Uniqueness Theorem for Static Dilaton $U(1)^2$ Black Holes

Marek Rogatko
Technical University of Lublin
20-618 Lublin, Nadbystrzycka 40, Poland
rogat@tytan.umcs.lublin.pl
rogat@akropolis.pol.lublin.pl
(April 27, 1999)

Using a sigma model formulation of the field equations as on a two-dimensional manifold we provide the proof of a black hole uniqueness solutions in $N = 4, d = 4$ supergravity subject to certain boundary conditions. We considered the black hole solutions both in $SU(4)$ and $SO(4)$ versions of the underlying theory.

I. INTRODUCTION

During the last years there has been a considerable resurgence of mathematical studies of topics related to the black hole equilibrium states. Not only restricted to the pure vacuum Einstein or Einstein-Maxwell theory but also including nonlinear matter models, general sigma models or fields occuring in the low energy limits of the superstring theories.

In his pioneering investigations [1,2] Israel established the uniqueness of the Schwarzschild metric and its Reissner-Nordström generalization as static asymptotically flat solutions of the Einstein and Einstein-Maxwell vacuum field equations subject to the condition of regularity outside a well behaved ergosurface where the static Killing vector becomes null. In view of the subsequent demonstration [3,4] that in the static case (though not more generally) the ergosurface will be an event horizon, it follows that the Schwarzschild and Reissner-Nordström solutions are the only Einstein or Einstein-Maxwell (non-extreme) solutions that satisfy the condition of being static black hole metrics in the strict modern sense of the term.

Israel’s article [1] was the foundation of the next works of Müller zum Hagen et al. [5] and Robinson [6] establishing the generalization of Israel’s theorem of the uniqueness of the Schwarzschild black hole solution.

The uniqueness results for rotating configurations, i.e., for stationary and axisymmetric black hole spacetimes were achieved by Carter in [7] and completed by Hawking and Ellis [26] and the next works of Carter [4,9], Robinson [10] and Wald [11]. These works were devoted to the vacuum case. Bunting [12] and Mazur [13] generalize these results to the electromagnetic case. Bunting’s approach was based on using a general class of harmonic mapping between Riemannian manifolds. In Mazur’s proof the key point was the observation that the Ernst Eqs. described a nonlinear sigma model on a symmetric space. The review of the new methods presented by Bunting and Mazur was discussed in [14].

Bunting and Masood-ul-Alam [15] used the positive mass theorem [16,17] to prove the uniqueness, the spherical symmetry or the nonexistence for several static black hole solutions of the Einstein’s Eqs. In Ref. [15] the aforementioned technique was explored to prove the nonexistence of multiple black holes in the asymptotically Euclidean static vacuum spacetime. They found the conformal transformation which caused that the mass of the spatial part of
the metric was equal to zero, but the scalar curvature tensor was non-negative. Ruback [18] applied this technique to prove the uniqueness theorem for charged black holes in static Einstein-Maxwell spacetime. Masood-ul-Alam [19] gave an alternative, much simpler, a rigorous proof that the unique non-degenerate electrovac static black hole metrics are the Reissner-Nordström family. It was done without assuming the connectedness of the event horizon. The further generalization of the uniqueness proof for the static electrovac black holes including the case of a non-vanishing magnetic charge was proposed by Heusler [20]. He used a generalization of the conformal factor and established the nonexistence of multiple black hole solutions of Einstein-Maxwell system with electric and magnetic fields in a static, asymptotically flat spacetime. Heusler [21] demonstrated also the uniqueness of multiple black hole solutions for any self-coupled, stationary scalar mapping (sigma-model) with nonrotating horizon.

Using the fact that Einstein-Abelian gauge field Eqs. can be formulated as a sigma model on the adequate Kähler manifold, Gürses [22] found that \((n - 1)\) Abelian gauge charged Kerr black hole is a unique stationary black hole solution of Einstein-Abelian gauge field Eqs. In Ref. [23], he also proved that, the boundary value problems of some sigma models in a non-Riemannian background have unique solutions.

Quite recently a uniqueness theorem was extended to the case of the Ernst solution and C-metric [24].

Uniqueness theorems for black holes are closely related to the problem of staticity. Lichnerowicz [25] was the first who considered the idea of staticity for a stationary perfect fluid, locally static in the sense that its flow vector was connected with the Killing vector. The next extensions was attributed to Hawking [26] (vacuum case) and the Hawking’s strong rigidity theorem [8] emphasized that the event horizon of a stationary black hole had to be a Killing horizon. Recently, Sudarsky and Wald [27], by means of the notion of an asymptotically flat maximal slice with a compact interior, obtained the conditions that the solution of Einstein-Yang-Mills Eqs. is static when it had a vanishing Yang-Mills electric field on static hypersurfaces. They also reached to the conclusion that nonrotating Einstein-Maxwell black hole must be static when it has a vanishing magnetic field on static slices [28].

For a recent review concerning various aspects of uniqueness theorems for nonrotating and rotating black holes see [29], while the mathematical rigor of the afore mentioned problems has been studied in the review articles provided by Chruściel [30,31].

Recently, there has been an active period for constructing black hole solutions in the string theory which seems the most promising for a theory of quantum gravity (see [32] for a recent review of the subject).

The uniqueness of static, charged dilaton black hole solutions in the low energy string theory was certified by Masood-ul-Alam [33]. He found a conformal spatial metric which had the sufficient properties for existing suitable Dirac spinors. By means of the Lichnerowicz Eq. it was shown that these spinors are constant.

The alternative proof of a uniqueness of a static charged dilaton black hole was provided by Gürses and Sermutlu [34]. They used a sigma model formulation of equations of motion. The problem of black hole solutions and their uniqueness in axion-dilaton gravity was studied by Bowick et al. [35]. They managed to find uniqueness theorem in the case of the minimal coupling of axion field to gravity. Cambell et al. [36] showed the existence of axion hair for a Kerr black hole and calculated it explicitly in the case of a slow motion. They considered axion fields with a Lorentz Chern-Simons coupling to gravity. A dilaton coupling to axion fields strengths were considered in [37], where the authors calculated dilaton hair arising from the specific axion source.
Wells [38] wrote down the analogue of the Robinson’s identity for dilatonic black holes which allowed him to prove uniqueness theorem for a class of accelerating stringy black holes.

The problem of staticity theorems in Einstein-Maxwell axion-dilaton gravity was studied by Rogatko in Ref. [39], where the modified Carter arguments were used to find staticity conditions for fields and the metric. It was found, in Ref. [40], that static black hole solutions in the above theory had vanishing electric and axion-electric fields on static slices.

Our paper is organized as follows. In Sec.II we present the equations of motion for the bosonic sector of $N=4, d=4$ supergravity in a static axially symmetric spacetime. Introducing the adequate forms of the pseudopotentials and complex scalars enables us to write Eqs. of motion as two complex Ernst-like Eqs. Then, using the matrix formulation of Ernst Eqs., conceived by Gürses and Xanthopoulos [41], we reached the conclusion that two metrics satisfying the Eqs. of motion and having the same boundary conditions must be equal to each other in all points of the region of the two-dimensional manifold. Which implies in turns, that all black hole solutions in the theory under consideration subject to the same boundary conditions are the same everywhere in the spacetime. We considered the $SU(4)$ and $SO(4)$ versions of the underlying theory. We conclude in Sec.III with a brief summary of our researches and their implications.

II. UNIQUENESS OF BLACK HOLE SOLUTIONS

A. Doubly Charged Black Holes in $SU(4)$ version of $N=4, d=4$ supergravity

Superstring theories provide interesting generalizations of the Einstein-Maxwell theory in the so-called low energy limit. A dimensionally reduced superstring theory can be described in terms of $N=4, d=4$ supergravity. It turned out, that one can refer to the $SO(4)$ version [42] or $SU(4)$ one [43]. In our paper, we shall consider bosonic sectors of these theories, taking into account two $U(1)$ gauge fields and a dilaton field $\phi$, called $U(1)^2$ theories. We begin with the $SU(4)$ version of $N=4, d=4$ supergravity, which the action is of the form [42,44]

$$I_{SU(4)} = \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - e^{-2\phi} \left( F_{\alpha \beta} F^{\alpha \beta} + G_{\gamma \delta} G^{\gamma \delta} \right) \right],$$  

where the strengths of the gauge fields are descibed by $F_{\mu \nu} = 2\nabla_{[\mu} A_{\nu]}$ and $G_{\alpha \beta} = 2\nabla_{[\alpha} B_{\beta]}$. The resulting equations of motion, derived from the variational principle, are as follows:

$$R_{\mu \nu} = e^{-2\phi} \left( 2F_{\mu \rho} F_{\nu}^{\rho} - \frac{1}{2} g_{\mu \nu} F^2 \right) + e^{-2\phi} \left( 2G_{\mu \rho} G_{\nu}^{\rho} - \frac{1}{2} g_{\mu \nu} G^2 \right) + 2\nabla_{\rho} \phi \nabla_{\nu} \phi,$$  

$$\nabla_{\mu} \nabla^{\mu} \phi + \frac{1}{2} e^{-2\phi} F^2 + \frac{1}{2} e^{-2\phi} G^2 = 0,$$  

$$\nabla_{\mu} (e^{-2\phi} F^{\mu \nu}) = 0,$$  

$$\nabla_{\mu} (e^{-2\phi} G^{\mu \nu}) = 0.$$  

The black hole solutions in the theory under consideration were widely discussed by Kallosh et al. in Ref. [44].

Our main task will be to prove the uniqueness of the obtained results. We want to provide some continuity with the researches of Gürses [23,34], in some respects to generalize them, we shall present our analysis of the problem in a
form and notation similar to theirs. First, we shall formulate the corresponding Eqs. of motion as a two-dimensional sigma model and prove the uniqueness of the static solution under the same boundary conditions. In order to do so we introduce the static axially symmetric line element expressed as

$$ds^2 = -e^{2\psi} dt^2 + e^{-2\psi} \left[ e^{2\gamma} (dr^2 + dz^2) + r^2 d\phi^2 \right], \quad (6)$$

where $\psi$ and $\gamma$ depended only on $r$ and $z$ coordinates. The components of the $U(1)$ gauge strength tensors will be in the direction of time $F_{\mu\nu} = (A_0, 0, 0, 0)$ and in the azimuthal angle $G_{\alpha\beta} = (0, 0, 0, B_\phi)$. In our further considerations we assume that the components of the $U(1)$ gauge fields are functions of $r$ and $z$. Then, the resulting Eqs. of motion are as follows:

$$\nabla^2 \phi - e^{-2\psi-2\phi} \left( A_{0,r}^2 + A_{0,z}^2 \right) + \frac{e^{2\psi-2\phi}}{r^2} \left( B_{\phi,r}^2 + B_{\phi,z}^2 \right) = 0, \quad (7)$$

$$\nabla^2 \psi - e^{-2\psi-2\phi} \left( A_{0,r}^2 + A_{0,z}^2 \right) - \frac{e^{-2\psi-2\phi}}{r^2} \left( B_{\phi,r}^2 + B_{\phi,z}^2 \right) = 0, \quad (8)$$

$$\nabla^2 A_0 - 2 \left( \psi, r + \phi, r \right) A_{0,r} - 2 \left( \psi, z + \phi, z \right) A_{0,z} = 0, \quad (9)$$

$$\triangle B_\phi + 2 \left( \psi, r - \phi, r \right) B_{\phi,r} + 2 \left( \psi, z - \phi, z \right) B_{\phi,z} = 0, \quad (10)$$

$$e^{-2\psi-2\phi} \left( A_{0,r}^2 - A_{0,z}^2 \right) + \frac{1}{r^2} e^{2\psi-2\phi} \left( B_{\phi,r}^2 - B_{\phi,z}^2 \right) + \left( \phi, z - \phi, r \right) = \psi, r - \psi, z - \frac{\gamma, r}{r}, \quad (11)$$

$$\gamma, z - 2\psi, r, z = -2e^{-2\psi-2\phi} A_{0,r} A_{0,z} + \frac{2}{r^2} e^{2\psi-2\phi} B_{\phi,r} B_{\phi,z} + 2\phi, r, \phi, z, \quad (12)$$

where $\nabla^2$ is the Laplacian operator in the $(r, z)$ coordinates, namely, $\nabla^2 = \partial_r^2 + \partial_z^2 + \frac{1}{r} \partial_r$ and $\triangle = \partial_r^2 + \partial_z^2 - \frac{1}{r} \partial_r$. From Eq.(11) or (12) one can determine the function $\gamma$ if $\psi, \phi, A_0, B_\phi$ are known. Hence, the essential part of the Eqs. of motion consists of Eqs.(7-10).

Now, let us define the quantities

$$E = -\phi - \psi, \quad M = \psi - \phi, \quad (13)$$

and

$$\tilde{A}_0 = \frac{A_0}{\sqrt{2}}, \quad \tilde{B}_\phi = \frac{iB_\phi}{\sqrt{2}}, \quad (14)$$

Consistently with the above definitions, Eqs.(7-10) can be rewritten in the forms

$$\nabla^2 E + e^{2E} \nabla^2 \tilde{A}_0 \nabla \tilde{A}_0 = 0, \quad (15)$$

$$\nabla^2 \tilde{A}_0 + 2\nabla E \nabla \tilde{A}_0 = 0, \quad (16)$$

$$\nabla^2 M + e^{2M} \nabla \tilde{B}_\phi \nabla \tilde{B}_\phi = 0, \quad (17)$$

4
\[ \Delta \tilde{B}_\phi + 2 \nabla M \nabla \tilde{B}_\phi = 0. \] (18)

Thus, we obtain two pairs of Eqs., one described in terms of \( \tilde{A}_0, E \) and the other for \( \tilde{B}_\phi \) and \( M \).

We observe that Eq. (16) allows us to define a pseudopotential \( \omega_{(A)} \), given by

\[ \omega_{(A)}(r) = re^{2E} \tilde{A}_{0,z}, \quad \omega_{(A)}(z) = -re^{2E} \tilde{A}_{0,r}, \] (19)

while from Eq. (18), one can establish the following \( \omega_{(B)} \) pseudopotential:

\[ \omega_{(B)}(r) = -\frac{e^{2M} \tilde{B}_{\phi,z}}{r}, \quad \omega_{(B)}(z) = \frac{e^{2M} \tilde{B}_{\phi,r}}{r}. \] (20)

Then, we want to rewrite the Eqs. (15-18) in the forms similar to the Ernst ones. In order to do this, we introduce two complex scalars, determined by

\[ \epsilon_1 = re^{E} + i\omega_{(A)} \], \hspace{1cm} (21)

\[ \epsilon_2 = e^{M} + i\omega_{(B)} \]. \hspace{1cm} (22)

Now, Eqs. (15-16) and (17-18) can be arranged into the following two complex Eqs.:

\[ (\bar{\epsilon}_1 + \epsilon_1) \nabla^2 \epsilon_1 = 2 \nabla \epsilon_1 \nabla \epsilon_1, \] (23)

\[ (\bar{\epsilon}_2 + \epsilon_2) \nabla^2 \epsilon_2 = 2 \nabla \epsilon_2 \nabla \epsilon_2, \] (24)

where a bar denotes complex conjugation.

Eqs. (23) and (24) are two Ernst Eqs., which each of them combine in a convenient and a symmetric fashion the two Eqs. governing \( \phi, \psi \) and the adequate gauge field. Each of them defined a sigma model on \( SU(2)/U(1) \).

It was shown in Ref. [41], that the various combinations of Ernst’s Eqs. were included in the single matrix Eq. In our case, the matrix Eq. is determined by

\[ \partial_r \left[ P_{(i)}^{-1} \partial_r P_{(i)} \right] + \partial_z \left[ P_{(i)}^{-1} \partial_z P_{(i)} \right] = 0, \] (25)

where the subscript \((i)\) in \( P \) matrix refers respectively to \( \tilde{A}, \tilde{B} \) gauge fields. The explicit form of the matrices are given by

\[ P_{(A)} = \frac{1}{re^{E}} \left( \begin{array}{c} 1 \\ \omega_{(A)} \\ r^2 e^{2E} + \omega_{(A)}^2 \end{array} \right), \quad P_{(B)} = \frac{1}{e^{M}} \left( \begin{array}{c} 1 \\ \omega_{(B)} \\ e^{2M} + \omega_{(B)}^2 \end{array} \right). \] (26)

One can check that, the above matrix Eqs. when written out explicitly in terms of its elements constitute four Eqs., all of them are various combinations of Eqs. (23-24) and Eqs. the complex conjugate of them.

In order to prove a uniqueness theorem we shall follow the line described by Gürses [23]. To begin with, one should assume enough differentiability for the matrices components in a region \( D \) of the two-dimensional manifold \( M \) with boundary \( \partial D \). Let \( P_{(i)1} \) and \( P_{(i)2} \) will be two different solutions of Eqs. (25) respectively for the cases of \( \tilde{A} \) and \( \tilde{B} \) gauge fields, than the difference of their Eqs. will have be given by

\[ \nabla \left( P_{(i)1}^{-1} \left( \nabla Q_{(i)} \right) P_{(i)2} \right) = 0, \] (27)

5
where $Q_{(i)} = P_{(i)1}^{-1} P_{(i)2}^{-1}$. Multiplying Eq.(27) by $Q_{(i)}^\dagger$, one arrives at the expression

$$\nabla^2 q_{(i)} = \text{Tr} \left[ \left( \nabla Q_{(i)}^\dagger \right) P_{(i)1}^{-1} \left( \nabla Q_{(i)} \right) P_{(i)2} \right],$$  

(28)

where $q = \text{Tr} Q$. Taking into account hermiticity and positive definiteness of the matrices $P_{(i)1}$ and $P_{(i)2}$, we can postulate the form of the above matrices, satisfying

$$P_{(A)\alpha} = A_\alpha A^\dagger_\alpha, \quad P_{(B)\alpha} = B_\alpha B^\dagger_\alpha,$$

(29)

where $\alpha = 1, 2$. The explicit forms of the matrices $A_\alpha$ and $B_\alpha$ can be established as follows:

$$A_\alpha = \frac{1}{\sqrt{r e^{E_1/2}}} \begin{pmatrix} 1 & 0 \\ \omega_{(A)\alpha} & r e^{E_1} \end{pmatrix}, \quad B_\alpha = \frac{1}{e^{M_2/2}} \begin{pmatrix} 1 & 0 \\ \omega_{(B)\alpha} & e^{M_2} \end{pmatrix}.$$  

(30)

Using relation (29) one can rewrite Eq.(28) in the form

$$\nabla^2 q_{(i)} = \text{Tr} \left( J_{(i)}^\dagger J_{(i)} \right),$$

(31)

where $J_{(A)} = A_1^{-1}(\nabla Q_{(A)}) A_2$ and $J_{(B)} = B_2^{-1}(\nabla Q_{(B)}) B_2$. Thus, the explicit versions of $q_{(i)}$ are given by

$$q_{(A)} = 2 + \frac{1}{r^2 e^{E_1+E_2}} \left[ (\omega_{(A)1} - \omega_{(A)2})^2 + r^2 (e^{E_1} - e^{E_2})^2 \right],$$

(32)

$$q_{(B)} = 2 + \frac{1}{e^{M_1+M_2}} \left[ (\omega_{(B)1} - \omega_{(B)2})^2 + (e^{M_1} - e^{M_2})^2 \right].$$

(33)

It is evident that on the boundary $q_{(A)} = q_{(B)} = 2$ and their first derivatives disappear there. Taking into account the boundary conditions on $\partial D$, one can integrate Eq.(31) to obtain the relation

$$\int_{\partial D} \text{Tr} \left( J_{(i)}^\dagger J_{(i)} \right) = 0.$$  

(34)

The expression (34) implies vanishing of $J_{(A)}$ and $J_{(B)}$, which in turns causes that $Q_{(i)} = \text{const}$ in all region $D$. Because of this fact, $Q_{(i)} = I$ matrix on $\partial D$, then $Q_{(i)} = I$ in $D$. Therefore we reach to the conclusion that, $P_{(i)1} = P_{(i)2}$ at all points of the region $D$ of the two-dimensional manifold $\mathcal{M}$.

As was mentioned in Ref. [34], the other way of reaching these conclusions is to observe that, vanishing of (34) implies the harmonicity of $q$ in $D$ region. Since $q_{(i)} = 2$ on the boundary $\partial D$, then it must be equal to the same constant value in the region $D$. Thus, $P_{(i)1} = P_{(i)2}$ in $D$.

The above considerations enables us to formulate the main result, the following.

**Theorem:** Consider a two-dimensional manifold $\mathcal{M}$ equipped with a local coordinates $(r, z)$. Suppose that, $D$ is a region in $\mathcal{M}$ with boundary $\partial D$. Let $P_{(i)}$ be hermitian, positive definite two-dimensional matrices with unit determinants, respectively for $i = \hat{A}, \hat{B}$ gauge fields. Suppose further that, matrices $P_{(i)1}$ and $P_{(i)2}$ satisfy Eq.(27), namely

$$\nabla \left( P_{(i)1}^{-1} (\nabla Q_{(i)}) P_{(i)2} \right) = 0,$$

(35)

in the region $D$ and have the same boundary conditions on $\partial D$. Then, in all points of the region $D$, one has that $P_{(i)1} = P_{(i)2}$.

The doubly charged dilaton black holes in $SU(4)$ version of $N = 4, d = 4$ supergravity are characterized by mass $M$, the $F$-field electric charge, the $G$-field magnetic charge and the dilaton charge $\Sigma$. The above theorem envisages that all black holes subject to the same boundary conditions, as the solution obtained by Kallosh et al. in Ref. [44], are the same everywhere in the spacetime.
B. Doubly Charged Black Holes in $SO(4)$ version of $N = 4, d = 4$ supergravity

The bosonic part of the $SO(4)$ version of $N = 4$ supergravity in four dimensions can be described by the action [42,44]

$$I_{SO(4)} = \int d^4x \sqrt{-g} \left[ R - 2(\nabla \phi)^2 - \left( e^{-2\phi} F_{\alpha \beta} F^{\alpha \beta} + e^{2\phi} \tilde{G}_{\gamma \delta} \tilde{G}^{\gamma \delta} \right) \right],$$

(36)

where the strengths of the gauge fields are described by $F_{\mu \nu} = 2\nabla_{[\mu} A_{\nu]}$ and $\tilde{G}_{\alpha \beta} = 2\nabla_{[\alpha} G_{\beta]}$. The equations derived from the variational principle are as follows:

$$R_{\mu \nu} = e^{-2\phi} \left( 2F_{\mu \nu} F^\rho - \frac{1}{2} g_{\mu \nu} F^2 \right) + e^{2\phi} \left( 2\tilde{G}_{\mu \nu} \tilde{G}^\rho - \frac{1}{2} g_{\mu \nu} \tilde{G}^2 \right) + 2\nabla_{\mu} \phi \nabla_{\nu} \phi,$$

(37)

$$\nabla_{\mu} \nabla^\mu \phi + \frac{1}{2} e^{-2\phi} F^2 - \frac{1}{2} e^{2\phi} \tilde{G}^2 = 0,$$

(38)

$$\nabla_{\mu} \left( e^{-2\phi} F^{\mu \nu} \right) = 0,$$

(39)

$$\nabla_{\mu} \left( e^{2\phi} \tilde{G}^{\mu \nu} \right) = 0.$$

(40)

The components of the $U(1)$ gauge fields are in the time direction [44], namely $F_{\mu \nu} = (A_0, 0, 0, 0)$ and $\tilde{G}_{\mu \nu} = (G_0, 0, 0, 0)$. As in the preceding paragraph the components of the $U(1)$ gauge fields are functions depending only on $(r, z)$ coordinates. The Eqs. of motion in metric (6) satisfy

$$\nabla^2 \phi - e^{-2\psi - 2\phi} \left( A^2_{0,r} + A^2_{0,z} \right) + \frac{e^{-2\psi + 2\phi}}{r^2} \left( G^2_{0,r} + G^2_{0,z} \right) = 0,$$

(41)

$$\nabla^2 \psi - e^{-2\psi - 2\phi} \left( A^2_{0,r} + A^2_{0,z} \right) - e^{-2\psi + 2\phi} \left( G^2_{0,r} + G^2_{0,z} \right) = 0,$$

(42)

$$\nabla^2 A_0 - 2 \left( \psi_r + \phi_r \right) A_{0,r} - 2 \left( \psi_z + \phi_z \right) A_{0,z} = 0,$$

(43)

$$\nabla^2 G_0 + 2 \left( -\psi_r + \phi_r \right) G_{0,r} + 2 \left( -\psi_z + \phi_z \right) G_{0,z} = 0,$$

(44)

$$e^{-2\psi - 2\phi} \left( A^2_{0,r} - A^2_{0,z} \right) + e^{-2\psi + 2\phi} \left( G^2_{0,r} - G^2_{0,z} \right) + \left( \phi^2_{r} - \phi^2_{z} \right) = \psi^2_{r} - \psi^2_{z} - \frac{\gamma^r}{r},$$

(45)

$$\frac{\gamma}{r} = 2\psi_r \psi_z = -2 e^{-2\psi - 2\phi} A_{0,r} A_{0,z} - 2 e^{-2\psi + 2\phi} G_{0,r} G_{0,z} + 2 \phi_r \phi_z.$$

(46)

Thus, with the substitution

$$E = -\phi - \psi, \quad N = \phi - \psi,$$

(47)

and

$$\tilde{A}_0 = \frac{A_0}{\sqrt{2}}, \quad \tilde{G}_0 = \frac{G_0}{\sqrt{2}},$$

(48)

we find that, Eqs. of motion can be rewritten as follows:
\[ \nabla^2 E + e^{2E} \nabla \tilde{A}_0 \nabla \tilde{A}_0 = 0, \quad (49) \]
\[ \nabla^2 \tilde{A}_0 + 2 \nabla E \nabla \tilde{A}_0 = 0, \quad (50) \]
\[ \nabla^2 N + e^{2N} \nabla \tilde{G}_0 \nabla \tilde{G}_0 = 0, \quad (51) \]
\[ \nabla^2 \tilde{G}_0 + 2 \nabla N \nabla \tilde{G}_0 = 0. \quad (52) \]

Now, the pseudopotentials for gauge fields \( \tilde{A} \) and \( \tilde{G} \) have the same form as in Eq. (19), where one should substitute the adequate values of \( E, (N) \) and derivatives of the gauge fields under consideration. Then, using Eq. (21) to introduce the complex scalars, enables to rewrite the system of Eqs. (49–52) as the decoupled two Ernst’s like Eqs. Then, the proof follows the same line as that of the preceding subsection. Finally one can reach the conclusion, that all points of the region \( D \) equipped with the boundary \( \partial D \) on two-dimensional manifold \( M \), \( P_{(1)1} = P_{(1)2} \).

In view of the foregoing uniqueness theorem, we conclude that all doubly charged black hole solutions characterized by mass \( M \), the \( F \)-field electric charge, the \( G \)-field electric charge and the dilaton charge \( \Sigma \), with the same boundary conditions as found in Ref. [44] are the same everywhere in the spacetime.

### III. CONCLUSIONS

In our work we were studying the doubly charged dilaton black holes in the bosonic sector of \( N = 4, d = 4 \) supergravity, being the low energy limit of the superstring theories. We were interested in the uniqueness of these solutions. Using the method proposed in Ref. [23] and finding the adequate forms of pseudopotentials and complex scalars for \( SU(4) \) and \( SO(4) \) versions of the theory, we were able to find that Eqs. of motion could be arranged in the two Ernst’s Eqs. for each of the gauge fields appearing in the theory. These Eqs. give the Ernst’s formulation of the generalized Einstein Eqs. for the bosonic sector of \( N = 4, d = 4 \) supergravity. Then, using the idea of G"urses and Xantopoulous [41] that Ernst Eqs. are included in the single matrix Eq., we prove the uniqueness of the previously [44] obtained black hole solutions.

It will be interesting to find the exact form of the Ernst Eqs. for more complicated version of the low energy string theory, for instance Einstein-Maxwell axion-dilaton gravity or \( N = 8 \) black holes now intensively studied [45]. We hope to return to these problems elsewhere.

---
