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An elegant and fast method to solve QCD evolution equations, application to the determination of the gluon content of the Pomeron

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Abstract

We present an original implementation of a very elegant method to solve Q^2 evolution equations of perturbative QCD at NLO : the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations. Based on a seminal idea of Furmanski and Petronzio, our numerical treatment consists in expanding parton distributions and splitting functions on Laguerre polynomials, which reduces DGLAP integro-differential equations to a set of ordinary differential equations defined by recurrence. Then, we apply our evolution method to determine the gluon content of the Pomeron by realizing QCD fits on diffractive structure function.

1 Introduction

DGLAP equations are fundamental equations of perturbative QCD [1], they quantify the size of scaling violations in DIS and, due to factorization theorem at high energy, they are used extensively for the determinations of parton densities in hadrons.

Of course, to achieve these determinations, the first step is to get a high level of accuracy in numerical treatment of DGLAP equations.

Different methods can be used : one can solve DGLAP equations by iterating the evolution over infinitesimal steps in the fractional momentum x [2], another technique is based on the use of Mellin moments and their inversion [3]. The accuracy of the above methods can be very good, however, the difficult point for them is the time consuming which is an important numerical limitation.

That is why we have developed a polynomial method formulated some time ago by Furmanski and Petronzio [4]. As we show in the following, the idea is to use Laguerre expansions of parton distributions and DGLAP kernels.

We have extended this method to NLO in the non-singlet and singlet sectors of DGLAP equations. This enables to get a very elegant and fast solution of these evolution equations starting from arbitrary initial parton distributions.

Then, we exemplify our treatment on the determination of the gluon content of the Pomeron by fitting diffractive structure function $F_2^{D(3)}$.

2 Laguerre method to solve DGLAP equations

2.1 Laguerre polynomials

Laguerre polynomials are defined by a recurrence :

$$\begin{aligned}(n+1)L_{n+1}(x) &= (2n+1-x)L_n(x) - nL_{n-1}(x) \\ L_0(x) &= 1 \\ L_1(x) &= 1-x\end{aligned}\tag{1}$$

with :

$$\int_0^\infty dx' e^{-x'} L_n(x') L_m(x') = \delta_{n,m}\tag{2}$$

Any integrable function f , with $\int_0^\infty dx' e^{-x'} L_n(x') f(e^{-x'}) < \infty$ for $n \in \mathbb{N}$ can be developed such that :

$$f(e^{-x'}) = \sum_{n=0}^\infty f(n) L_n(x')\tag{3}$$

$$f(n) = \int_0^\infty dx' e^{-x'} L_n(x') f(e^{-x'})\tag{4}$$

Moreover, Laguerre polynomials verify two important properties :

$$\int_0^{x'} dy' L_n(x'-y') L_m(y') = L_{n+m}(x') - L_{n+m+1}(x')\tag{5}$$

$$L_{n \gg 1}(x) = \pi^{-\frac{1}{2}} e^{\frac{x}{2}} x^{-\frac{1}{4}} n^{-\frac{1}{4}} \cos \left[2(nx)^{\frac{1}{2}} - \frac{\pi}{4} \right] + O(n^{-\frac{3}{4}})\tag{6}$$

The relation (5) justifies the choice of this particular set of orthogonal polynomials for Laguerre expansions of DGLAP equations as we show in the following and the identity (6) justifies that we can expand integrable functions till a finite order ($N \simeq 30$) :

$$f(e^{-x'}) = \sum_{n=0}^N f_n L_n(x') \quad (7)$$

2.2 DGLAP equations in perturbative QCD

With the notation $\tilde{f}(z) = zf(z)$, DGLAP equations can be written (N_f is the number of flavors) [1] :

$$\begin{aligned} Q^2 \frac{\partial \tilde{q}_{NS}^\pm(x, Q^2)}{\partial Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \tilde{q}_{NS}^\pm\left(\frac{x}{z}, Q^2\right) \tilde{P}_{NS}^\pm(z) \\ Q^2 \frac{\partial \tilde{q}_S(x, Q^2)}{\partial Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[\tilde{q}_S\left(\frac{x}{z}, Q^2\right) \tilde{P}_S(z) + 2N_f \tilde{g}\left(\frac{x}{z}, Q^2\right) \tilde{P}_{qg}(z) \right] \\ Q^2 \frac{\partial \tilde{g}(x, Q^2)}{\partial Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \int_x^1 \frac{dz}{z} \left[\tilde{q}_S\left(\frac{x}{z}, Q^2\right) \tilde{P}_{gq}(z) + \tilde{g}\left(\frac{x}{z}, Q^2\right) \tilde{P}_{gg}(z) \right] \end{aligned}$$

where the NLO splitting functions can be expressed as [5] :

$$P_{ij}(z) = P_{ij}^{(0)}(z) + \frac{\alpha_s(Q^2)}{2\pi} P_{ij}^{(1)}(z)$$

q_S designed the singlet distribution : $q_S = \sum_{i=1}^{N_f} q_i + \bar{q}_i$ and $q_{NS}^\pm$ referred to a non-singlet distribution : $q_{NS}^\pm = \sum_j a_j (q_j \pm \bar{q}_j)$ where a_j are constants.

Also, $\alpha_s(Q^2)$ is given by the Renormalization Group Equation [6] :

$$Q^2 \frac{d\alpha_s(Q^2)}{dQ^2} = -\frac{\beta_0}{4\pi} \alpha_s^2 - \frac{\beta_1}{(4\pi)^2} \alpha_s^3 \quad (8)$$

In the renormalization scheme $\overline{MS}$, β_0 and β_1 are given by :

$$\beta_0 = 11 - \frac{2}{3}N_f \quad , \quad \beta_1 = 102 - \frac{38}{3}N_f$$

In our implementation, we take $\alpha_s(M_Z^2) = 0.113$ and we calculate $\alpha_s(Q^2)$ by solving numerically the differential equation (8).

2.3 The Laguerre method

We explain the method on the non-singlet sector of DGLAP equations.

After the variable transformation : $x = e^{-x'}$, $z = e^{-z'}$, the non-singlet evolution equation can be written :

$$Q^2 \frac{\partial \tilde{q}_{NS}^\pm(e^{-x'}, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_0^{x'} dz' \tilde{q}_{NS}^\pm(e^{-(x'-z')}, Q^2) \tilde{P}_{NS}^\pm(e^{-z'})$$

Then, we expand each term of this equation on Laguerre polynomials according to relation (3), we get :

$$Q^2 \frac{\partial \sum_n \tilde{q}_{NS}^\pm(n, Q^2) L_n(x')}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \int_0^{x'} dz' \sum_{n_1} \tilde{q}_{NS}^\pm(n_1, Q^2) L_{n_1}(x'-z') \sum_{n_2} \tilde{P}_{NS}^\pm(n_2) L_{n_2}(z')$$

where :

$$\tilde{P}_{NS}^\pm(n_{1,2}) = \tilde{P}_{NS}^{(0)\pm}(n_{1,2}) + \frac{\alpha_s(Q^2)}{2\pi} \tilde{P}_{NS}^{(1)\pm}(n_{1,2})$$

$\tilde{P}_{NS}^{(0)\pm}(n_{1,2})$ and $\tilde{P}_{NS}^{(1)\pm}(n_{1,2})$ can be calculated numerically for all $n_{1,2} = 0, 1, 2, \dots$ from splitting functions $\tilde{P}_{NS}^{(0)\pm}(z)$ and $\tilde{P}_{NS}^{(1)\pm}(z)$ (see relation (4)).

Here, we see that the property (5) is very useful. With the help of this relation the above equation can be rewritten :

$$Q^2 \frac{\partial \sum_n \tilde{q}_{NS}^\pm(n, Q^2) L_n(x')}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \sum_{n_1, n_2} \left[\tilde{q}_{NS}^\pm(n_1, Q^2) \tilde{P}_{NS}^\pm(n_2) (L_{n_1+n_2}(x') - L_{n_1+n_2+1}(x')) \right]$$

We can identify each order of this equation by using the orthogonality of Laguerre polynomials (2) :

$$Q^2 \frac{\partial \tilde{q}_{NS}^\pm(n, Q^2)}{\partial Q^2} = \frac{\alpha_s(Q^2)}{2\pi} \sum_{m=0}^n \left[\tilde{q}_{NS}^\pm(m, Q^2) \left(\tilde{P}_{NS}^\pm(n-m) - \tilde{P}_{NS}^\pm(n-m-1) \right) \right] \quad (9)$$

Equation (9) is a recursion relation which defines a set ($n = 0, 1, 2, \dots$) of ordinary Q^2 differential equations on $\tilde{q}_{NS}^\pm(n, Q^2)$.

Thus, we can evaluate the non-singlet distribution by solving this recursion relation :

$$\tilde{q}_{NS}^\pm(e^{-x'}, Q^2) = \sum_{n=0}^{\infty} \tilde{q}_{NS}^\pm(n, Q^2) L_n(x') \quad (10)$$

which gives :

$$\tilde{q}_{NS}^{\pm}(x, Q^2) = \sum_{n=0}^{\infty} \tilde{q}_{NS}^{\pm}(n, Q^2) L_n(\ln \frac{1}{x}) \quad (11)$$

With a similar treatment, singlet DGLAP equations can be expanded on Laguerre polynomials, we get :

$$\begin{aligned} Q^2 \frac{\partial \tilde{q}_S(n, Q^2)}{\partial Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \sum_{m=0}^n \left[\tilde{q}_S(m, Q^2) \left(\tilde{P}_S(n-m) - \tilde{P}_S(n-m-1) \right) \right. \\ &\quad \left. + 2N_f \tilde{g}(m, Q^2) \left(\tilde{P}_{qg}(n-m) - \tilde{P}_{qg}(n-m-1) \right) \right] \\ Q^2 \frac{\partial \tilde{g}(n, Q^2)}{\partial Q^2} &= \frac{\alpha_s(Q^2)}{2\pi} \sum_{m=0}^n \left[\tilde{q}_S(m, Q^2) \left(\tilde{P}_{gq}(n-m) - \tilde{P}_{gq}(n-m-1) \right) \right. \\ &\quad \left. + \tilde{g}(m, Q^2) \left(\tilde{P}_{gg}(n-m) - \tilde{P}_{gg}(n-m-1) \right) \right] \end{aligned} \quad (12)$$

Then :

$$\tilde{q}_S(x, Q^2) = \sum_{n=0}^{\infty} \tilde{q}_S(n, Q^2) L_n(\ln \frac{1}{x}) \quad (13)$$

$$\tilde{g}(x, Q^2) = \sum_{n=0}^{\infty} \tilde{g}(n, Q^2) L_n(\ln \frac{1}{x}) \quad (14)$$

2.4 Conclusions

We have shown that, after expansions on Laguerre polynomials, DGLAP equations can be reduced to a set of ordinary differential equations defined by recursion relation (9) and (12) for non-singlet and singlet sectors respectively. Thus, for all $n = 0, 1, 2, \dots$, $\tilde{f}(n, Q^2)$ whatever Q^2 ($f \equiv q_S, q_{NS}^{\pm}, g$) can be deduced from initial values $\tilde{f}(n, Q_0^2)$.

Then, we have deduced distributions with respect to fractional momentum x :

$$\tilde{f}(x, Q^2) = \sum_{n=0}^{\infty} \tilde{f}(n, Q^2) L_n(\ln \frac{1}{x})$$

In fact, as we have explained above, we can limit this sum to a finite order ($N \simeq 30$) :

$$\tilde{f}(x, Q^2) = \sum_{n=0}^{30} \tilde{f}(n, Q^2) L_n(\ln \frac{1}{x})$$

Also, we can deduce structure functions from parton densities (we keep the notation $\tilde{f}(z) = z f(z)$). In fractional momentum space, structure functions

are given by convolutions of Wilson coefficients with parton densities [5] :

$$F_{NS,(i)}^{\pm}(x, Q^2) = \int_x^1 \frac{dz}{z} \tilde{C}_i^q(z, \alpha_s) \tilde{q}_{NS}^{\pm}\left(\frac{x}{z}, Q^2\right)$$

$$F_{S,(i)}(x, Q^2) = \int_x^1 \frac{dz}{z} \tilde{C}_i^q(z, \alpha_s) \tilde{q}_S\left(\frac{x}{z}, Q^2\right) + \int_x^1 dz N_f \tilde{C}_i^g(z, \alpha_s) \tilde{g}\left(\frac{x}{z}, Q^2\right)$$

which gives after expansions on Laguerre polynomials :

$$F_{NS,(i)}^{\pm}(n, Q^2) = \sum_{m=0}^n \left[\tilde{q}_{NS}^{\pm}(m, Q^2) \left(\tilde{C}_i^q(n-m) - \tilde{C}_i^q(n-m-1) \right) \right]$$

$$F_{S,(i)}(n, Q^2) = \sum_{m=0}^n \left[\tilde{q}_S(m, Q^2) \left(\tilde{C}_i^q(n-m) - \tilde{C}_i^q(n-m-1) \right) \right. \\ \left. + N_f \tilde{g}(m, Q^2) \left(\tilde{C}_i^g(n-m) - \tilde{C}_i^g(n-m-1) \right) \right] \quad (15)$$

In the following, we exemplify this method to solve DGLAP equations on the determination of the gluon content of the pomeron by fitting diffractive structure function.

3 Gluon content of the Pomeron

3.1 Introduction

From Ingelman-Schlein model [7], the diffractive structure function $F_2^{D(3)}$, measured from DIS events with large rapidity gaps, can be investigated in the framework of Regge phenomenology and $F_2^{D(3)}$ can be factored into a pomeron flux ($f_{\mathbb{P}/p}$) and pomeron structure function ($F_2^{\mathbb{P}}$) :

$$F_2^{D(3)}(Q^2, \beta, x_{\mathbb{P}}) = f_{\mathbb{P}/p}(x_{\mathbb{P}}) F_2^{\mathbb{P}}(Q^2, \beta) \quad (16)$$

The pomeron flux is taken to follow a Regge behaviour with a linear trajectory $\alpha_{\mathbb{P}}(t) = \alpha_{\mathbb{P}}(0) + \alpha'_{\mathbb{P}} t$, such that :

$$f_{\mathbb{P}/p}(x_{\mathbb{P}}) = \int_{t_{cut}}^{t_{min}} \frac{e^{B_{\mathbb{P}} t}}{x_{\mathbb{P}}^{2\alpha_{\mathbb{P}}(t)-1}} dt \quad (17)$$

where $|t_{min}|$ is the minimum kinematically allowed value of $|t|$ and $t_{cut} = -1 \text{ GeV}^2$ is the limit of the measurement.

In fact, it has been shown that a leading ($\mathbb{P}$) and a sub-leading meson ($\mathbb{R}$) trajectory must be considered to describe the breaking of factorization (violation of equation (16)) shown by $F_2^{D(3)}$ experimental values [8] :

$$F_2^{D(3)}(Q^2, \beta, x_{\mathbb{P}}) = f_{\mathbb{P}/\mathbb{P}}(x_{\mathbb{P}})F_2^{\mathbb{P}}(Q^2, \beta) + f_{\mathbb{R}/\mathbb{P}}(x_{\mathbb{P}})F_2^{\mathbb{R}}(Q^2, \beta) \quad (18)$$

Moreover, it has been suggested that the Q^2 evolution of these structure functions may be understood in terms of parton dynamics and therefore perturbative QCD where parton densities are evolved according to DGLAP equations [7].

In the next sections, we test this idea with our method to solve DGLAP equations. In fact, it has already been tested in reference [8] with an iterating method over infinitesimal steps in x space to solve DGLAP equations. A leading gluon content was found carrying about 90% of the momentum fraction at low Q^2 : we will see in the following if these conclusions can be verified with our very efficient method.

3.2 Parton densities of the Pomeron

We assign parton distribution functions to the pomeron and to the meson. A simple prescription is adopted in which the parton distributions of both the pomeron and the meson are parameterised in terms of non-perturbative input distributions at some low scale $Q_0^2 = 3 \text{ GeV}^2$.

The pion structure function [9] is assumed for the sub-leading meson trajectory with a free global normalisation to be determined by the data.

For the pomeron, a quark flavour singlet distribution ($zS_q(z, Q^2) = u + \bar{u} + d + \bar{d} + s + \bar{s} + c + \bar{c}$) and a gluon distribution ($zG(z, Q^2)$) are parameterized in terms of coefficients $C_j^{(S)}$ and $C_j^{(G)}$ at $Q_0^2 = 3 \text{ GeV}^2$ such that :

$$zS(z, Q^2 = Q_0^2) = \left[\sum_{j=1}^n C_j^{(S)} \cdot P_j(2z-1) \right]^2 \cdot e^{\frac{a}{z-1}} \quad (19)$$

$$zG(z, Q^2 = Q_0^2) = \left[\sum_{j=1}^n C_j^{(G)} \cdot P_j(2z-1) \right]^2 \cdot e^{\frac{a}{z-1}} \quad (20)$$

where $z = x_{i/\mathbb{P}}$ is the fractional momentum of the pomeron carried by the struck parton. If the photon couples directly to a quark ($i = q$) then $z = \beta$, whilst if the photon interacts with a gluon ($i = g$) then z is the fractional momentum carried by the gluon and $0 < \beta < z$. $P_j(\zeta)$ is the j^{th} member in

a set of Chebyshev polynomials, which are chosen such that $P_1 = 1$, $P_2 = \zeta$ and $P_{j+1}(\zeta) = 2\zeta P_j(\zeta) - P_{j-1}(\zeta)$.

A sum of $n = 3$ orthonormal polynomials is used so that the input distributions are free to adopt a large range of forms for a given number of parameters. Any bias towards a particular solution due to the choice of the functional form of the input distribution is therefore minimized.

The series are squared to ensure positive definite parameterizations.

In the following fits, a is set to 0.01 such that this term only influences the parametrizations in the region $z > 0.9$.

3.3 QCD fits of $F_2^{D(3)}$

The trajectory intercepts are fixed to $\alpha_{\mathbb{P}} = 1.20$ and $\alpha_{\mathbb{P}} = 0.57$ [8]. Only data points of H1 (1994) with $Q^2 \geq 3 \text{ GeV}^2$, $\beta \leq 0.65$, $M_X > 2 \text{ GeV}$ and $y \leq 0.45$ are included in the fit in order to avoid large higher twist effects and the region that may be most strongly affected by a non zero value of R . Thus, we have 161 data points with 7 free parameters (3 for the quark singlet density, 3 for the gluon density and 1 for the normalization of the pion structure function), which makes 154 degrees of freedom.

3.4 Results

We realize QCD fits on the above data solving DGLAP equations with our Laguerre method.

All QCD fits are performed using only the statistical errors. We get : $\chi^2/ndf = 173.8/154$ (which is consistent with the previous analysis of reference [8] : $\chi^2/ndf = 176.3/154$).

The resulting parton densities of the Pomeron are presented on fig. 1 ; a large gluonic content is obtained, with significant contributions at large fractional momenta.

The result of the fit is presented on fig. 2 with experimental values, we see on this figure the good agreement of the QCD prediction and data points, which is a proof of the validity of description of the Pomeron in terms of partons following a QCD dynamics.

Fig. 1. Quark flavour singlet (zS) and gluon (zG) distributions of the pomeron deduced from the fit.

Fig. 2. The H1 data points on $x_P F_2^{D(3)}$ (1994) are shown with the result of the QCD fit described in section (3) : $\chi^2/ndf = 173.8/154$; the result of the fit is drawn only in bins included in the minimization procedure.

4 Conclusion

We have shown that the Laguerre expansion is a significant method to solve integro-differential equations very efficiently. We have described our implementation of this technique on QCD evolution equations (the DGLAP equations).

On the numerical side, the short time consuming of our treatment is a real advantage to realize fits of perturbative QCD.

We have exemplified this last point on the determination of the gluon content of the Pomeron. We have obtained a good agreement between experimental values of $F_2^{D(3)}$ and QCD predictions ($\chi^2/ndf = 173.8/154$). Then, a large gluon content in the Pomeron is confirmed by our analysis.

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