ x_{(2p+1)/2}^{(2)} \equiv |0, 2p+1, 0 \rangle, x_{-(2p+1)/2}^{(2)} \equiv |0, -2p+1, 0 \rangle \}
\end{align*}

The appropriate identifications can be read off from (4.37)

\begin{align*}
x_{-(2p+1)/2}^{(2)} &= x_{(2p+1)/2}^{(2)} = x_{(2p+1)/2}^{(3)} \\
x_{(2p+1)/2}^{(1)} &= x_{-(2p+1)/2}^{(2)} = x_{-(2p+1)/2}^{(3)} \\
p_{(2p+1)/2}^{(1)} &= p_{-(2p+1)/2}^{(2)} = p_{-(2p+1)/2}^{(3)} \\
p_{-(2p+1)/2}^{(1)} &= p_{(2p+1)/2}^{(2)} = p_{(2p+1)/2}^{(3)}
\end{align*}

\begin{align*}
x_{-(2p+1)/2}^{(2)} &= x_{(2p+1)/2}^{(2)} = x_{(2p+1)/2}^{(3)} \\
x_{(2p+1)/2}^{(1)} &= x_{-(2p+1)/2}^{(2)} = x_{-(2p+1)/2}^{(3)} \\
p_{(2p+1)/2}^{(1)} &= p_{-(2p+1)/2}^{(2)} = p_{-(2p+1)/2}^{(3)} \\
p_{-(2p+1)/2}^{(1)} &= p_{(2p+1)/2}^{(2)} = p_{(2p+1)/2}^{(3)}
\end{align*}
So, finally we obtain the explicit expression for the primitive (associate to the simple roots of $so(6)$) generators of $V(so(6))$ for the $\mathcal{D}_{\mu_\nu} = \oplus_p \mathcal{D}_p^\mu$, representation.

$$L_n^{(i)} = \sum_{r \in \mathbb{Z}_{+1/2}} (n - \frac{r}{2}) \left( x_{n+r}^{(i)} (p_{-r}^{(i)} + x_{n+r}^{(i)} p_{r}^{(i)}) \right). \quad (4.39)$$

**The $\mathcal{D}_{\mu_\nu}^+$ representation of $so(6)$**

In this example, we show how one can obtain an explicit realization of the $\hat{\mathcal{D}}_{\mu_\nu}^+$ representation without giving any differential realization. Starting with the highest weight state $|\mu_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F\rangle = |\mu_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F\rangle$. The interesting point with such a representation is that the states of the third $sl(2)$ belong to $\mathcal{D}_{i/F}^+$ and of the first and second $sl(2)$ to finite dimensional representations. Now, to obtain the whole representation $\hat{\mathcal{D}}_{\mu_\nu}^+$ it is enough to find the primitive vector associated to $(L_0^{(i)} / (2F + 1), L_0^{(i)} / (2F + 1))$ (there are no other states annihilated by some $L_n$, see (4.15) and (4.16)). This vector, $|\mu_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F, p_{\frac{1}{2}|2}^F\rangle$ induces a $\mathcal{D}_{\mu_\nu}^+$ representation. With the analogous identifications as for the vectorial and spinorial representations we get $\hat{\mathcal{D}}_{\mu_\nu}^+$.

**Application to $F$–Lie algebras with $so(6)$**

Having constructed the representations of $V(so(6))$, one can easily prove that $\hat{\mathcal{D}}_{\mu_\nu}^+$ is included in $S^F \left( \hat{\mathcal{D}}_{\mu_\nu}^+ \right)_\text{red}$ ($S^F \left( \hat{\mathcal{D}}_{\mu_\nu}^+ \right)_\text{red}$ is defined as in section 3.2). We just have to notice that

$$\hat{\mu}_F \left( S^F \left( |p_1 + \frac{1}{2F}, p_1 + \frac{1}{2F}, p_1 + \frac{1}{2F}, \cdots, p_F + \frac{1}{2F}, p_F + \frac{1}{2F}, p_F + \frac{1}{2F}\rangle \right) \right)$$

with $\sum p_i = p$ is a primitive vector which induces the spinorial representations (4.33) obtained with $|\mu_p\rangle$ in $\mathcal{D}_{\mu_\nu}^+$. This leads to the $F$–Lie algebra $(V(so(6)) \oplus (\mathcal{D}_{\mu_\nu}^+)_\text{red}) \oplus \hat{\mathcal{D}}_{\mu_\nu}^+$.

To conclude, the results of this sub-section probably extend, along the same lines, to any Lie algebra $g$.

**5 Conclusion**

Supersymmetry is a well established mathematical structure ($\mathbb{Z}_2$ graded algebras) and beyond its purely formal aspects it has found a wide range of applications in field theories and
particle physics. In this article, we have defined a mathematical structure generalizing the concept of super-Lie algebras ($F$-Lie algebras). These algebras seem to be appropriate if one wants to generalize supersymmetry in the sense of $F$th-roots of representations. Indeed, we have shown that, within the framework of $F$-Lie algebras, it is possible to take the $F$th-root of any representation of any (complex semi-simple) Lie algebra. In addition, $F$-Lie algebras naturally lead to the infinite dimensional Lie algebra $V(g)$ containing $g$ as a sub-algebra. For the special case $g = sl(2, \mathbb{IR})$, $V(g)$ is the centerless Virasoro algebra. As a consequence, one may wonder whether or not central extensions of $V(g)$ exist. Furthermore, one has a geometrical interpretation of the Virasoro algebra (as vector fields on the circle) so is there a similar interpretation for $V(g)$?

Unitary representations of FSUSY, for $g = so(1, 2)$ have also been constructed. It has also been checked that it is a symmetry acting on relativistic anyons [12]. In the same way, since the Lorentz group in higher dimensions is just $SO(1, d - 1)$, what is the interpretation of FSUSY for $g = so(1, d - 1)$ when the $F$-Lie algebra induces the $F$-root of the spin or the vector representations?

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References


