Abstract

By using the fact that Polychronakos-like models can be obtained through the ‘freezing limit’ of related spin Calogero models, we calculate the exact spectrum as well as partition function of $SU(m|n)$ supersymmetric Polychronakos (SP) model. It turns out that, similar to the non-supersymmetric case, the spectrum of $SU(m|n)$ SP model is also equally spaced. However, the degeneracy factors of corresponding energy levels crucially depend on the values of bosonic degrees of freedom ($m$) and fermionic degrees of freedom ($n$). As a result, the partition functions of SP models are expressed through some novel $q$-polynomials. Finally, by interchanging the bosonic and fermionic degrees of freedom, we obtain a duality relation among the partition functions of SP models.

Keywords: Polychronakos model, supersymmetry, partition function, duality, $q$-polynomials

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1 Introduction

Integrable one-dimensional models with long ranged interactions have recently attracted a lot of interest due to their close connection with diverse subjects like fractional statistics, random matrix theory, level statistics for disordered systems, Yangian algebra, q-polynomials etc. [1-20]. The $SU(M)$ Polychronakos spin chain [6-8] is a well known example of such integrable models with the Hamiltonian given by

$$H_P = \sum_{1 \leq i < j \leq N} \frac{(1 - \epsilon P_{ij})}{(\bar{x}_i - \bar{x}_j)^2},$$

(1.1)

where $\epsilon = 1(-1)$ represents the ferromagnetic (anti-ferromagnetic) case and $P_{ij}$ is the exchange operator interchanging the ‘spins’ (taking $M$ possible values) of $i$-th and $j$-th lattice sites. Moreover the positions of corresponding lattice sites $(\bar{x}_i)$, which are inhomogeneously distributed on a line, are given by the zero points of the $N$-th order Hermite polynomial and may also be obtained as a solution of the following set of equations:

$$x_i = \sum_{k \neq i} \frac{2}{(x_i - x_k)^2},$$

(1.2)

where $i \in [1, 2, \cdots, N]$. Though the Polychronakos spin chain (1.1) does not enjoy translational invariance, one can find out its exact spectrum as well as partition function by considering the ‘freezing limit’ of the Calogero Hamiltonian which possesses both spin and particle degrees of freedom [8]:

$$H_C = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{l(l - \epsilon P_{ij})}{(x_i - x_j)^2},$$

(1.3)

$\omega$ and $l$ being some positive coupling constants. Thus, it is revealed that the Hamiltonian (1.1) generates an equidistant spectrum, where the energy levels are highly degenerate in general. Such high degeneracy of energy levels can be explained through the ‘motif’ picture originating from the $Y(gl_M)$ Yangian symmetry of the Polychronakos spin chain (1.1) and Calogero model (1.3) [13,21]. Moreover, the partition function of the Hamiltonian (1.1) is found to be closely related [13,22] to the $SU(M)$ Rogers-Szegő (RS) polynomial, which have appeared earlier in different contexts like the theory of partitions [23] and the character formula for the Heisenberg XXX spin chain [24].

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Now, for the purpose of obtaining a supersymmetric extension of the Polychronakos spin chain, we consider a set of operators like $C_{i\alpha} \dagger \left( C_{i\alpha} \right)$, which creates (annihilates) a particle of species $\alpha$ on the $i$-th lattice site. Such creation (annihilation) operators are defined to be bosonic if $\alpha \in [1, 2, \cdots, m]$ and fermionic if $\alpha \in [m + 1, m + 2, \cdots, m + n]$ (where, according to our notation, $m + n = M$). Next, we focus our attention only to a subspace of the related Fock space, for which the total number of particles per site is always one:

$$\sum_{\alpha=1}^{M} C_{i\alpha} \dagger C_{i\alpha} = 1,$$

for all $i$. On the above mentioned subspace, one can define a supersymmetric exchange operator as [18]

$$\hat{P}^{(m|n)}_{ij} = \sum_{\alpha, \beta=1}^{M} C_{i\alpha} \dagger C_{j\beta} \dagger C_{i\beta} C_{j\alpha},$$

and show that this $\hat{P}^{(m|n)}_{ij}$ yields a realisation of the permutation algebra given by

$$P_{ij}^2 = 1, \quad P_{ij} P_{jl} = P_{ji} P_{il}, \quad [P_{ij}, P_{lm}] = 0,$$

($i, j, l, m$ being all distinct). This supersymmetric exchange operator (1.5) was used earlier for constructing the $SU(m|n)$ supersymmetric Haldane-Shastry (HS) model [18]. So, in analogy with the case of supersymmetric HS model, we may use the exchange operator (1.5) for constructing a Hamiltonian of the form

$$\mathcal{H}^{(m|n)}_P = \sum_{1 \leq i < j \leq N} \frac{\left( 1 - \hat{P}^{(m|n)}_{ij} \right)}{(\bar{x}_i - \bar{x}_j)^2}.$$

It should be noted that, at the special case $m = M, n = 0$, i.e. when all degrees of freedom are bosonic, $\hat{P}^{(m|n)}_{ij}$ (1.5) becomes equivalent to the spin exchange operator $P_{ij}$ which appears in the Hamiltonian (1.1). Therefore, at this pure bosonic case, (1.7) would reproduce the ferromagnetic Polychronakos spin chain (1.1) where $\epsilon = 1$. Similarly, for the case $m = 0, n = M$, i.e., when all degrees of freedom are fermionic, (1.7) would reproduce the anti-ferromagnetic Polychronakos spin chain (1.1) where $\epsilon = -1$. So, when both bosonic and fermionic degrees of freedom are involved (i.e., when both $m$ and $n$ take nonzero values), we may say that $\mathcal{H}^{(m|n)}_P$ (1.7) represents the Hamiltonian of $SU(m|n)$ supersymmetric Polychronakos (SP) model.
In this article, our aim is to study the spectrum and partition function of the above defined $SU(m|n)$ SP model. To this end, in Sec.2, we introduce an appropriate extension of spin Calogero model (1.3), which would generate the SP model (1.7) at the ‘freezing limit’. In this section, we also demonstrate that the spectrum of such extended spin Calogero model is essentially the same with the spectrum of decoupled $SU(m|n)$ supersymmetric harmonic oscillators, where each oscillator has $m$ bosonic as well as $n$ fermionic spin degrees of freedom. In this way we are able to show that, the equidistant spectrum of $SU(m|n)$ SP model (1.7) can be obtained by simply ‘modding out’ the spectrum of $SU(m|n)$ harmonic oscillators through the spectrum of spinless bosonic harmonic oscillators. Subsequently, in Sec.3, we derive the partition function of SP model (1.7) and interestingly observe that such partition functions can be expressed through some novel $q$-polynomials. Finally, we obtain a duality relation between the partition functions of $SU(m|n)$ and $SU(n|m)$ SP models. Sec.4 is the concluding section.

2 Spectra of $SU(m|n)$ SP model and related spin Calogero model

For obtaining the spectrum of SP model, we wish to follow here the approach of Ref.8 and find out at first a suitable extension of spin Calogero model (1.3), which would reproduce the SP model (1.7) at the ‘freezing limit’. However, one immediate problem which arises at present is that a spin Calogero model like (1.3) is a first quantised system, while the SP model (1.7) is a second quantised system. So, for applying the above mentioned approach, it is convenient to transform the second quantised SP model (1.7) to a first quantised spin system.

To this end, we now consider some special cases of ‘anyon like’ representations of permutation algebra (1.6), which were used earlier for constructing integrable extensions of $SU(M)$ Calogero-Sutherland model as well as HS spin chain [25-28]. Such
special cases of ‘anyon like’ representations are defined by their action on a spin state like \(|\alpha_1\alpha_2 \cdots \alpha_N\rangle\) (with \(\alpha_i \in [1, 2, \ldots, M]\)) as

\[
\tilde{P}^{(m|n)}_{ij} |\alpha_1\alpha_2 \cdots \alpha_i \cdots \alpha_j \cdots \alpha_N\rangle = e^{i\Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j)} |\alpha_1\alpha_2 \cdots \alpha_j \cdots \alpha_i \cdots \alpha_N\rangle,
\]

(2.1)

where \(\Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j) = 0\) if \(\alpha_i, \alpha_j \in [1, 2, \ldots, m]\), \(\Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j) = \pi\) if \(\alpha_i, \alpha_j \in [m+1, m+2, \ldots, m+n]\), and \(\Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \cdots, \alpha_j) = \pi \sum_{\tau=m+1}^{m+n} \sum_{p=i+1}^{j-1} \delta_{\tau,\alpha_p}\) if \(\alpha_i \in [1, 2, \ldots, m]\) and \(\alpha_j \in [m+1, m+2, \ldots, m+n]\) or vice versa. It is clear that \(\tilde{P}^{(M|0)}_{ij}\) reproduces the original spin exchange operator \(P_{ij}\) and \(\tilde{P}^{(0|M)}_{ij}\) reproduces \(-P_{ij}\). Next we notice that, due to the constraint (1.4), the Hilbert space associated with SP Hamiltonian (1.7) can be spanned through the following orthonormal basis vectors: \(C^i_{1\alpha_1} C^j_{2\alpha_2} \cdots C^k_{N\alpha_N} |0\rangle\), where \(|0\rangle\) is the vacuum state and \(\alpha_i \in [1, 2, \ldots, M]\).

So, it is possible to define a one-to-one correspondence between the state vectors of the above mentioned Hilbert space and the state vectors associated with a spin chain as

\[
|\alpha_1\alpha_2 \cdots \alpha_N\rangle \leftrightarrow C^i_{1\alpha_1} C^j_{2\alpha_2} \cdots C^k_{N\alpha_N} |0\rangle.
\]

(2.2)

However it should be noted that, the matrix elements of \(\tilde{P}^{(m|n)}_{ij}\) (2.1) and \(\tilde{P}^{(m|n)}_{ij}\) (1.5) are related as [28]

\[
\langle \alpha'_1\alpha'_2 \cdots \alpha'_N | \tilde{P}_{ij} | \alpha_1\alpha_2 \cdots \alpha_N \rangle = \langle 0 | C_{N\alpha_N'} C_{N-1}\alpha_{N-1}' \cdots C_1\alpha_1' \tilde{P}^{(m|n)}_{ij} C^i_{1\alpha_1} C^j_{2\alpha_2} \cdots C^k_{N\alpha_N} |0\rangle,
\]

(2.3)

where \(\alpha_i's\) and \(\alpha'_i's\) may be chosen in all possible ways. Thus one finds that, the ‘anyon like’ representation (2.1) is in fact equivalent to the supersymmetric realisation (1.5).

Consequently, the first quantised spin Hamiltonian given by

\[
H_P^{(m|n)} = \sum_{1 \leq i < j \leq N} \left( \frac{1 - \tilde{P}^{(m|n)}_{ij}}{(x_i - x_j)^2} \right),
\]

(2.4)

would also be completely equivalent to the second quantised SP model (1.7). In particular, the Hamiltonians (1.7) and (2.4) would share the same spectrum and their eigenfunctions can be related through the correspondence (2.2).

Next, we define a spin Calogero model as

\[
H_C^{(m|n)} = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{l(1 - \tilde{P}^{(m|n)}_{ij})}{(x_i - x_j)^2},
\]

(2.5)
where $\tilde{P}_{ij}^{(m|n)}$ is the ‘anyon like’ representation (2.1). At the special case $m = M$, $n = 0$ ($m = 0$, $n = M$), the above Hamiltonian reproduces the $SU(M)$ spin Calogero model (1.3) with $\epsilon = 1$ ($\epsilon = -1$). Notice that the Hamiltonian (2.5) can be rewritten as

$$H_{C}^{(m|n)} = H_0 + lH_1^{(m|n)},$$

(2.6)

where $H_0$ is the Hamiltonian for spinless Calogero model:

$$H_0 = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{l(l-1)}{4} \frac{(x_i - x_j)^2}{(x_i - x_j)^2},$$

(2.7)

and $H_1^{(m|n)}$ is obtained from $H_P^{(m|n)}$ (2.4) by replacing $\tilde{x}_i$ and $\tilde{x}_j$ with $x_i$ and $x_j$ respectively. Thus, the operator $H_1^{(m|n)}$ possesses both spin and particle degrees of freedom.

However, in analogy with the case of usual spin Calogero model (1.3) [8], we may now consider the ‘freezing limit’ of extended spin Calogero model (2.5). Such ‘freezing limit’ is obtained by setting $l = \omega$ in the Hamiltonian (2.5) and finally taking its $\omega \to \infty$ limit. It is evident that, at this ‘freezing limit’, particle and spin degrees of freedom of Hamiltonian (2.5) would decouple and the operator $H_1^{(m|n)}$ would be transformed to a pure spin Hamiltonian where the fixed values of $x_i$s are determined through the minima of potential energy associated with spinless Calogero Hamiltonian (2.7). Since the solution of eqn.(1.2) leads to the minima of such potential energy, the operator $H_1^{(m|n)}$ will exactly reproduce the SP Hamiltonian (2.4) at the ‘freezing limit’. Consequently, the eigenfunctions of spin Calogero model (2.5) will factorise into eigenfunctions of spinless Calogero model (2.7) containing only particle degrees of freedom and $SU(m|n)$ SP model (2.4) containing only spin degrees of freedom. Moreover, at this ‘freezing limit’, the energy eigenvalues (denoted by $E_{p,s}(\omega)$, where the subscripts $p$ and $s$ represent the particle and spin degrees of freedom respectively) of the full system (2.5) can be expressed as

$$E_{p,s}(\omega) = E_p(\omega) + \omega E_s,$$

(2.8)

where $E_p(\omega)$ and $E_s$ denote the energy eigenvalues of spinless Calogero model (2.7) and SP spin chain (2.4) respectively.

It is clear from eqn.(2.8) that, by ‘modding out’ the spectrum of spin Calogero model (2.5) through the spectrum of spinless Calogero model (2.7), one can construct
the spectrum of SP model (2.4) or (1.7). So, for the purpose of obtaining the spectrum of SP model, it is essential to find out at first the spectrum of spin Calogero model (2.5). To this end, we consider the spinless Calogero model of distinguishable particles, which is described by a Hamiltonian of the form

\[ H_0 = \frac{1}{2} \sum_{i=1}^{N} \left( -\frac{\partial^2}{\partial x_i^2} + \omega^2 x_i^2 \right) + \sum_{1 \leq i < j \leq N} \frac{l(l - K_{ij})}{(x_i - x_j)^2}, \tag{2.9} \]

where \( K_{ij} \) is the coordinate exchange operator: \( K_{ij} \psi(\cdots,x_i,\cdots,x_j,\cdots) = \psi(\cdots,x_j,\cdots,x_i,\cdots). \) By restricting the action of the above Hamiltonian on completely symmetric wave functions, we may recover the spinless Calogero model (2.7) of indistinguishable particles. It has been recently found that, by using some similarity transformations, one can completely decouple all particle degrees of freedom of Calogero Hamiltonians (2.7) as well as (2.9) [29-33]. Such similarity transformations naturally lead to a very efficient method of calculating the eigenfunctions of spinless Calogero models. In particular, the spectrum and eigenfunctions of \( H_0 \) (2.9) can be obtained through a similarity transformation which maps this interacting Hamiltonian to a system of decoupled harmonic oscillators like

\[ H_{\text{free}} = \omega \sum_{k=1}^{N} a_k^\dagger a_k, \tag{2.10} \]

where \( a_k^\dagger = \frac{1}{\sqrt{2\omega}} \left( \omega x_k - \frac{\partial}{\partial x_k} \right) \) and \( a_k = \frac{1}{\sqrt{2\omega}} \left( \omega x_k + \frac{\partial}{\partial x_k} \right). \) The above mentioned similarity transformation is explicitly given by [32,33]

\[ S^{-1} (H_0 - E_g) S = H_{\text{free}}, \tag{2.11} \]

where \( E_g = \frac{1}{2} N\omega \left( Nl + (1 - l) \right) \) and

\[ S = \phi_g e^{-\frac{1}{4\omega}O_L} e^{\frac{1}{4\omega} \nabla^2} e^{\frac{1}{2\omega}X^2}, \tag{2.12} \]

with

\[ X^2 = \sum_{j=1}^{N} x_j^2, \quad \nabla^2 = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2}, \quad \phi_g = \prod_{1 \leq j < k \leq N} |x_j - x_k|^l \exp \left( -\frac{\omega}{2} \sum_{j=1}^{N} x_j^2 \right), \]

\[ O_L = \sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + l \sum_{j \neq k} \left\{ \frac{1}{(x_j - x_k)^2} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k} \right) + \frac{K_{jk} - 1}{(x_j - x_k)^2} \right\}. \]
The operator $O_L$ is called the Lassalle operator. As is well known, the eigenfunctions of decoupled oscillators (2.10) can be written in the form: $|n_1n_2\cdots n_N\rangle = \prod_{j=1}^{N} (a_j^\dagger)^{n_j} |0\rangle$, where $|0\rangle$ is the corresponding vacuum state. The coordinate representation of such eigenfunctions, having eigenvalues $\omega \sum_{j=1}^{N} n_j$, is given by

$$\psi_{n_1,n_2,\ldots,n_N}(x_1,x_2,\ldots,x_N) = e^{-\frac{1}{2} \omega x^2} \prod_{j=1}^{N} H_{n_j}(x_j), \quad (2.13)$$

where $H_{n_j}(x_j)$ is the Hermite polynomial of order $n_j$. It should be noted that, the above eigenfunctions do not generally obey any symmetry property under the exchange of coordinates and, therefore, they are free from any statistics. Due to the similarity transformation (2.11), the eigenfunctions of spinless Calogero model (2.9) may be obtained through the eigenfunctions (2.13) of free oscillators as

$$\chi_{n_1,n_2,\ldots,n_N}(x_1,x_2,\ldots,x_N) = S \left( e^{-\frac{1}{2} \omega x^2} \prod_{j=1}^{N} H_{n_j}(x_j) \right), \quad (2.14)$$

with $E_{\{n_i\}} = E_g + \omega \sum_{j=1}^{N} n_j$ representing the corresponding eigenvalues. So, apart from a constant energy shift ($E_g$) for all levels, the spectrum of spinless Calogero model (2.9) is exactly the same as the spectrum of decoupled Hamiltonian (2.10) containing distinguishable particles [32,33]. A comment might be in order. The exponentiation of the Lassalle operator $e^{-\frac{1}{2} \omega O_L}$ yields essential singularities at $x_i = x_j$, $i,j = 1,2,\ldots,N$, when it operates on general multivariable functions. However, such essential singularities are not generated when the exponentiation of the Lassalle operator operates on multivariable polynomials. Our using of the Lassalle operator in the expression (2.14) is such a safe one. Details on the property of the Lassalle operators are given in refs. 32, 33.

In the following, we want to show that the eigenfunctions of spin Calogero model (2.5) can also be obtained through the eigenfunctions of decoupled harmonic oscillators, provided they possess nondynamical spin degrees of freedom and obey some definite statistics. To this end, we introduce a projection operator $\Lambda_N^{(m|n)}$ which satisfies the relations like

$$K_{ij} P_{ij}^{(m|n)} \Lambda_N^{(m|n)} = \Lambda_N^{(m|n)} K_{ij} P_{ij}^{(m|n)} = \Lambda_N^{(m|n)}, \quad (2.15)$$
where \( i, j \in [1, 2, \ldots, N] \). Such a projector can be formally written through ‘transposition’ operator \( t_{ij}^{(m|n)} (= \tilde{P}_{ij}^{(m|n)} K_{ij}) \) as

\[
\Lambda_N^{(m|n)} = \sum_P \sum_{\{i_k, j_k\}} t_{i_1,j_1}^{(m|n)} t_{i_2,j_2}^{(m|n)} \cdots t_{i_p,j_p}^{(m|n)},
\]

(2.16)

where the series of transposition \( t_{i_1,j_1}^{(m|n)} t_{i_2,j_2}^{(m|n)} \cdots t_{i_p,j_p}^{(m|n)} \) represents an element of the permutation group \( (P_N) \) associated with \( N \) objects and, due to the summations on \( p \) and \( \{i_k, j_k\} \), each element of \( P_N \) would appear only once in the r.h.s. of the above equation. For example, \( \Lambda_2^{(m|n)} \) and \( \Lambda_3^{(m|n)} \) are given by:

\[
\Lambda_2^{(m|n)} = 1 + \tau_{12}^{(m|n)},
\]

\[
\Lambda_3^{(m|n)} = 1 + \tau_{12}^{(m|n)} + \tau_{23}^{(m|n)} + \tau_{13}^{(m|n)} + \tau_{12}^{(m|n)} \tau_{23}^{(m|n)} + \tau_{13}^{(m|n)} \tau_{23}^{(m|n)}.
\]

It should be noted that at the special case \( m = M, \ n = 0 \) (\( m = 0, \ n = M \)), the projector \( \Lambda_N^{(m|n)} \) (2.16) completely symmetrises (antisymmetrises) any wave function under simultaneous interchange of particle as well as spin degrees of freedom, and thus projects the wave function to bosonic (fermionic) subspace. Next, we multiply the eigenfunction (2.14) of Calogero Hamiltonian (2.9) by an arbitrary spin state \( |\alpha_1 \alpha_2 \cdots \alpha_N\rangle \) and subsequently apply the projector (2.16) for obtaining a wave function like

\[
\Psi_{\alpha_1, \alpha_2, \cdots, \alpha_N}^{n_1, n_2, \cdots, n_N} (x; \gamma) = \langle \gamma_1 \gamma_2 \cdots \gamma_N | \Lambda_N^{(m|n)} \left\{ \mathcal{S} \left( e^{-\frac{1}{2} \omega X^2} \prod_{j=1}^{N} H_{n_j} (x_j) \right) |\alpha_1 \alpha_2 \cdots \alpha_N\rangle \right\},
\]

(2.17)

where \( x \equiv x_1, x_2, \cdots, x_N \) and \( \gamma \equiv \gamma_1, \gamma_2, \cdots, \gamma_N \). Since the operator \( \mathcal{H}_0 (2.9) \) commutes with \( K_{ij} \) as well as \( \Lambda_N^{(m|n)} \) (2.16), the expression (2.17) again gives an eigenfunction of the Calogero Hamiltonian (2.9), where each particle possesses some nondynamical spin degrees of freedom. Consequently, with the help of relation (2.15) which allows one to ‘replace’ \( K_{ij} \) by \( \tilde{P}_{ij}^{(m|n)} \), one can show that \( \Psi_{\alpha_1, \alpha_2, \cdots, \alpha_N}^{n_1, n_2, \cdots, n_N} (x; \gamma) \) (2.17) also gives an eigenfunction of spin Calogero model (2.5) with eigenvalue \( E_{\{n_i\}, \{\alpha_i\}} = E_g + \omega \sum_{j=1}^{N} n_j \).

Thus we interestingly find that, eqn.(2.17) produces all eigenfunctions of spin Calogero model (2.5) through the known eigenfunctions of decoupled harmonic oscillators (2.10). However it is important to notice that, due to the existence of projector \( \Lambda_N^{(m|n)} \) in eqn.(2.17), a definite correlation is now imposed among the eigenfunctions of decoupled harmonic oscillators. To demonstrate this point in an explicit way, we
observe first that the operator $S$ (2.12) commutes with $K_{ij}$ and the projector $\Lambda^{(m|n)}_N$ (2.16). By using such commutation relations, eqn.(2.17) can be rewritten as

$$
\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma) = S \Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma), \tag{2.18}
$$

where

$$
\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma) = \langle \gamma_1 \gamma_2 \cdots \gamma_N | \Lambda^{(m|n)}_N \left\{ \left( e^{-\frac{1}{2} \omega X^2} \prod_{j=1}^N H_{n_j}(x_j) \right) | \alpha_1 \alpha_2 \cdots \alpha_N \right\} \right\rangle. \tag{2.19}
$$

This $\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma)$ represents a correlated eigenfunction (with eigenvalue $\omega \sum_{j=1}^N n_j$) for decoupled harmonic oscillators (2.10), where each oscillator carries $M$ number of nondynamical spin degrees of freedom. To determine the precise nature of the above mentioned correlation, we use the relation (2.15) and replace $\Lambda^{(m|n)}_N$ by $\Lambda^{(m|n)}_N K_{ij} \tilde{P}^{(m|n)}_j$ in the r.h.s. of eqn.(2.19). Finally, by acting $\tilde{P}^{(m|n)}_j$ on $| \alpha_1 \alpha_2 \cdots \alpha_N \rangle$ and $K_{ij}$ on $e^{-\frac{1}{2} \omega X^2} \prod_{j=1}^N H_{n_j}(x_j)$, we find that the eigenfunction (2.19) of free oscillators must satisfy the following symmetry condition under simultaneous interchange of related particle and spin quantum numbers:

$$
\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma) = e^{i \Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \ldots, \alpha_j)} \Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma), \tag{2.20}
$$

where $\Phi^{(m|n)}(\alpha_i, \alpha_{i+1}, \ldots, \alpha_j)$ being the same phase factor which appeared in eqn.(2.1). It is clear from eqn.(2.20) that, for the case $\alpha_i, \alpha_j \in [1, 2, \ldots, m]$, the eigenfunction $\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma)$ remains completely unchanged under simultaneous interchange of particle and spin quantum numbers. Thus $\alpha_i$ may be treated as a ‘bosonic’ quantum number, if it takes any value ranging from 1 to $m$. Next, by using eqn.(2.20) for the case $\alpha_i, \alpha_j \in [m+1, m+2, \ldots, m+n]$, it is easy to see that $\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma)$ would pick up a minus sign under simultaneous interchange of particle and spin quantum numbers. Therefore, the eigenfunction $\Psi^{a_1, a_2, \ldots, a_N}_{n_1, n_2, \ldots, n_N}(x; \gamma)$ must be trivial if we choose $n_i = n_j$ and $\alpha_i = \alpha_j \in [m+1, m+2, \ldots, m+n]$. Thus $\alpha_i$ may be treated as a ‘fermionic’ quantum number, if it takes any value ranging from $m+1$ to $m+n$. Notice that if we simultaneously interchange a ‘bosonic’ quantum number $\alpha_i$ with a ‘fermionic’ quantum number $\alpha_j$ and $n_i$ with $n_j$, then the eigenfunctions satisfying
transformation relation (2.20) would either remain invariant or pick up a minus sign depending on whether even or odd number of fermionic spin quantum numbers are present in the configuration: \( \alpha_{i+1}\alpha_{i+2}\cdots\alpha_{j-1} \). It is obvious that, at the special case \( m = M, n = 0 \) \( (m = 0, n = M) \), \( \tilde{\Psi}^{\alpha_1,\alpha_2,\cdots,\alpha_N}(x;\gamma) \) (2.19) represents completely symmetric (antisymmetric) eigenfunctions of \( SU(M) \) bosonic (fermionic) oscillators. So, for the case \( m, n \neq 0 \), we may say that the correlated state vectors (2.19) would represent the eigenfunctions of \( N \) number of \( SU(m|n) \) supersymmetric harmonic oscillators.

Due to the existence of symmetry condition (2.20), we find that all independent eigenfunctions of \( SU(m|n) \) supersymmetric harmonic oscillators can be obtained uniquely through the following occupation number representation. Since, at present, spins behave as nondynamical degrees of freedom, all ‘single particle states’ for this occupation number representation may be constructed by taking \( m+n \) copies of each energy eigenstate associated with a spinless harmonic oscillator – the first \( m \) copies being bosonic in nature and last \( n \) copies being fermionic in nature. As usual, any bosonic single particle state can be occupied with arbitrary number of particles and any fermionic single particle state can hold at most one particle. By filling up such bosonic and fermionic single particle states through \( N \) number of particles, we can easily identify all independent eigenfunctions of the form (2.19). So, this occupation number representation of states (2.19) gives us a very convenient way of analysing the spectrum of \( SU(m|n) \) supersymmetric harmonic oscillators. In particular we may verify that, similar to pure bosonic or fermionic case, the spectrum of \( SU(m|n) \) supersymmetric harmonic oscillators is also equally spaced. But, it is interesting to note further that, the ground state energy of \( SU(m|n) \) supersymmetric harmonic oscillators coincides with that of the pure bosonic case, instead of pure fermionic case. Since \( M \) number of fermionic single particle states with zero energy can hold at most \( M \) number of particles, a nonzero ground state energy is obtained for a pure fermionic system when \( N > M \). However, if at least one bosonic single particle state with zero energy is available, we can fill up that state through all available particles. Consequently we find that, irrespective of the values of \( m \) and \( n \), the ground state energy of \( SU(m|n) \) supersymmetric harmonic
oscillators would always be zero. Moreover, by using the above mentioned occupation number representation, we obtain the degeneracy ($D_g$) of ground state for $SU(m|n)$ supersymmetric harmonic oscillators as

$$D_g = \sum_{k=0}^{n} \frac{(N + m - k - 1)! n!}{(N - k)! (m - 1)! k! (n - k)!}.$$  \hspace{1cm} (2.21)

It is worth observing that this degeneracy factor crucially depends on the values of $m$ and $n$, and reproduces the degeneracy of the ground state for $SU(M)$ bosonic oscillators [8] at the special case $m = M, n = 0$. Similarly, one can demonstrate that the degeneracy of higher energy levels, appearing in the spectrum of $SU(m|n)$ supersymmetric harmonic oscillators, would also depend on the values of $m$ and $n$.

Due to the relation (2.18), where $S$ acts as a nonsingular operator, there exists a one-to-one correspondence between the independent eigenfunctions of spin Calogero model (2.5) and $SU(m|n)$ supersymmetric oscillators. Consequently, up to a constant energy shift ($E_g$) for all energy levels, the spectrum of spin Calogero model (2.5) exactly coincides with the spectrum of $SU(m|n)$ supersymmetric harmonic oscillators. However it is well known that, up to the same constant energy shift for all energy levels, the spectrum of spinless Calogero model (2.7) exactly coincides with the spectrum of $N$ number of spinless bosonic harmonic oscillators. As the eigenvalues of both $SU(m|n)$ supersymmetric oscillators and spinless bosonic oscillators depend linearly on the coupling constant $\omega$, it is clear that eqn.(2.8) can be used even for any finite value of $\omega$ (though the eigenfunctions of spin Calogero model (2.5) will factorise only at the ‘freezing limit’). Moreover, since the spectra of both $SU(m|n)$ supersymmetric oscillators and spinless bosonic oscillators are equally spaced, it automatically follows from eqn.(2.8) that the spectrum of $SU(m|n)$ SP model (1.7) would also be equally spaced for any choice of $m$ and $n$. But it is natural to expect that, similar to the case of $SU(m|n)$ supersymmetric oscillators, the degeneracy of energy levels for $SU(m|n)$ SP model (1.7) would crucially depend on the values of $m$ and $n$. So, it should be interesting to derive the partition function of SP model (1.7), where all information about these degeneracy factors is encoded.
3 Partition function of SU($m|n$) SP model

In the previous section we have observed that, similar to the pure bosonic case, the ground state energy of SU($m|n$) supersymmetric harmonic oscillators would always be zero. So, in analogy with this pure bosonic (i.e., ferromagnetic) case [8], we may now put $\omega = 1$ in eqn.(2.8) and subsequently use this equation to obtain a relation like

$$Z^{(m|n)}_{N}(q) = \hat{Z}^{(1|0)}_{N}(q)^{N}, \quad (3.1)$$

where $q = e^{-\frac{1}{kT}}$, $Z^{(m|n)}_{N}(q)$ and $\hat{Z}^{(m|n)}_{N}(q)$ denote the canonical partition functions of SU($m|n$) SP model (1.7) and SU($m|n$) supersymmetric harmonic oscillators (where $\omega = 1$) respectively. Due to the above mentioned notations, $\hat{Z}^{(1|0)}_{N}(q)$ and $\hat{Z}^{(0|1)}_{N}(q)$ denote the canonical partition functions of $N$ number of spinless bosonic and fermionic harmonic oscillators respectively. It is well known that, the partition functions of such spinless bosonic and fermionic oscillators are given by [8]

$$\hat{Z}^{(1|0)}_{N}(q) = \frac{1}{(q)_{N}}, \quad \hat{Z}^{(0|1)}_{N}(q) = q^{N(N-1)} \frac{1}{(q)_{N}}, \quad (3.2)$$

where the standard notation: $(q)_{N} = (1-q)(1-q^{2})\cdots(1-q^{N})$ (and $(q)_{0} = 1$) is used. So, for calculating $Z^{(m|n)}_{N}(q)$ with the help of eqn.(3.1), we have to find out only the partition function of $N$ number of SU($m|n$) supersymmetric oscillators.

To this end, however, we consider at first the grand canonical partition function of SU($m|n$) supersymmetric oscillators. Such grand canonical partition function may be denoted by $\hat{Z}^{(m|n)}(q,y)$, where $y = e^{-\mu}$ and $\mu$ is the chemical potential of the system. So, according to our notation, $\hat{Z}^{(1|0)}(q,y)$ and $\hat{Z}^{(0|1)}(q,y)$ denote the grand canonical partition functions of spinless bosonic and fermionic harmonic oscillators respectively. As usual, the grand canonical partition function of SU($m|n$) supersymmetric oscillators can be related to the corresponding partition function through the following power series expansion in variable $y$:

$$\hat{Z}^{(m|n)}(q,y) = \sum_{N=0}^{\infty} y^{N} \hat{Z}^{(m|n)}_{N}(q) \quad (3.3)$$
where it is assumed that $\hat{Z}_0^{(m|n)}(q) = 1$. In the previous section we have found that, all independent eigenfunctions of $SU(m|n)$ supersymmetric harmonic oscillators can be obtained through an occupation number representation, where the corresponding single particle states are constructed by taking $m + n$ copies of each energy eigenstate associated with a spinless harmonic oscillator — the first $m$ copies being bosonic in nature and last $n$ copies being fermionic in nature. By exploiting this result, it is easy to prove that the grand canonical partition function of $SU(m|n)$ supersymmetric oscillators can be expressed through those of spinless bosonic and fermionic oscillators as

$$\hat{Z}^{(m|n)}(q, y) = \left[ \hat{Z}^{(1|0)}(q, y) \right]^m \left[ \hat{Z}^{(0|1)}(q, y) \right]^n.$$  \hfill (3.4)

Next, we substitute the power series expansion (3.3) to the place of all grand canonical partition functions appearing in the above equation. Subsequently, we compare the coefficients of $y^N$ from both sides of eqn.(3.4), and readily find that the canonical partition function of $SU(m|n)$ supersymmetric harmonic oscillators may also be related to those of spinless bosonic and fermionic oscillators:

$$\hat{Z}^{(m|n)}_N(q) = \sum_{a_1 + \sum_{j=1}^m b_j = N}^m \prod_{i=1}^m \hat{Z}_{a_i}^{(1|0)}(q) \prod_{j=1}^n \hat{Z}_{b_j}^{(0|1)}(q),$$  \hfill (3.5)

where $a_i$s and $b_j$s are some nonnegative integers. By substituting the known partition functions (3.2) of spinless bosonic and fermionic oscillators to the above equation, one may now obtain an explicit expression for the partition function of $SU(m|n)$ supersymmetric oscillators as

$$\hat{Z}^{(m|n)}_N(q) = \sum_{a_1 + \sum_{j=1}^m b_j = N}^m \frac{q^{\sum_{j=1}^n b_j(b_j-1)/2}}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_m} \cdot (q)_{b_1} (q)_{b_2} \cdots (q)_{b_n}}.$$  \hfill (3.6)

Finally, by using eqns.(3.1), (3.2) and (3.6), we derive the exact canonical partition function of $SU(m|n)$ SP model (1.7) as

$$Z^{(m|n)}_N(q) = \sum_{a_1 + \sum_{j=1}^m b_j = N}^m \frac{(q)_N \cdot q^{\sum_{j=1}^n b_j(b_j-1)/2}}{(q)_{a_1} (q)_{a_2} \cdots (q)_{a_m} \cdot (q)_{b_1} (q)_{b_2} \cdots (q)_{b_n}}.$$  \hfill (3.7)
Since, there exists an upper bound on the highest energy eigenvalue of SP model (1.7), the partition function $Z_N^{(m|n)}(q)$ (3.7) evidently yields some new $q$-polynomials which are characterised by the values of $m$ and $n$. The coefficients of various powers of $q$, appearing in such novel $q$-polynomial, would represent the degeneracy factors of corresponding energy levels associated with SP model (1.7). It is interesting to note that, by putting $m = M$, $n = 0$ to the expression (3.7), one can exactly reproduce the partition function of ferromagnetic Polychronakos spin chain [8]. On the other hand, for the limiting case $m = 0$, $n = M$, eqn.(3.7) would reproduce the partition function of anti-ferromagnetic Polychronakos spin chain up to a constant (i.e., $q$ independent) multiplicative factor. This is because, in pure fermionic case, a system of harmonic oscillators with $M$ spin degrees of freedom gives some nonzero ground state energy when $N > M$. So, eqn.(3.1) must be modified through a constant multiplicative factor for taking into account such nonzero ground state energy [8]. Consequently the expression (3.7) for partition function, which we have derived for the supersymmetric case (i.e., when both $m$ and $n$ are nonzero), should also be modified by the same constant factor at the pure fermionic limit.

With the help of a relation: $(q^{-1})_l = (-1)^l q^{\frac{l(l+1)}{2}} (q)_l$, we find that the partition function (3.7) of SP model satisfies a remarkable duality condition given by

$$Z_N^{(m|n)}(q) = q^{\frac{N(N-1)}{2}} Z_N^{(n|m)}(q^{-1}) \quad \text{(3.8)}$$

where $m$ and $n$ may be chosen as any nonzero integer. Due to this duality condition, one can write down a relation of the form

$$D_N^{(m|n)}(E) = D_N^{(n|m)} \left( \frac{N(N-1)}{2} - E \right) \quad \text{(3.9)}$$

where $D_N^{(m|n)}(E)$ denotes the degeneracy factor associated with energy eigenvalue $E$ of $SU(m|n)$ SP model. Moreover, by using the duality condition (3.8), along with the fact that the ground state energy of Hamiltonian (1.7) is zero, it is easy to show that

$$E_{\text{max}} = \frac{N(N-1)}{2} \quad \text{(3.10)}$$
represents the highest energy eigenvalue of the $SU(m|n)$ SP model. In this context one may notice that, the highest energy eigenvalue of ferromagnetic or anti-ferromagnetic $SU(M)$ Polychronakos spin chain (1.1) is given by [8]

$$E_{\text{max}} = \frac{M - 1}{2M} N^2 - \frac{t(M - t)}{2M},$$

(3.11)

where $t = N \mod M$. Thus we curiously find that, in contrast to the case of $SU(M)$ Polychronakos spin chain (1.1), the highest energy eigenvalue of $SU(m|n)$ SP model (1.7) does not depend at all on the values of $m$ or $n$.

For obtaining some insight about the above mentioned difference between the highest energy eigenvalues (3.10) and (3.11), we finally consider the ‘motif’ picture, which was used to analyse the degeneracy of eigenfunctions for $SU(M)$ HS as well as Polychronakos spin chain through the corresponding symmetry algebra [11,13]. As is well known, motifs of these spin chains are made of binary digits like ‘0’ and ‘1’. Moreover, for a spin chain with $N$ number of lattice sites, these binary digits would form motifs of length $N - 1$. So we can write a motif as $(a_1 a_2 \cdots a_{N-1})$, where $a_i \in [0, 1]$ and each motif represents a class of degenerate eigenfunctions which yield an irreducible representation of $Y(gl_M)$ Yangian algebra [11,18]. For the case of $SU(M)$ Polychronakos spin chain, the energy eigenvalue corresponding to $(a_1 a_2 \cdots a_{N-1})$ motif is given by [13]

$$E_{(a_1 a_2 \cdots a_{N-1})} = \sum_{r=1}^{N-1} r a_r.$$

(3.12)

However, for the case of $SU(M)$ Polychronakos and HS spin chain, there exists a ‘selection rule’ which forbids the occurrence of $M$ number of consecutive ‘1’s in any motif. By combining this selection rule along with eqn.(3.12), one can derive the highest energy eigenvalue (3.11) of $SU(M)$ Polychronakos model and also understand why this eigenvalue crucially depends on the value of $M$. It is worth noting that, the above mentioned motif picture can also be used to analyse the spectrum of $SU(m|n)$ supersymmetric HS spin chain [18,28]. But, for this supersymmetric case, there exists no ‘selection rule’ and binary digits like ‘0’ and ‘1’ can be chosen freely for constructing a motif of length $N - 1$. Being motivated by the case of supersymmetric HS spin chain, we
may now conjecture that the motifs corresponding to SP model (1.7) are also free from any ‘selection rule’ and eqn.(3.12) again gives the corresponding energy eigenvalues. So, the spectrum of $SU(m|n)$ SP model contains $2^{N-1}$ number of motifs, which can be constructed by filling up $N - 1$ positions with ‘0’ or ‘1’ in all possible ways. By using eqn.(3.12), it is easy to check that the motif $(11 \cdots 1)$ yields the highest energy eigenvalue (3.10). Thus the absence of any ‘selection rule’, for the motifs corresponding to SP model, turns out to be the main reason behind the remarkably simple expression (3.10).

4 Concluding Remarks

In this paper we have investigated the spectrum as well as partition function of $SU(m|n)$ supersymmetric Polychronakos (SP) model (1.7) and the related spin Calogero model (2.5). The similarity transformation (2.11), which maps the spinless Calogero model of distinguishable particles to decoupled oscillators, and the projection operator (2.16) have played a key role in our derivation for the spectrum of spin Calogero model (2.5). Thus we have found that, up to a constant energy shift for all states, the spectrum of this spin Calogero model is exactly the same with the spectrum of decoupled $SU(m|n)$ supersymmetric harmonic oscillators. Furthermore, by using the occupation number representation associated with $SU(m|n)$ supersymmetric harmonic oscillators, we have obtained the exact partition function for this spin Calogero model.

It turned out that, the above mentioned spin Calogero model reproduces the SP model (1.7) at the ‘freezing limit’. Consequently, by factoring out the contributions due to dynamical degrees of freedom from the spectrum as well as partition function of spin Calogero model (2.5), one can compute the spectrum and partition function for $SU(m|n)$ SP model. By following this procedure we have found that, similar to the non-supersymmetric case, the spectrum of $SU(m|n)$ SP model is also equally spaced. However, the degeneracy factors of corresponding energy levels crucially depend on the
values of \( m \) and \( n \). As a result, we get some novel \( q \)-polynomials which represent the partition functions of SP models. Moreover, by interchanging the bosonic and fermionic degrees of freedom, we obtain a duality relation among the partition functions of SP models.

As a future study, it should be interesting to find out the Lax operators and conserved quantities for the \( SU(m|n) \) SP model (1.7). Moreover, in parallel to the case of \( SU(m|n) \) Haldane-Shastry model [18], one might be able to show that the \( SU(m|n) \) SP model also exhibits the \( Y(gl(m|n)) \) super-Yangian symmetry. Such super-Yangian symmetry of SP model may turn out to be very helpful in analysing the degeneracy patterns for its spectrum. However, information about these degeneracy patterns is also encoded in our expression of partition function (3.7). So, there should exist an intriguing connection between the partition function (3.7) and the motif representations for super-Yangian algebra. In particular, the partition function (3.7) might lead to a supersymmetric generalisation of well-known Rogers-Szegö (RS) polynomial. The recursion relation among these supersymmetric RS polynomials may then be used to find out the motifs representations and degenerate multiplets associated with super-Yangian algebra. We hope to report about such supersymmetric RS polynomials and related motifs in a forthcoming publication [34].

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