Constraining the equation of state of the Universe from Distant Type Ia Supernovae and Cosmic Microwave Background Anisotropies

G. Efstathiou
Institute of Astronomy, Madingley Road, Cambridge, CB3 0HA.

27 April 1999

ABSTRACT

The magnitude-redshift relation for Type Ia supernovae is beginning to provide strong constraints on the equation of state 

\[ w = p/\rho \]

of the Universe. The recent supernovae observations (Perlmutter et al. 1998, Riess et al. 1999) have led to a resurgence of interest in the possibility that the Universe is dominated by a cosmological constant \( \Lambda \). A number of authors have shown how observations of distant Type Ia supernovae can be combined with observations of CMB anisotropies to constrain the cosmological constant and matter density of the Universe (White 1998, Tegmark 1999). For example, Efstathiou et al. (1999, hereafter E99) have shown that for a wide class of potentials, the evolution of a quintessence-like scalar field follows ‘tracking solutions’, in which the late-time evolution is almost independent of initial conditions. The observational evidence for an accelerating Universe has stimulated interest in more general models containing a component with an arbitrary equation of state, \( p/\rho = w_Q \).

1 INTRODUCTION

The possible discovery of an accelerating Universe from observations of Type Ia supernovae (Perlmutter et al. 1998, Riess et al. 1999) has led to a resurgence of interest in the possibility that the Universe is dominated by a cosmological constant (for a recent review see Turner 1999). A number of authors have shown how observations of distant Type Ia supernovae (SN) can be combined with observations of CMB anisotropies to constrain the cosmological constant and matter density of the Universe (White 1998, Tegmark et al. 1998, Lineweaver 1998, Garnavich et al. 1998, Efstathiou and Bond 1999, Tegmark 1999, Efstathiou et al. 1999). For example, Efstathiou et al. (1999, hereafter E99) combine the large SN sample of the Supernova Cosmology Project (Perlmutter et al. 1998, hereafter P98; we will refer to these supernovae as the SCP sample) with a compilation of CMB anisotropy measurements and find \( \Omega_m = 0.25^{+0.18}_{-0.12} \) and \( \Omega_\Lambda = 0.63^{+0.17}_{-0.23} \) (95% confidence errors) for the cosmic densities contributed by matter and a cosmological constant respectively. These results are consistent with a number of other measurements, including dynamical measurements of \( \Omega_m \), the large-scale clustering of galaxies and the abundances of rich clusters of galaxies (Turner 1999, Bridle et al. 1999, Wang et al. 1999).

The observational evidence for an accelerating Universe has stimulated interest in more general models containing a component with an arbitrary equation of state, \( p/\rho = w_Q \). Examples include a dynamically evolving scalar field (see e.g. Ratra and Peebles 1988 and Caldwell, Dave and Steinhardt 1998, who have dubbed such a component ‘quintessence’; we will refer to this as a ‘Q’ component hereafter) and a frustrated network of topological defect (Spergel and Pen 1997, Bucher and Spergel 1999). In particular, Steinhardt, Wang and Zlatev 1998, have pointed out that for a wide class of potentials, the evolution of a Q-like scalar field follows ‘tracking solutions’, in which the late-time evolution is almost independent of initial conditions.

The purpose of this paper is three-fold. Firstly, to illustrate how the constraints on \( \Omega_m \) and \( \Omega_\Lambda \) can be improved...
The predicted peak magnitude-redshift relation is given by

$$m_{\text{pred}}(z) = M + 5\log D_L(z, \Omega_m, \Omega_\Lambda),$$

(1)

where $M$ is related to the peak absolute magnitude by $M = M_0 - 5\log H_0 + 25$, and $D_L = d_L + 5\log H_0$ is the Hubble constant-free luminosity distance. To compute the luminosity distance, we ignore gravitational lensing and use the standard expression for a Universe with uniform density by extending the redshift range of the supernovae samples. At low redshifts, the magnitude-redshift relation is degenerate for models with the same value of the deceleration parameter $q_0 (\equiv \frac{1}{2}(\Omega_m - 2\Omega_\Lambda))$. This degeneracy can be broken by observing supernovae at redshifts $z > 1$ (see, for example, Goobar and Perlmutter, 1995). Thus, by extending the redshift range of the current supernovae samples it should be possible to set tighter limits on $\Omega_m$ and $\Omega_\Lambda$ independently. This is important because there are significant worries that the SN data may be affected by grey extinction, evolution, or some other systematic effect. The consistency of SN constraints on $\Omega_m$ and $\Omega_\Lambda$ with those derived from the CMB anisotropy measurements would provide an important consistency check of systematic errors in the SN data and the interpretation of the CMB data. Secondly, we estimate the accuracy with which a more general Q-like equation of state can be constrained by high redshift SNP and CMB data. Thirdly, we use the current SN and CMB anisotropy data to constrain Q-like models in a spatially flat universe and in a universe with arbitrary spatial curvature.

### Table 1: Fisher Matrix Errors, $\Omega_m$ and $\Omega_\Lambda$.

<table>
<thead>
<tr>
<th></th>
<th>SCP</th>
<th>SCP + 20 SN</th>
<th>$2\times$SCP + 40SN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$&lt;z&gt;$</td>
<td>1.0</td>
<td>1.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$\delta\Omega_m$</td>
<td>0.53</td>
<td>0.130</td>
<td>0.081</td>
</tr>
<tr>
<td>$\delta\Omega_\Lambda$</td>
<td>0.71</td>
<td>0.265</td>
<td>0.218</td>
</tr>
<tr>
<td>$\delta M_0$</td>
<td>0.056</td>
<td>0.053</td>
<td>0.049</td>
</tr>
<tr>
<td>$\delta M$</td>
<td>0.046</td>
<td>0.042</td>
<td>0.039</td>
</tr>
</tbody>
</table>

(see, e.g. Peebles 1993).

$$d_L(z, \Omega_m, \Omega_\Lambda) = \frac{c}{H_0} \sqrt{\frac{\Omega_k}{1 + z}} \sinh \left[ \frac{\Omega_k}{1 + z} x(z, \Omega_m, \Omega_\Lambda) \right],$$

(2)

$$x(z, \Omega_m, \Omega_\Lambda) = \int_0^z \frac{dz'}{[\Omega_m(1 + z')^3 + \Omega_\Lambda(1 + z')^2 + \Omega_k]^{\frac{1}{2}}}$$

where $\Omega_k = 1 - \Omega_m - \Omega_\Lambda$ and $\sinh = \sin$ if $\Omega_k > 0$ and $\sin = \sinh$ for $\Omega_k < 0$.

We assume that we observe $N$ supernovae, with peak magnitude $m_i$, (corrected for k-term, decline-rate-luminosity relation, reddening etc), magnitude error $\sigma_i$ and redshift $z_i$, from which we want to determine a set of parameters $s_k$ by maximising the likelihood function,

$$L = \prod_{i=1}^{N} \frac{1}{\sqrt{(2\pi\sigma_i}) \exp \left( - \frac{(m_i - m_{\text{pred}})^2}{2\sigma_i^2} \right).}$$

(3)

In this section we assume that the parameters $s_k$ are $\Omega_m$, $\Omega_\Lambda$ and $M$ (defined in equation 1). An estimate of the co-

\[ Figure 1. \] The dashed lines in each panel show 1, 2 and 3σ likelihood contours in the $\Omega_\Lambda - \Omega_m$ plane for the SCP distant supernova sample as analysed by E99. The solid contours are derived from the Fisher matrix (equation 4) for the SCP sample supplemented by 20 SN with a mean redshift of $<z>=1$ (Figure 1a) and for twice the SCP sample and 40 SN with $<z>=1.5$ (Figure 1b). The points show maximum likelihood values of $\Omega_\Lambda$ and $\Omega_m$ for Monte-Carlo realizations of these samples, as described in the text.
variance matrix, $C_{ij}$, for these parameters for a given SN data set is given by the inverse of the Fisher matrix

$$F_{ij} = \sum_k \frac{1}{\sigma_k^2} \frac{\partial m^\text{pred}_k}{\partial s_i} \frac{\partial m^\text{pred}_k}{\partial s_j}$$  \hspace{1cm} (4)

(Kendall and Stewart 1979). The marginalized error on each parameter (given by $\sqrt{C_{ii}}$) is listed in Table 1 for several assumed supernova datasets. The column labelled SCP gives the Fisher matrix errors on $\Omega_m$, $\Omega_A$ and $M$ derived for sample C (56 supernovae) of P98, i.e. assuming the magnitude errors, intrinsic magnitude scatter and redshift distribution of the real sample. The next two columns give the expected errors for the SCP sample supplemented by 20 supernovae with a peak magnitude error of $\Delta m = 0.25$ magnitudes and a Gaussian redshift distribution of dispersion $\Delta z = 0.5$ and mean redshift $\langle z \rangle = 1$ and $1.5$. The upper redshift limit is close to the maximum for feasible spectroscopic measurements with 10 metre-class telescopes (see Goobar and Perlmutter 1995). As these authors comment, ground based spectroscopy at optical wavelengths becomes prohibitively expensive for supernovae at higher redshifts because of the strong K-correction. The last two columns give the errors for a sample twice as large as the SCP sample supplemented by 40 supernovae with mean redshift of 1.0 and 1.5. We adopt a background cosmology with $\Omega_A = 0.63$ and $\Omega_m = 0.25$ as indicated by the joint likelihood analysis of the SCP sample and CMB anisotropies described in E99.

From Table 1 we see that the Fisher matrix analysis of the SCP sample gives relatively large errors on $\Omega_m$ and $\Omega_m$, in agreement with the likelihood analysis presented by P98. However, by adding 20 SN at $z \sim 1$, the errors on $\Omega_A$ and in particularly $\Omega_m$ are reduced significantly. The last column shows that an enhanced SCP sample together with 40 SN at $z \sim 1.5$ (a formidable, but feasible observing programme) can provide a tight constraint on $\Omega_m$. The parameters $\Omega_A$ and $\Omega_m$ are, of course, highly correlated. This is illustrated in Figure 1 which shows 1, 2 and 3$\sigma$ error ellipses in the $\Omega_A$-$\Omega_m$ plane after marginalization over $s_3 = M$ assuming a uniform prior distribution. (The components of the new Fisher matrix after marginalization are given by $F'_{ij} = F_{ij} - F_{ik} F_{kj} / F_{kk}$, $F'_{12} = F_{12} - F_{13} F_{23} / F_{33}$, $F'_{13} = F_{13} - F_{12} F_{32} / F_{22}$.) The points in the Figure show the results of Monte-Carlo calculations, where we have simulated the observational samples and determined the parameters $s_i$ by maximising the likelihood function (equation 2). By diagonalizing the matrix $F'$ we can find the orthogonal linear combinations $\Omega_{ij} = a \Omega_m + b \Omega_A$ and $\Omega_{i2} = b \Omega_m - a \Omega_A$ defining the major and minor axes of the likelihood contours shown in Figure 1. The distributions in these orthogonal directions are shown in Figure 2 and compared with the distributions determined from the Monte-Carlo simulations. The Monte-Carlo distributions are very close to Gaussians and show that the Fisher matrix gives an extremely accurate description of the errors in the $\Omega_m$-$\Omega_A$ plane.

Although the errors in $\Omega_m$ and $\Omega_A$ are significantly reduced by the addition of high redshift supernovae over those of the SCP sample, they are still quite large in the parallel direction $\Omega_A$. This means that it is difficult to set tight limits on $\Omega_A$ from SN measurements alone. The constraints on the spatial curvature $\Omega_k$ are even weaker. For example, for the larger sample shown in Figures 1 and 2, the 1$\sigma$ error on $\Omega_k$ is $\delta\Omega_k = 0.19$. This can be reduced by extending the range to even higher redshifts (see Section 2.2) or by combining the SN data with cosmic microwave background anisotropies, as has been done by several authors (White 1998, Lineweaver 1998, Garnavich et al. 1998, Tegmark 1999, E99).

CMB anisotropy measurements, especially with future satellites such as MAP and Planck, are capable of setting tight constraints on the locations of the acoustic peaks in the CMB power spectrum. Following E99, we define an acoustic peak location parameter $\gamma_D (\Omega_m, \Omega_A)$ to be the ratio of the peak position in a model with arbitrary cosmology compared to that in a spatially flat model with zero cosmological constant. (This parameter depends weakly on the matter content of the Universe and on the spectral index of the fluctuations, but we ignore these small dependences in what follows). CMB measurements are therefore capable of fixing $\gamma_D$, defining a degeneracy direction in the $\Omega_A$-$\Omega_m$ plane given by
Observations of very distant supernovae at redshifts \(z > 3\) (see text) illustrate that by extending the redshift range one can determine \(\Omega_m\) independently of \(\Omega_\Lambda\).

\[
\Delta \Omega = -\frac{\partial \gamma_D / \partial \Omega_m}{\partial \gamma_D / \partial \Omega_\Lambda} \Delta \Omega_m, \tag{5}
\]

(see Efstathiou and Bond 1998). The results in the lower panel of Table 1 show the Fisher matrix analysis of the SN samples including the constraint imposed by equation (5). As is well known, the combination of SN and CMB measurements can break the degeneracy between \(\Omega_\Lambda\) and \(\Omega_m\) and it should be possible to determine these parameters with an error of less than 0.04 with an enlarged supernova sample assuming, of course, that systematic errors are unimportant.

Although the errors on \(\Omega_\Lambda\) from SN measurements alone converge relatively slowly as the redshift range is increased, consistency of the cosmological parameter estimates provides a strong test of systematic errors in the SN data. If we believe that systematic errors are unimportant, and that our interpretation of the CMB anisotropies (in terms of adiabatic CDM-like models) are correct, then current data already constrain \(\Omega_m\) and \(\Omega_\Lambda\) to high precision (see Fig 5 of E99). Consistency requires that the likelihood contours for a high redshift supernova sample converge to the same answer.

### 2.2 Constraining \(\Omega_m\) with NGST

Observations of very distant supernovae at \(z \gtrsim 3\) may be possible with a Next Generation Space Telescope (e.g., Miralda-Escude and Rees 1997, Madau 1998, Livio 1999). We will not analyse the feasibility of such observations here. Rather, we note from Figures 1 and 2 that the major axis of the error ellipses in the \(\Omega_\Lambda - \Omega_m\) tilt and become more vertical as the redshift range of the SN sample is increased. This is because the magnitude redshift relations for models with very different values of \(\Omega_\Lambda\) and the same \(\Omega_m\) converge at higher redshifts. The convergence redshift depends on \(\Omega_m\) and lies between \(z \approx 2-4\) for \(\Omega_m\) in the range 0.2–1 (see Figure 1 of Melnick, Terlevich and Terlevich, 1999).

This is illustrated by Figure 3, which shows the 1, 2 and 3\(\sigma\) likelihood contours from the Fisher matrix for a sample consisting of twice the SCP sample, 100 SN with \(\langle z \rangle = 1.5\), \(\Delta z = 0.5\), and 40 SN with \(\langle z \rangle = 3\), \(\Delta z = 1\). As expected, these contours are almost vertical in the \(\Omega_\Lambda - \Omega_m\) plane. A sample of supernovae (or some other distance indicator such as HII galaxies, Melnick et al. 1999) at redshifts \(z \sim 3\) can therefore produce a tight constraint on \(\Omega_m\) independently of the value of \(\Omega_\Lambda\).

### 3 CONSTRAINTS ON AN ARBITRARY EQUATION OF STATE

In this Section, we analyse the constraints that SN can place on an arbitrary equation of state. We first consider a constant equation of state. Models of this type (see Bucher and Spergel) include a frustrated network of cosmic strings \(p/\rho = -1/3\) and a frustrated network of domain walls \(p/\rho = -2/3\). A constant equation of state is also a good approximation to a Q component obeying tracker solutions. Tracker solutions are discussed in Section 3.2. Constraints on generalised forms of dark matter with anisotropic stress are discussed by Hu et al. (1999) and will not be considered here.

#### 3.1 Constant equation of state

If we include a Q-like component with equation of state \(p/\rho = w Q\), the expression for the term \(x\) in the luminosity distance (equation 2) is modified to

\[
x(z; \Omega_m, Q, w_Q) = \int_0^z \frac{dz'}{\Omega_m(1 + z')^3 + \Omega_\Lambda(1 + z')^2 + \Omega_Q(1 + z')^{3(1+w_Q)}}. \tag{6}
\]

The addition of the parameter \(w_Q\) means that it is not possible to constrain all of the parameters \(\Omega_m, Q, w_Q\) to high accuracy from the supernova data alone (see Section 4.2). Thus, Garnavich et al. (1998) analyse the High-z Supernovae Search (HJS) sample (Riess et al. 1998) assuming a spatially flat universe and find that \(w_Q < -0.55\) at 95% confidence. A similar analysis of the SCP sample byPerlmuter, Turner and White (1999) yields \(w_Q \approx -0.5\).

Table 2 lists the results of a Fisher matrix analysis for a Q-like component with a constant \(w_Q\). Here we have applied the constraints \(w_Q \geq -1\) and \(\Omega_G \geq 0\). The upper table gives results for the supernovae magnitude-redshift relation alone assuming a spatially flat Universe with \(\Omega_m = 0.25\) and \(w_Q = -1\). The constraints on \(w_Q\) from a sample such as the SCP data are quite poor and improve relatively slowly as the sample is extended to higher redshift because of a strong degeneracy between \(w_Q\) and \(\Omega_m\) in the magnitude-redshift relation. This is illustrated in Figure 4, which shows the analogue of Figure 2 for Q-like models. As the supernovae sample is extended to higher redshift, the likelihood contours narrow but \(w_Q\) and \(\Omega_m\) remain strongly degenerate.

The situation is dramatically improved by the addition of constraints from CMB anisotropies. The addition of a Q-like component affects the location of the Doppler peaks (see Caldwell et al. 1998, White 1998) and, in analogy with equation (5), an accurate determination of the CMB power spectrum imposes the constraint
improve relatively slowly as the SN sample is extended to
$\Delta \Omega_1, \Delta \Omega_2$ and $\Delta \Omega_3$. Figure 4. As Figure 2, but for an arbitrary constant equation of state in a spatially flat Universe. The dashed lines in each panel show maximum likelihood derived from Monte-Carlo realizations of these samples. The solid contours are derived from the Fisher matrix for enhanced samples of high redshift supernovae and the points show maximum likelihood derived from Monte-Carlo realizations of these samples.

Table 2: Fisher Matrix Errors, $\Omega_m$, $\Omega_Q$ and $w_Q$.

<table>
<thead>
<tr>
<th></th>
<th>Supernovae Alone, $\Omega_k = 0$</th>
<th>$\langle z \rangle$ SCP + 20 SN</th>
<th>$\langle z \rangle$ 2SCP + 40SN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Omega_m$</td>
<td>0.14</td>
<td>0.12</td>
<td>0.097</td>
</tr>
<tr>
<td>$\delta w_Q$</td>
<td>0.36</td>
<td>0.35</td>
<td>0.32</td>
</tr>
<tr>
<td>$\delta M$</td>
<td>0.051</td>
<td>0.051</td>
<td>0.037</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Supernovae + CMB, $\Omega_k = 0$</th>
<th>$\langle z \rangle$ SCP + 20 SN</th>
<th>$\langle z \rangle$ 2SCP + 40SN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Omega_m$</td>
<td>0.027</td>
<td>0.022</td>
<td>0.0210</td>
</tr>
<tr>
<td>$\delta w_Q$</td>
<td>0.10</td>
<td>0.085</td>
<td>0.081</td>
</tr>
<tr>
<td>$\delta M$</td>
<td>0.048</td>
<td>0.046</td>
<td>0.045</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Supernovae + CMB, $\Omega_k \neq 0$</th>
<th>$\langle z \rangle$ SCP + 20 SN</th>
<th>$\langle z \rangle$ 2SCP + 40SN</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta \Omega_m$</td>
<td>0.14</td>
<td>0.12</td>
<td>0.095</td>
</tr>
<tr>
<td>$\delta w_Q$</td>
<td>0.10</td>
<td>0.083</td>
<td>0.066</td>
</tr>
<tr>
<td>$\delta M$</td>
<td>0.051</td>
<td>0.051</td>
<td>0.036</td>
</tr>
</tbody>
</table>

The second panel of Table 2 shows the constraints derived on an arbitrary equation of state by combining supernovae data with the CMB constraint of equation (7). For spatially flat models, the combination of SN and CMB anisotropies constrains $w_Q$ to an accuracy of better than 0.1, sufficient to set tight constraints on the physical parameters of Q-like models (for example, whether one requires contrived potentials, see Section 4). However, the constraints on $w_Q$ improve relatively slowly as the SN sample is extended to higher redshift. Similar conclusions apply if the assumption of a spatially flat universe is relaxed (see the lower panel of Table 2). In that case, the parameters $\Omega_m$ and $\Omega_Q$ can be determined to high precision, but the constraints on $w_Q$ improve slowly as the SN sample is increased. This implies that it is worth analysing the constraints on Q-like models with arbitrary spatial curvature using current SN and CMB data (see Section 4.2).

3.2 Time varying equation of state: tracker solutions

In the previous section we have investigated the simplified case of a constant $w_Q$. If, in fact, the Q-like component arises from a slowly rolling scalar field evolving in a potential $V(Q)$, the equation of state of the Q component will vary as a function of time. The equations of motion of the Q field can be written in the following compact form (Steinhardt, Wang and Zlatev, 1998)

$$\frac{V''V}{(V')^2} = \frac{1 + w_B - w_Q}{2(1 + w_Q)} - \frac{1 + w_B - 2w_Q}{2(1 + w_Q)} \frac{\dot{x}}{6 + \dot{x}}$$

$$\frac{2}{(1 + w_Q)(6 + \dot{x})^2}, \quad x \equiv \frac{(1 + w_Q)}{(1 - w_Q)}$$

(8)

where primes denote derivatives with respect to Q, $\dot{x} = dnx/dna$, $x = d^2nx/dn^2a$ and $a$ is the scale factor of the cosmological model. For a wide class of potentials, and almost independently of the initial conditions, the evolution of Q locks on to a tracking solution in which Q and $w_Q$ vary slowly (see Zlatev, Wang and Steinhardt 1998, Steinhardt et al. 1998). Examples of the evolution of $w_Q$ and $\Omega_Q$ at late times are shown in Figure 5 for three forms of the potential $V(Q)$. In each case, the evolution of $w_Q$ at $z \lesssim 4$ is well approximated by

$$w_Q = w_Q(a_0) + a\ln(a/a_0)$$

(9)
Figure 5. The evolution of the equation of state $w_Q$ and its contribution to the cosmic density parameter $\Omega_Q$ as a function of redshift derived from solutions to the tracker equation (8) for three potentials: $V(Q) = M^4(\exp(1/Q) - 1)$ (figures 5a and 5b); $V(Q) = M^4(M/Q)^2$ (figures 5c and 5d); $V(Q) = M^4(M/Q)^6$ (figures 5e and 5f). The curves in each figure are computed by varying the parameter $M$, with more negative values of $w_Q$ corresponding to higher values of $\Omega_Q$.

Figure 6. The left hand panels show the tracker solution relations between $\Omega_Q$, $w_Q$ at the present day for the three potentials used in Figure 5. The right hand panels show the derivative $\alpha \equiv \partial w_Q/\partial \ln a$ for the tracker solutions as a function of $w_Q$.

where $\alpha$ is a small number determined from the value of $\dot{x}$ at the present time.

Figure 7 shows the relations between $\Omega_Q$, $w_Q$ and $\alpha$ at the present time derived from the solutions to equation 4 for the three potentials considered in Figure 5. The minimum value of $\alpha$ is about $-0.14$, reflecting the fact that $Q$ is evolving relatively slowly even at late times.

With the approximation of equation (9), the energy density of the $Q$ component evolves according to

$$\rho_Q(a) = \left(\frac{a}{a_0}\right)^{-3(1+w_Q(a_0))} \exp\left(-\frac{3}{2}\alpha[\ln(a/a_0)]^2\right). \quad (10)$$

Note also that with the approximation of equation (9), the tracker equation (8) becomes an algebraic equation relating $\frac{V''}{V'}$ to $w_Q$, $\Omega_Q$ and $\alpha$ ($w_B = w_Q\Omega_Q$ in the matter dominated era).

A small value of $\alpha \sim -0.1$ to $-0.2$ cannot be determined accurately from SN and CMB observations because it is highly degenerate with $w_Q$ and $\Omega_m$. As we will show in the next Section, the introduction of the parameter $\alpha$ provides a convenient way of testing the sensitivity of constraints on Q-like models to the time evolution of $w_Q$.

We note that Huterer and Turner (1998) have recently proposed a prescription for reconstructing the potential of a $Q$-like component directly from the magnitude-redshift relation of Type Ia supernovae. This approach may produce interesting constraints if the field $Q$ is rapidly evolving at late times. For tracker solutions, however, the equation of state changes so slowly that it would be difficult to distinguish the true potential from a perfectly flat one.
4 LIMITS ON THE EQUATION OF STATE FROM TYPE I A SUPERNOVAE AND THE COSMIC MICROWAVE BACKGROUND

4.1 Spatially flat models

In this Section, we use current SN and CMB data to constrain the equation of state of the Universe. The analysis closely follows that presented in E99. We use the sample of 56 Type Ia SN of fit C of P98 and adopt the likelihood analysis described by E99 (including a parametric fit to the luminosity-decline rate correlation), modifying the expression for luminosity distance to incorporate the parameters of the Q-like model. The CMB data that we use are plotted in Figure 1 of E99. We perform a likelihood analysis for these data assuming scalar adiabatic perturbations, varying the amplitude of the fluctuation spectrum, the scalar spectral index, the physical densities of the CDM and baryons $\omega_c = \Omega_c h^2$, $\omega_b = \Omega_b h^2$, and the Doppler peak location parameter $\gamma_D$. Modifications to the CMB power spectrum arising from spatial fluctuations in the Q component are ignored as these are negligible in the slowly evolving Q models considered here (see Caldwell et al. 1998, Huey et al. 1998). We integrate over the CMB likelihood assuming uniform prior distributions of the parameters to compute a marginalized likelihood for $\gamma_D$ as described in E99. The likelihood functions for the parameters $w_Q$, $\Omega_m$ and $\Omega_Q$ presented below are constructed from the expression for the angular diameter distance to the last scattering surface and the probability distribution of $\gamma_D$.

Figure 7 shows the constraints on $w_Q$ and $\Omega_m$ for spatially flat universes. The different line types show the constraints for three different values of the parameter $\alpha$ characterising the evolution of $w_Q$, $\alpha = 0$ (solid lines), $\alpha = -0.1$ (dotted lines) and $\alpha = -0.2$ (dashed lines). As described in the previous section, these values span the range found for tracker solutions for a variety of potentials. These rates of evolution are so low that they have very little effect on the likelihood contours. The constraints plotted in Figure 7 are in very good agreement with those derived by Garnavich et al. (1998) from an analysis of the HZS sample, and with the analysis of the SCP sample (Perlmutter, Turner and White 1999) and of the combined HZS and SCP samples (Wang et al. 1999). The fact that the constraints are weakly dependent on the size of the SN sample is a consequence of the strong degeneracy between $w_Q$ and $\Omega_m$ discussed in Section 3.2.

Figure 7b shows the results of combining the SN likelihoods with those determined from the CMB. The likelihood peaks at $w_Q = -1$, $\Omega_m = 0.29$. Qualitatively, these results are similar to those of Perlmutter et al. (1999); the favoured cosmology has an equation of state $w_Q = -1$ and $w_Q$ is constrained to be less than $-0.6$ at the 2$\sigma$ level. However, in detail, the constraints in Figure 7b are somewhat less stringent than those of Perlmutter et al., allowing a broader range in $\Omega_m$ ($0.15 \lesssim \Omega_m \lesssim 0.5$ at the 2$\sigma$ level). This is because Perlmutter et al. include constraints on the power spectrum of galaxy clustering based on the data compiled by Peacock and Dodds (1994). In our view this is dangerous because it requires a specific assumption concerning the distribution of galaxies relative to the mass. Qualitatively, for nearly scale-invariant adiabatic models, galaxy clustering imposes a constraint on the parameter combination $\Gamma = \Omega_m h$ of $0.2 \lesssim \Gamma \lesssim 0.3$, if galaxies are assumed to trace the mass fluctuations on large scales (Efstathiou, Bond and White 1992, Maddox, Efstathiou and Sutherland 1996). Combined with

* $h$ is the Hubble constant in units of 100 kms$^{-1}$Mpc$^{-1}$.

† Perlmutter et al. (1999) do not combine the SN and CMB likelihoods but analyse the SN data assuming a spatially flat Universe.
Figure 8. Analogue of Figure 5, but for quintessence models with arbitrary spatial curvature. Figures 7a and 7b show marginalized likelihoods in the $w_Q$–$\Omega_m$ and $\Omega_Q$–$\Omega_m$ planes derived from Type Ia supernovae. Figures 7c and 7d show the combined likelihoods for the Type Ia and CMB anisotropies. As in Figure 5, the solid contours are derived for $\alpha = 0$ and dashed contours for $\alpha = -0.2$.

The constraints of Figure 7b place strong limits on $Q$-like models. For tracking solutions, the constraint $w_Q \lesssim -0.6$ excludes steep potentials (e.g. $V(Q) \propto Q^{-\beta}$ with $\beta \gtrsim 2$) and the data clearly favour a standard cosmological term ($w_Q = -1$). These limits on $w_Q$ are very close to the lower limit ($w_Q \gtrsim -0.7$) allowed for ‘physically well motivated’ tracker solutions (Steinhardt, Wang and Zlatev, 1998, i.e. smooth potentials with simple functional forms). With a slight improvement of the observations one may be forced to fine-tune the shape of the potential to construct a viable quintessence model.

The constraints of Figure 7b are somewhat stronger than those of Wang et al. (1999), who perform a ‘concordance analysis’ of Q-like models using a number of observational constraints including those from Type Ia supernovae and CMB anisotropies. These authors conclude limits of $-1 \lesssim w_Q \lesssim -0.4$. The difference is caused by the different methods of statistical analysis. The concordance analysis of Wang et al. leads, by construction, to more conservative limits than the maximum likelihood analysis and is more robust to systematic errors in any particular data set. However, provided systematic errors are negligible in the CMB and SN datasets, then the constraints of Figure 7b derived by combining likelihoods should be realistic. These small differences in the upper limits on $w_Q$ are important because they can place significant restrictions on the physics. As stressed in the previous paragraph, the upper limit of $w_Q \approx -0.6$ places strong constraints on tracker models with simple potentials.

4.2 Models with arbitrary spatial curvature

Figure 8 shows the results of a likelihood analysis of the SN and CMB data, but now allowing arbitrary spatial curvature. We show two projections of the likelihood distribu-
Equation of state from Supernovae and CMB Observations

9

5 CONCLUSIONS

Observations of distant Type Ia supernovae have provided important evidence that the Universe may be dominated by a cosmological constant (P98, Riess et al. 1999). However, the constraints in the $\Omega_L-\Omega_m$ plane from current data are degenerate along a line defined by $\Omega_L \approx 0.32 + 1.43\Omega_m$ (Figure 1). This degeneracy can be reduced significantly by extending the redshift range of the supernovae sample. For example, with 20 additional supernovae at redshift $z > 1.5$ the errors in $\Omega_m$ and $\Omega_L$ could be reduced to $\delta\Omega_m \approx 0.08$ and $\delta\Omega_L \approx 0.22$. A sample of supernovae at $z \approx 3$ could provide an accurate estimate of $\Omega_m$ that is independent of the value of $\Omega_L$.

The combination of supernovae and CMB anisotropy measurements can break the degeneracy between $\Omega_L$ and $\Omega_m$ if the initial fluctuations are assumed to be adiabatic and characterised by a smooth fluctuation spectrum. This method applied to recent supernovae and CMB data suggests a nearly spatially flat universe dominated by a cosmological constant ($\Omega_m = 0.12$, $\Omega_L = 0.73$ irrespective of the value of $\Omega_w$, suggesting that the Universe is almost spatially flat. The $2\sigma$ upper limit of $\Omega_w = 0.6$ for universes of arbitrary spatial curvature, the $2\sigma$ upper limit is $\Omega_w = 0.4$. The combined SN and CMB likelihood peaks at $\Omega_m = 0.12$ and $\Omega_L = 0.73$ irrespective of the value of $\Omega_w$, suggesting that the Universe is almost spatially flat. The $2\sigma$ upper limit of $\Omega_w = 0.6$ for spatially flat Universes is close to the minimum value of $\Omega_w \approx 0.7$ allowed for simple quintessence models. This suggests that some fine tuning of the potential may be required to construct a viable quintessence model.

Acknowledgements. I thank Richard Ellis, Paul Steinhardt and Roberto Terlevich for useful discussions and PPARC for the award of a Senior Fellowship. I also thank Sarah Bridle, Anthony Lasenby, Mike Hobson and Graca Rocha for allowing me to use their compilation of CMB anisotropy data.

REFERENCES


Ratra B., Peebles P.J.E., 1988, Phys Rev D, 37, 3406.


