Magnetized cosmological perturbations

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A large-scale cosmic magnetic field affects not only the growth of density perturbations, but also rotational instabilities and anisotropic deformation in the density distribution. We give a fully relativistic treatment of all these effects, incorporating the magneto-curvature coupling that arises in a relativistic approach. We show that this coupling produces a small enhancement of the growing mode on superhorizon scales. The magnetic field generates new nonadiabatic constant and decaying modes, as well as nonadiabatic corrections to the standard growing and decaying modes. Magnetized isocurvature perturbations are purely decaying on superhorizon scales. On subhorizon scales before recombination, magnetized density perturbations propagate as magneto-sonic waves, leading to a small decrease in the spacing of acoustic peaks. Fluctuations in the field direction induce scale-dependent vorticity, and generate precession in the rotational vector. On small scales, magnetized density vortices propagate as Alfvén waves during the radiation era. After recombination, they decay slower than non-magnetized vortices. Magnetic fluctuations are also an active source of anisotropic distortion in the density distribution. We derive the evolution equations for this distortion, and find a particular growing solution.

I. INTRODUCTION

Recent observations reveal the widespread existence of magnetic fields in the universe and are producing much firmer estimates of their strengths in interstellar and intergalactic space. They also appear to be a common property of the intracluster medium of galaxy clusters, extending well beyond the core regions (see [1] and references therein). Strengths of ordered magnetic fields in the intracluster medium of cooling flow clusters exceed those typically associated with the interstellar medium of the Milky Way, suggesting that galaxy formation and even cluster dynamics are, at least in some cases, influenced by magnetic forces. Furthermore, reports of Faraday rotation associated with high redshift Lyman-α absorption systems seem to imply that dynamically significant magnetic fields may be present in condensations at high redshift [2]. The more we look for extragalactic magnetic fields, the more ubiquitous we find them to be.

Large-scale magnetic fields introduce new ingredients into the standard, but nevertheless uncertain, picture of the early universe. They seem unlikely to survive an epoch of inflation, but it is conceivable that large-scale fields and magnetic inhomogeneities could be generated at the end of that era or in subsequent phase transitions (see, e.g., [3]). Studies of magnetogenesis are partly motivated by the need to explain the origin of large-scale galactic fields. Typical spiral galaxies have magnetic fields of the order of a few µG coherent over the plane of their disc. Such fields could arise from a relatively large primordial seed field, amplified by the collapse of the protogalaxy, or by a much weaker one that has been strengthened by the galactic dynamo. Provided that this mechanism is efficient, the seed can be as low as \( \sim 10^{-23}\text{G} \) at present. However, in the absence of nonlinear amplification, seeds of the order of \( 10^{-12}\text{G} \) or even \( 10^{-8}\text{G} \) are required [4].

Determining whether the origin of galactic and cluster magnetic fields is primordial or post-recombination is a difficult task, since strong amplification in these virialized systems overwhelms any traces of their earlier history. In contrast, magnetic effects on the cosmic microwave background (CMB) anisotropies, or any magnetic presence away from clusters and galaxies, can provide better insight into these early phases. If large-scale magnetic fields are present throughout the universe today, their structure and spectrum should bear clearer signatures of their past. Thus, improved direct observations, such as high resolution Faraday rotation maps and the study of extragalactic cosmic rays, may help in this respect [5]. For example, we would like to know whether or not the intergalactic voids are

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permeated by a widespread magnetic field, and whether there is magnetic field evolution in galaxies. If large-scale magnetic fields were present in the early universe, were they dynamically significant, and if so, how have they affected the formation and evolution of the observed structure? It is known that element abundances constrain the strength of a primordial field at the nucleosynthesis epoch [6]. Stronger limits on a primordial magnetic field are imposed via CMB anisotropies, since the field distorts the acoustic peaks and induces Faraday rotation in the polarization [7–9].

In this article we assume the existence of a large-scale ordered magnetic field of primordial origin a priori, and we investigate the magnetic effects on density inhomogeneities. Specifically, we analyze magnetized density perturbations, magnetized cosmic vortices (i.e., rotational instabilities), and magnetized shape distortion. Magnetized density perturbations were studied by Ruzmaikina and Ruzmaikin [10] in Newtonian theory, while Wasserman [11] looked at the rotational behavior of a magnetized fluid. Kim et al. [12] derive a magnetized Jeans length, assuming that there are no density perturbations in the absence of the field. In a relativistic treatment, Battaner et al. [13] investigated magnetized structure formation in the radiation era. Subramanian and Barrow [14] have considered magnetic dissipative effects at recombination.

We generalize aspects of these previous treatments by giving a fully relativistic analysis of the scalar and vector contributions of the magnetic field to the evolution of density inhomogeneity. We consider not only density perturbations and rotational instabilities, but also the shape-distortion effects of the field. Density perturbations are found explicitly in the radiation and dust eras, including a new solution that shows how the relativistic magneto-curvature coupling acts to enhance growth on superhorizon scales. The existence of the magneto-curvature coupling was first identified by Tsagas and Barrow [15]. The nonadiabatic magnetic effect on the modes is clearly identified, including the magnetized isocurvature modes. New solutions are also found for rotational instabilities, which are significantly affected by the field, and we show that magnetic effects actively generate shape distortion in the density distribution. We follow the relativistic analysis of cosmic electromagnetic fields given by Ellis [16], and we use the covariant and gauge-invariant approach to perturbations [17–19]. A covariant and gauge-invariant analysis of magnetized density perturbations was first developed by Tsagas and Barrow [15,20], whose results we extend.

We adopt the usual approximation that in the background, which is a spatially flat Friedmann-Robertson-Walker (FRW) model, the ordered large-scale magnetic field is too weak to destroy spatial isotropy. The weak field approximation is an acceptable physical approximation when the field energy density is a small fraction of the isotropically distributed dominant energy density (see [21] for further discussion). The standard assumption of very high conductivity is also made, so that we can ignore large-scale electric fields, while maintaining the desired coupling between the fluid and the magnetic field. We use a single perfect fluid model, which is reasonable in the radiation era, but does not apply during recombination, while after last scattering, it means that our solutions only apply to a baryon-dominated universe. (See [22] for a discussion of the effects of cold dark matter (CDM) potential wells on the field.)

In Sec. II, we outline the formalism and the main equations that govern the coupled evolution of density inhomogeneity, the magnetic field and the curvature. Sec. III considers magnetized density perturbations, identifying the nonadiabatic effects of the field. We find a new solution on superhorizon scales in the radiation era, showing how the magneto-curvature coupling slightly enhances growth. In the radiation era, a small damping effect is wrongly predicted when the magneto-curvature coupling is ignored. On subhorizon scales, magneto-sonic waves in the radiation era have a slightly increased frequency, leading to a decrease in the spacing of CMB acoustic peaks. In the dust era, the growing mode on small scales is slightly damped by magnetic effects. We also find the pure-magnetic density perturbations, i.e., the fluctuations created in a smooth plasma at magnetogenesis. These include growing modes. Magnetized isocurvature modes are characterized, and found explicitly on superhorizon scales. These modes all decay, in the radiation and dust eras. Magnetized cosmic vortices are considered in Sec. IV. We show that magnetized vorticity is scale-dependent, and that the field generates precession in the rotational vector. We solve exactly for the rotational instabilities, showing that they propagate as Alfvén (vector) waves on small scales during the radiation era. After recombination, such vortices persist for longer than non-magnetized vortices. Sec. V investigates magnetized shape distortion, showing that the field is as an active source of distortion. Purely magnetic distortion on superhorizon scales in the dust era is shown to have a growing mode. Conclusions are given in Sec. VI.

We use units with $c = 1 = 8\pi G$ and our signature is $(-+++)$; spacetime indices are $a, b, \cdots$ and (square) round brackets enclosing indices denote (anti-)symmetrization.

\footnote{Spatial flatness is necessary for the gauge-invariance of all the perturbative variables [15].}
II. COSMIC MAGNETOHYDRODYNAMICS

As noted above, the cosmic magnetic field \( B^a \) must have weak energy density \( \rho_{\text{mag}} = \frac{1}{2} B_a B^a \) to be consistent with observational limits, so that \( c_a \ll 1 \), where \( c_a \) is the Alfvén speed. The Alfvén speed, which effectively leads to a nonadiabatic increase in the sound speed, is given by

\[
\sqrt{\frac{c_s^2}{\rho}} = \frac{B^2}{\rho}, \tag{1}
\]

where \( \rho \) is the energy density of the cosmic fluid. In the limit of vanishing density inhomogeneity, i.e., in the background, the field is uniform, but its weak magnitude means that it does not disturb the background isotropy, so that the magnetic anisotropic stress is negligible in the background. In the actual inhomogeneous universe, magnetic effects arise from the following aspects of the field:

- the background energy density and pressure (\( \rho_{\text{mag}} = \frac{1}{2} \rho_{\text{mag}} \)) occur in terms of the form \( c_s^2 P \), where \( P \) is a perturbed quantity, and despite their weakness, they can have observable consequences (e.g., a change in the spacing of CMB acoustic peaks in the radiation era);
- spatial gradients of \( \rho_{\text{mag}} \) couple with gradients of \( \rho \) and thus alter the fluctuations of \( \rho \) (in particular, introducing nonadiabatic modes);
- the background direction of the field introduces anisotropy by picking out preferred directions in perturbed vector and tensor fields, and preferred directional derivatives of perturbed scalar/ vector/ tensor fields, leading to effects such as Faraday rotation;
- the background direction of the field is also the source of the magneto-curvature coupling, via terms of the form \( K_{abcd} B^d \), where \( K_{abcd} \) is the part of the curvature tensor which vanishes in the background;
- fluctuations in the direction of the field generate new anisotropies that can source magnetized vortices (leading in particular to Alfvén waves) and shape distortion.

We include all of these aspects in our analysis, so that we incorporate the full range of scalar (magnetic energy density and isotropic pressure) and vector (anisotropic stress) effects of the field, allowing for fluctuations in both the magnitude and direction. In order to provide a transparent relativistic generalization of Newtonian analysis, and to use variables that as far as possible have a direct physical interpretation, we adopt a covariant Lagrangian approach [16–19,23,24]. This continues and develops the work of [15]. In particular, we discuss in detail the physical meaning and implications of the density perturbation solutions, and we extend the investigation to cover magnetized vortices and shape distortion.

The cosmic perfect fluid defines a unique four-velocity \( u_a \) (with \( u_a u^a = -1 \)), and then \( h_{ab} = g_{ab} + u_a u_b \), where \( g_{ab} \) is the spacetime metric, projects into the local rest spaces of comoving observers. The projection of a vector is \( V_{(a)} = h_{ab} V_b \), and a projected second rank tensor \( S_{ab} \) splits irreducibly as

\[
S_{ab} = \frac{1}{2} h_{ac} S^c + S_{(ab)},
\]

where \( S \equiv h_{ac} S^{cb} \) is the spatial trace, \( S_a \equiv \frac{1}{2} \varepsilon_{abc} S^{bc} \) is the spatial vector dual to the skew part of \( S_{ab} \), and \( S_{(ab)} \equiv [h_{(a} c_{b)} d - \frac{1}{2} h^{cd} h_{ab}] S_{cd} \) is the projected symmetric tracefree (PSTF) part. Here \( \varepsilon_{abc} \) is the projection of \( \eta_{abcd} \), the spacetime alternating tensor.

The covariant derivative splits into a comoving time derivative \( J_{a...b} = u^c \nabla_c J_{a...b} \), and a covariant spatial derivative \( D_c J_{a...b} = h_c \dot{h}_a c ... h_b \nabla_c J_{a...b} \). Then we define a covariant spatial divergence and curl that generalize the Newtonian operators to curved spacetime [23]:

\[
\text{div} V = D^a V_a, \quad \text{(div} S)_{a} = D^b S_{ab}, \quad \text{curl} V_{a} = \varepsilon_{abc} D^b V^c, \quad \text{curl} S_{ab} = \varepsilon_{cd(a} D^c S_{b)} d. \tag{2}
\]

The fluid kinematics are described by the expansion \( \Theta = \text{div} u \), four-acceleration \( A_a = \dot{u}_a \), vorticity \( \omega_a = -\frac{1}{2} \varepsilon_{abc} u^b \) and shear \( \sigma_{ab} = D_{(a} u_{b)} \). Local curvature is described by the Ricci tensor \( R_{ab} \), while nonlocal tidal forces and gravitational radiation are described by the electric and the magnetic parts of the Weyl tensor, \( E_{ab} = E_{(ab)} = C_{abcd} u^c u^d \) and \( H_{ab} = H_{(ab)} = \frac{1}{2} \varepsilon_{abcd} C^{cd} u^e u^c \). The magnetized perfectly conducting fluid has energy density \( \rho \) and
isotropic pressure \( p \). The magnetic field is \( B_a = B_{(a)} \), with energy density, isotropic pressure and anisotropic stress given respectively by

\[
\rho_{\text{mag}} = \frac{1}{2} B_a B^a, \quad p_{\text{mag}} = \frac{1}{3} \rho_{\text{mag}}, \quad \pi_{ab} = -B_{(a} B_{b)}. \tag{2}
\]

Then the total energy-momentum tensor is

\[
T_{ab} = (\rho + \rho_{\text{mag}}) + (p + p_{\text{mag}}) + \pi_{ab}. \tag{3}
\]

Notice that the absence of an electric field means that there is no energy flux (Poynting vector). The magnetic field appears from Eq. (3) to behave like a radiation fluid with anisotropic stress. However, this fluid picture does not fully encompass the vector properties of the field, and in particular, its coupling to the curvature.

In the background, \( B_a \) is weak enough not to affect the isotropy, i.e. the anisotropic stress is negligible in the background, and \( \rho_{\text{mag}} \ll \rho \). The background expansion is \( \Theta = 3H \), where \( H = \dot{a}/a \) is the Hubble rate. The background is covariantly characterized by

\[
D_a \Theta = D_a \rho = D_a p = D_a B_a = 0, \tag{4}
\]

\[
A_a = \omega_a = 0, \tag{5}
\]

\[
\sigma_{ab} = E_{ab} = H_{ab} = R_{ab} = \pi_{ab} = 0, \tag{6}
\]

where

\[
R_{ab} = h_{a}^{c} h_{b}^{d} R_{cd} + R_{acbd} u^{c} u^{d} + D_{c} u_{a} D_{b} u^{c} - \Theta D_{b} u_{a},
\]

with \( R_{ab} \) the Ricci tensor and \( R_{abcd} \) the Riemann tensor. Note that \( R_{ab} \) is the intrinsic 3-Ricci tensor of spatial hypersurfaces only if \( \omega_{a} = 0 \); otherwise there are no such hypersurfaces orthogonal to \( u^{a} \) [16].

Quantities that vanish in the background are gauge-invariant, and they covariantly describe linear deviations from homogeneity and anisotropy. We collect below the linearized evolution and constraint equations given in [15], rewritten in the streamlined formalism of [23], which considerably simplifies the equations and facilitates analysis of their properties. The following covariant identities [19] are used in deriving the equations (assuming a flat background with vanishing cosmological constant):

\[
\text{curl} D_a f = -2 \dot{f} \omega_a, \tag{7}
\]

\[
(aD_a f) = a D_a f + \dot{a} f A_a, \tag{8}
\]

\[
D^2 (D_a f) = D_a (D^2 f) + 2 \dot{f} \text{curl} \omega_a, \tag{9}
\]

\[
(aD_a J_{b...}) = a D_a J_{b...}, \tag{10}
\]

\[
D_{[a} D_{b} V_{c} = 0 = D_{[a} D_{b]} S^{cd}, \tag{11}
\]

\[
\text{div curl} V = 0, \tag{12}
\]

\[
(\text{div curl} S)_{a} = \frac{1}{2} \text{curl} (\text{div} S)_{a}, \tag{13}
\]

\[
\text{curl curl} V_{a} = D_{a} (\text{div} V) - D^2 V_a, \tag{14}
\]

\[
\text{curl curl} S_{ab} = \frac{3}{2} D_{[a} (\text{div} S)_{b]} - D^2 S_{ab}, \tag{15}
\]

where the vectors and tensors vanish in the background, \( S_{ab} = S_{(ab)} \). The magnetic field itself does not vanish in the background, so that its projected derivatives do not commute at linear order: the vector identity in Eq. (11) is replaced by

\[
D_{[a} D_{b]} B_{c} = R_{dcba} B^{d} - \epsilon_{abcd} \omega^{d} B_{c},
\]

where \( R_{abcd} \) is formed from \( R_{abcd} \) and the kinematic quantities [15]. This non-commutativity is the root of the magneto-curvature coupling found in [15].

### A. Maxwell’s equations

In the infinite conductivity limit, Maxwell’s equations [16,24] provide three constraints,
\[
\text{div } B = 0, \tag{16}
\]
\[
\text{curl } B_a = \varepsilon_{abc} B^b A^c + j_a, \tag{17}
\]
\[
\omega^a B_a = \frac{1}{2} q, \tag{18}
\]
where \( j_a \) is the current and \( q \) the charge density generated by fluctuations, and one propagation equation
\[
\dot{B}_{(a)} = -\frac{2}{3} \Theta B^a + \sigma_{ab} B^b + \varepsilon_{abc} B^b \omega^c, \tag{19}
\]
which is the covariant form of the induction equation. Note that \( \dot{B}_{(a)} = \dot{B}_a - A_b B^b u_a \), and \( B^a A_a = 0 \) to first order only in the case of a pressure-free perfect fluid [15].

Contracting Eq. (19) with \( B^a \), and neglecting the second order term \( \sigma_{ab} \pi^{ab} \), we deduce the radiation-like evolution law of the magnetic energy density,
\[
(B^2)\dot{} + \frac{4}{3} \Theta (B^2) = 0. \tag{20}
\]
We can also derive the evolution of the anisotropic stress from Eq. (19):
\[
\dot{\pi}_{ab} = -4H \pi_{ab} - \frac{2}{3} c_s^2 \rho \sigma_{ab}, \tag{21}
\]

**B. Conservation laws**

Energy density conservation is expressed via the equation of continuity
\[
\dot{\rho} + \Theta (1 + w) \rho = 0, \tag{24}
\]
where \( w = p/\rho \). Notice the absence of magnetic terms in this equation, since field energy conservation holds separately as a consequence of Maxwell’s equations, as shown in Eq. (20). The two energy conservation equations imply
\[
(c_a^2)\dot{} = (w - \frac{1}{3}) \Theta c_a^2. \tag{25}
\]

On the other hand, the field does enter conservation of momentum density
\[
(1 + w) \rho A_a + c_s^2 D_a \rho + \varepsilon_{abc} B^b \text{curl } B^c = 0, \tag{22}
\]
where \( c_s^2 = \dot{p}/\dot{\rho} \) is the adiabatic sound-speed squared (with \( D_a p = c_s^2 D_a \rho \)). This equation reflects the momentum density exchange between the fluid and the field. The magnetic field is a source of acceleration (provided that curl \( B_a \) is not parallel to \( B_a \)); it can destroy the geodesic motion of the matter even in the absence of pressure.

**C. Kinematic equations**

Evolution of the expansion is governed by the Raychaudhuri equation
\[
\dot{\Theta} + \frac{1}{3} \Theta^2 + \frac{1}{2} (1 + 3w) \rho - \frac{c_a^2}{3(1 + w)} R + \frac{1}{2(1 + w) a^2} (2c_s^2 \Delta + c_a^2 B) - \Lambda = 0, \tag{23}
\]
where \( \Lambda \) is the cosmological constant, \( R = h^{ab} R_{ab} \) is the projected curvature scalar, and
\[
\Delta = a D^a \Delta_a \quad \Delta_a = \frac{a D_a \rho}{\rho} \quad B = \frac{a^2 D^a B_a}{B^2} \quad B_a = D_a B^2,
\]
describe perturbations in the fluid and field energy densities. Note that the overall magnetic effect includes a coupling to the projected curvature, \( c_s^2 \mathcal{R} \).

Magnetic influence on cosmic rotation is encoded in the vorticity propagation equation
\[
\dot{\omega}_a + 2H \omega_a = -\frac{1}{2} \text{curl } A_a, \tag{24}
\]
which may be rewritten, after eliminating the acceleration term via Eq. (22), as
\[
\dot{\omega}_a + (2 - 3c^2_s) H \omega_a = -\frac{1}{2(1+w)\rho} B^b D_b \text{curl} B_a .
\]  

(25)

Thus there is a magnetically induced vorticity component parallel to \text{curl} \(B_a\). The effect disappears if the directional derivative \(B^b D_b \text{curl} B_a\) vanishes, i.e., when \text{curl} \(B_a\) does not change along the magnetic force lines.

Kinematic anisotropies evolve via the shear propagation equation

\[
\dot{\sigma}_{ab} + 2H \sigma_{ab} = -\frac{c^2_s}{a(1+w)} D(a \Delta b) - \frac{1}{2(1+w)\rho} D(aB_b) + \frac{c^2_s}{3(1+w)} R_{(ab)} + \frac{1}{(1+w)\rho} B^c D_c D(aB_b) - E_{ab} .
\]

(26)

The direct magnetic effects propagate through the field’s anisotropic stress \((\pi_{ab})\), as well as via anisotropies in the distribution of magnetic energy density \((D_a B_b)\) and of the field vector itself \((D_a B_b)\). The latter effect vanishes when \(D_a B_b\) is invariant along the magnetic force lines. Also, the coupling between the field and the projected curvature has led to an extra magneto-geometrical contribution, \(c^2_s R_{(ab)}\).

The kinematic quantities also obey constraint equations:

\[
\begin{align*}
\text{(div} \sigma)_a &= \frac{2}{3} D_a \Theta + \text{curl} \omega_a , \\
\text{div} \omega &= 0 , \\
H_{ab} &= \text{curl} \sigma_{ab} + D(a\omega_b) .
\end{align*}
\]

(27-29)

D. Curvature

The electric and magnetic Weyl tensors obey Maxwell-like equations \([16,24]\):

\[
\begin{align*}
\dot{E}_{ab} + 3H E_{ab} - \text{curl} H_{ab} &= -\frac{1}{2} \rho(1+w) \sigma_{ab} + 3H \pi_{ab} , \\
\dot{H}_{ab} + 3H H_{ab} + \text{curl} E_{ab} &= \frac{1}{3} \text{curl} \pi_{ab} , \\
(\text{div} E)_a &= \frac{1}{3} D_a \rho + \frac{1}{6} B_a - \frac{1}{2} (\text{div} \pi)_a , \\
(\text{div} H)_a &= (1+w) \rho \omega_a ,
\end{align*}
\]

(30-33)

where we have used Eq. (21). For a magnetized fluid the projected curvature tensor \(R_{ab}\) is not in general symmetric, but has the form \(R_{ab} = R_{(ab)} + \frac{1}{3} R h_{ab} + \varepsilon_{abc} R^c\), where \([15]\)

\[
\begin{align*}
R_{(ab)} &= D(aB_b) - \frac{1}{a^3} (a^3 \sigma_{ab}) + \pi_{ab} , \\
R_a &= \text{curl} A_a - \frac{1}{a^3} (a^3 \omega_a) , \\
R &= 2 \left( \rho - \frac{1}{3} \Theta^2 + \Lambda \right)
\end{align*}
\]

(34-36)

Note that the background relation \(3H^2 = \rho + \Lambda\) ensures that \(R\) vanishes in the background, so that it is gauge-invariant. By Eq. (24), the vector part of \(R_{ab}\) is simply \(R_a = -H \omega_a\), which vanishes when the vorticity vanishes. The dimensionless curvature perturbation \(K = a^2 R\) has comoving gradient \([15]\)

\[
a D_a K = 2 \rho a^2 \Delta_a + a^3 B_a - 4H a^3 D_a \Theta ,
\]

(37)

which evolves as \([15]\)

\[
(a D_a K)' = \frac{2aH}{\rho(1+w)} D^2 (2a^2 \Delta_a + a B_a) + 24a^3 H^2 c^2_s \text{curl} \omega_a .
\]

(38)
E. Evolution of inhomogeneities

The key gauge-invariant quantities describing inhomogeneity are the comoving spatial gradients of the fluid density, the field density, the expansion and the field vector:

$$\Delta_\iota = \frac{aD_\iota \rho}{\rho}, \quad B_\iota = D_\iota B^2, \quad \Theta_\iota = aD_\iota \Theta, \quad B_{ab} = aD_\iota B_\alpha.$$  (39)

Their propagation equations are [15]

$$\ddot{\Delta}_\iota = 3wH\Delta_\iota + (1 + w)\Theta_\iota + \frac{3aH}{\rho} \varepsilon_{abc} B^b \text{curl} B^c,$$  (40)

$$\ddot{\Theta}_\iota = -2H\Theta_\iota - \frac{1}{2} \rho \Delta_\iota - \frac{c_s^2}{(1 + w)} D^2 \Delta_\iota - \frac{1}{2} aB_\iota - \frac{a}{2(1 + w)} D^2 B_\iota$$

$$+ \frac{3}{2} a \varepsilon_{abc} B^b \text{curl} B^c - \left[ 6c_s^2 + \frac{4c_s^2}{(1 + w)} \right] aH \text{curl} \omega_\iota,$$  (41)

$$\ddot{B}_{ab} = -2HB_{ab} + \frac{c_s^2 H}{(1 + w)} \{ 3B_{(a} \Delta_{b)} + B_{(a} \Delta_{b)} \} - B_{(a} \Theta_{b)} - B_{(a} \Theta_{b)}$$

$$+ aB^c \text{D}_c \rho_{ab} + a\varepsilon_{abc} B^b \text{curl} \omega_{(c)} - aB_{(a} \text{curl} \omega_{b)} - a\varepsilon_{abc} B^c H^d \omega^c.$$  (42)

Note how the magnetic field couples to the magnetic Weyl curvature via the last term in Eq. (42).

By eliminating the expansion gradients from the time derivative of Eq. (40), we arrive at [15]

$$\ddot{\Delta}_\iota = - (2 + 3c_s^2 - 6w) H \dot{\Delta}_\iota + \left[ \frac{1}{2} \left( 1 - 6c_s^2 + 8w - 3w^2 \right) \rho - 2 \left( 3c_s^2 - 5w \right) \Lambda \right] \Delta_\iota$$

$$+ c_s^2 D^2 \Delta_\iota + \frac{a}{2} \rho D^2 B_\iota + \frac{3a}{\rho} \left[ (c_s^2 - w) \rho + (1 + c_s^2) \Lambda \right] \varepsilon_{abc} B^b \text{curl} B^c$$

$$+ \frac{1}{2} (1 + w) aB_\iota + \left[ \frac{3aH}{\rho} \right] \varepsilon_{abc} B^b \text{curl} \dot{B}_{(c)} + \left[ 6(1 + w) c_s^2 + 4c_s^2 \right] aH \text{curl} \omega_\iota.$$  (43)

In the Newtonian limit, Eq. (43) recovers the results of [10]. The relativistic correction terms are the last three terms on the right hand side, i.e., the terms with $B_\iota, \varepsilon_{abc} B^b \text{curl} \dot{B}_{(c)}$, and $\text{curl} \omega_\iota$.

III. MAGNETIZED DENSITY PERTURBATIONS

The equations (40)–(42) provide the basis for a complete description of coupled density-magnetic inhomogeneities. We begin by isolating the evolution equations for the density perturbation scalars $\Delta$ (a covariant alternative to the density contrast $\delta \rho/\rho$) and $B$ (describing fluctuations in the magnetic energy density), and the curvature perturbation $K$:

$$\Delta = aD^a \Delta_a, \quad B = \frac{a^2 D^a B_a}{B^2}, \quad K = a^2 \mathcal{R}.$$  

The required evolution equations are [15]:

$$\ddot{\Delta} = - (2 + 3c_s^2 - 6w) H \dot{\Delta} + \frac{1}{2} \left[ (1 - 6c_s^2 + 8w - 3w^2) \rho - 4 (3c_s^2 - 5w) \Lambda \right] \Delta$$

$$+ c_s^2 D^2 \Delta - \frac{1}{2} \left[ (1 - 3c_s^2 + 2w) \rho - (1 + 3c_s^2) \Lambda \right] c_s^2 B + \frac{1}{2} c_s^2 D^2 B$$

$$+ \frac{1}{2} \left[ (2 - 3c_s^2 + 3w) \rho - (1 + 3c_s^2) \Lambda \right] c_s^2 K,$$  (44)

$$\ddot{B} = \frac{4}{3(1 + w)} \dot{\Delta} - \frac{4(c_s^2 - w)H}{1 + w} \Delta,$$  (45)

$$\ddot{K} = \frac{4c_s^2 H}{(1 + w)} \Delta + \frac{2c_s^2 H}{(1 + w)} B.$$  (46)

This system of equations governs the coupling between density fluctuations $\Delta$, magnetic fluctuations $B$ and curvature fluctuations $K$. Eq. (46) shows that $K$ grows if $\Delta$ and $B$ are growing, while Eq. (45) shows that if $c_s^2 \geq w$ (which holds in the radiation and dust eras), then $B$ grows in concert with growing $\Delta$. 

7
The magnetic field introduces a direct effect, via the term $c_a^2 K$ in Eq. (44), of the curvature on the density perturbations. In the non-magnetized case, there are two modes of $\Delta$, which is governed by the single second-order equation (44) with $c_a = 0$. In this case, the evolution of $\Delta$ is independent of $K$, and $K$ is determined once $\Delta$ is known, via Eq. (46). Magnetism introduces two additional modes, since the system has four degrees of freedom. These modes are nonadiabatic, and can source density perturbations, i.e., even when $\Delta(t_0) = 0 = \dot{\Delta}(t_0)$, magnetic effects will lead to $\Delta \neq 0$ for $t > t_0$. If one omits the magneto-curvature effect, then the evolution equation for $K$, Eq. (46), is uncoupled from the system, which can then be decoupled via a third-order equation in $\Delta$. Neglecting the magneto-curvature effect thus removes one of the additional nonadiabatic modes.

For zero cosmological constant, we can solve the system analytically in the radiation and dust eras, treating super- and sub-horizon scales separately. Some solutions were given in [15]. There, however, magneto-curvature effects were neglected in three out of the four cases. Here, we generalize some of the solutions to incorporate the magneto-curvature coupling, and we show that the magneto-curvature coupling cannot in general be neglected, since it leads to important qualitative differences in the behavior of $\Delta$.

A. Radiation era

During the radiation era, $w = c_s^2 = \frac{1}{3}$, $\rho = \rho_0(a_0/a)^4$, and the Alfvén speed does not change along the fluid flow, i.e., $c_a = 0$, reflecting the radiation-like evolution of the magnetic energy density, as given by Eq. (20). For the Fourier modes with comoving wave-number $k$, we get

$$
\left(\frac{a}{a_0}\right)^2 \Delta'' = \left[2 - \frac{1}{2} \left(\frac{k}{k_{h0}}\right)^2 \left(\frac{a}{a_0}\right)^2\right] \Delta - \left[1 + \frac{1}{2} \left(\frac{k}{k_{h0}}\right)^2 \left(\frac{a}{a_0}\right)^2\right] c_a^2 B + 2c_a^2 K,
$$

(47)

$$
B' = \Delta',
$$

(48)

$$
K' = \Delta + \frac{3}{2} c_a^2 B,
$$

(49)

where a prime denotes $d/d(a/a_0)$ and $k_{h0} = a_0 H_0$ is the comoving wavenumber of the horizon at $a_0$.

1. Superhorizon scales and the curvature coupling

In the long wavelength limit $k \ll k_{h0}$, the system has the power-law solution

$$
\Delta = C_{(0)} + \sum_{\alpha} C_{(\alpha)} \left(\frac{a}{a_0}\right)^\alpha, \quad (50)
$$

$$
B = - \left(\frac{2}{3c_s^2}\right) C_{(0)} + \sum_{\alpha} C_{(\alpha)} \left(\frac{a}{a_0}\right)^\alpha, \quad (51)
$$

$$
K = - \left(\frac{4}{3c_s^2}\right) C_{(0)} + \left(1 + \frac{3}{2} c_a^2\right) \sum_{\alpha} \frac{C_{(\alpha)}}{\alpha} \left(\frac{a}{a_0}\right)^\alpha, \quad (52)
$$

where $C_{(0)}$ and $C_{(\alpha)}$ are constants, and the parameter $\alpha$ satisfies the cubic equation

$$
\alpha^3 - \alpha^2 - (2 - c_s^2) \alpha - (2 + 3c_a^2) c_a^2 = 0. \quad (53)
$$

The cubic has one positive and two negative roots. One of the negative roots corresponds to a decaying nonadiabatic mode. The other nonadiabatic mode is the $C_{(0)}$-mode, which is constant. The remaining cubic roots correspond to the magnetized versions of the standard adiabatic modes, one growing and one decaying. Since $c_a^2$ is small, we can find the roots perturbatively. The zero-order roots are $0, -1, 2$ (the $\alpha = 0$ solution is spurious in the non-magnetized case). To lowest order, we find that:

$$
\alpha = \begin{cases} 0 - c_s^2 + O(c_s^4) & , \\ -1 + c_s^2 + O(c_s^4) & , \\ 2 + \frac{1}{2} c_s^4 + O(c_s^6) & . \end{cases}
$$

(54)
Thus the adiabatic growing mode of the non-magnetized case is slightly enhanced by magnetic effects (the enhancement is not felt to lowest order in $c_a^2$); the adiabatic decaying mode decays less rapidly by virtue of magnetic effects; the decaying nonadiabatic mode decays very slowly; and the final, nonadiabatic, mode is constant. To lowest order

$$\Delta = C_{(+)} \left( \frac{a}{a_0} \right)^2 + C_{(1-)} \left( \frac{a}{a_0} \right)^{-1+e_a^2} + C_{(0)} + C_{(2-)} \left( \frac{a}{a_0} \right)^{-e_a^2}. \tag{55}$$

The magnetic and curvature fluctuations are given by equations (51) and (52), with $\alpha$ given by Eq. (54).

This new solution in Eq. (55) can be compared with the solution that arises when the magneto-curvature coupling term $c_a^2 K$ is ignored in Eq. (50) [15]. Then the last term in Eq. (53) falls away, leading to the quadratic

$$\alpha^2 - \alpha - \left( 2 - c_a^2 \right) = 0.$$

To lowest order

$$\alpha = \begin{cases} 
-1 + \frac{1}{3} c_a^2, \\
2 - \frac{1}{3} c_a^2.
\end{cases}$$

so that the density perturbation is given by

$$\Delta = C_{(+)} \left( \frac{a}{a_0} \right)^{2 - \frac{1}{3} c_a^2} + C_{(-)} \left( \frac{a}{a_0} \right)^{-1 + \frac{1}{3} c_a^2} + C_{(0)},$$

and the magnetic fluctuations are

$$B = \Delta - \left[ 1 - \left( \frac{2}{c_a^2} \right) \right] C_{(0)}.$$

Clearly, omitting the magneto-curvature coupling has a significant qualitative impact. Not only is one of the nonadiabatic modes ($C_{(2-)}$) removed, as expected, but we also find that the growing mode is slightly damped, at odds with the correct solution in Eq. (55). Thus the magneto-curvature coupling, which was identified in general in [15], turns out to have a crucial role in increasing (even though it is only by a small amount) the standard adiabatic modes of density perturbations on large scales in the radiation era. It is not reasonable to omit the magneto-curvature coupling in this case.

2. Subhorizon scales and magneto-sonic waves

At the opposite end of the wavelength spectrum, when $k \gg k_{10}$, we differentiate Eq. (47) and use Eq. (49) to decouple the system. Integrating once we get

$$6 \left( \frac{a}{a_0} \right)^2 \Delta'' + 2 \left( \frac{k}{k_{10}} \right)^2 \left( \frac{a}{a_0} \right)^2 \left( 1 + \frac{3}{2} c_a^2 \right) \Delta = 6 C_K - 3 C_B c_a^2 \left( \frac{k}{k_{10}} \right)^2 \left( \frac{a}{a_0} \right)^2,$$

where $C_K$ is an additional constant associated with curvature effects. (We have ignored higher order terms in $c_a^2$, given the weakness of the magnetic field.) This has solution (to lowest order in $c_a^2$)

$$\Delta = \left[ C_{(1)} - C_K Si \left( \frac{k}{k_{10}} \frac{a}{a_0} \right) \right] \sin \left( \frac{k}{k_{10}} \frac{a}{a_0} \right)$$

$$+ \left[ C_{(2)} - C_K Ci \left( \frac{k}{k_{10}} \frac{a}{a_0} \right) \right] \cos \left( \frac{k}{k_{10}} \frac{a}{a_0} \right) - C_B c_a^2,$$ \tag{56}

where $C_{(1)}$ are constants, Si and Ci are the sine and cosine integral functions,\(^2\) and

$$\beta = c_a \left( 1 + \frac{3}{4} c_a^2 \right). \tag{57}$$

\(^2\)Si($x$) = $\int_0^x t^{-1} \sin t \, dt$ and Ci($x$) = $\gamma + \ln x + \int_0^x t^{-1} (\cos t - 1) \, dt$, where $\gamma = 0.578 \cdots$ is Euler’s constant [26].
where $c_s = 1/\sqrt{3}$ is the adiabatic sound speed. Thus $\beta$ is the magnetized (nonadiabatic) sound speed of magneto-sonic waves. These waves differ slightly in amplitude and frequency from the adiabatic acoustic waves.

The magneto-curvature coupling, reflected in the nonadiabatic $C_K$ mode, has the effect of slightly modulating the amplitude of acoustic oscillations, with the effect decreasing as $a/a_0$ increases. The main magnetic effect is on the frequency. Comparing our result in Eq. (57) to the standard solutions of magnetic-free models (see, e.g., [25]), we see that the field has increased the frequency of acoustic oscillations. Since $a \propto \sqrt{t}$, the magnetized acoustic frequency is

$$\nu_{ac, mag} = \nu_{ac} \left(1 + \frac{3}{2} c_a^2 \right)$$

(58)

This magnetic correction results from the “tensioning” effect of magnetic force lines in the plasma, which produces a nonadiabatic increase of the sound speed via a contribution from the Alfvén speed. As a result, the magnetic influence brings the acoustic peaks of short-wavelength radiation density oscillations closer, producing in principle an observable signature on CMB anisotropies [7]. An additional effect comes from the nonadiabatic constant mode in Eq. (56). Its presence suggests that the average value of the density contrast is nonzero, unlike the magnetic-free case.

B. Dust era

After recombination, in a baryon-dominated cold matter background, $w = 0 = c_s^2$, $a = a_0 (t/t_0)^{2/3}$, $H = 2/3 t$ and $\rho = 4/3 t^2$. The Alfvén speed is no longer constant, but by Eq. (20) varies as

$$c_a^2 = (c_a^2)_0 \left(\frac{t_0}{t} \right)^{2/3},$$

reflecting the fact that the magnetic energy density drops faster than that of nonrelativistic matter. Thus magnetic effects grow weaker as the expansion of the universe proceeds beyond recombination.

The equations for the Fourier modes become

$$\Delta'' = -\frac{4}{3} \left(\frac{t_0}{t} \right) \Delta' + \frac{2}{3} \left(\frac{t_0}{t} \right)^2 \Delta$$

$$- \frac{2}{3} (c_a^2)_0 \left(\frac{t_0}{t} \right)^{8/3} \left[1 + \frac{1}{3} \left(\frac{k}{k_{0}} \right)^2 \left(\frac{t}{t_0} \right)^{2/3} \right] \mathcal{B} + \frac{8}{3} (c_a^2)_0 \left(\frac{t_0}{t} \right)^{8/3} \mathcal{K},$$

(59)

$$\mathcal{B}' = \frac{4}{3} \Delta',$$

(60)

$$\mathcal{K}' = \frac{4}{3} (c_a^2)_0 \left(\frac{t_0}{t} \right)^{5/3} \mathcal{B},$$

(61)

where a prime denotes $d/d(t/t_0)$. Thus

$$\mathcal{B} = \frac{4}{3} (\Delta + C_B),$$

(62)

$$\mathcal{K}' = \frac{16}{9} (c_a^2)_0 \left(\frac{t_0}{t} \right)^{5/3} [\Delta + C_B],$$

(63)

where $\dot{C}_B = 0$. We can now decouple the system.

1. Superhorizon scales

For long wavelength fluctuations, we get

$$9 \left(\frac{t}{t_0} \right)^3 \Delta''' + 36 \left(\frac{t}{t_0} \right)^2 \Delta'' + 14 \left(\frac{t}{t_0} \right) \Delta' - 4 \Delta = 0,$$

(64)

to lowest order in $c_a^2$. Note how $c_a^2$ and the constant $C_B$ do not appear in Eq. (64), but are indirectly linked to $\Delta$ via equations (62) and (63). The density perturbations admit 3 modes, one nonadiabatic, and the fourth mode (constant,
nonadiabatic) arises in $B$. Thus there are still 4 degrees of freedom, but they are distributed differently compared to the radiation era.

We can solve Eq. (64), which is of Euler-type:

$$\Delta = C_{(+)} \left( \frac{t}{t_0} \right)^{2/3} + C_{(-)} \left( \frac{t}{t_0} \right)^{-1} + C_{(2-)} \left( \frac{t}{t_0} \right)^{-2/3}. \quad (65)$$

Thus the relativistic magneto-curvature coupling, i.e., the term $(c_a^2)_0 K$ in Eq. (59), produces a nonadiabatic decaying mode $C_{(2-)}$ in the evolution of superhorizon density perturbations, while the non-magnetized adiabatic modes are unchanged (compare [15]). The growth of large-scale matter aggregations proceeds virtually unaffected by the presence of the field or by curvature complexities. The solution is completed by equations (62) and (63).

Magnetic effects on superhorizon scales in the dust era do not change the adiabatic growing mode to lowest order in $c_a^2$. The adiabatic decaying mode is also unchanged (unlike the radiation case). However, a new nonadiabatic decaying mode arises, which decays less rapidly than the adiabatic mode.

2. Subhorizon scales

On subhorizon scales

$$9 \left( \frac{t}{t_0} \right)^3 \Delta''' + 36 \left( \frac{t}{t_0} \right)^2 \Delta'' + 14 \left( \frac{t}{t_0} \right) \left[ 1 + \frac{4}{27} (c_a^2)_0 \left( \frac{k}{k_{h0}} \right)^2 \right] \Delta' - 4 \left[ 1 - \frac{4}{27} (c_a^2)_0 \left( \frac{k}{k_{h0}} \right)^2 \right] \Delta = -\frac{16}{9} (c_a^2)_0 \left( \frac{k}{k_{h0}} \right)^2 C_B, \quad (66)$$

where again we have ignored terms of higher order in $(c_a^2)_0$. The solution is

$$\Delta = C_{(+)} \left( \frac{t}{t_0} \right)^{\alpha_+} + C_{(-)} \left( \frac{t}{t_0} \right)^{\alpha_-} + C_K \left( \frac{t}{t_0} \right)^{-2/3} + C_B \left[ \frac{4 (c_a^2)_0 k^2}{4 (c_a^2)_0 k^2 - 9 k_{h0}^2} \right], \quad (66)$$

with

$$\alpha_+ = \frac{1}{6} \left[ -1 \pm 5 \sqrt{1 - \frac{22}{27} (c_a^2)_0 \left( \frac{k}{k_{h0}} \right)^2} \right]. \quad (67)$$

The magnetic influence is expressed in two ways: additional decaying $(C_K)$ and constant $(C_B)$ nonadiabatic modes; and modification of the non-magnetized adiabatic modes $(C_{(+)})$. The net effect is to inhibit the growth of matter aggregations, as noted also in the Newtonian case [10]. The relativistic magneto-curvature coupling contributes only an additional decaying mode, leaving the evolution of $\Delta$ essentially unaffected. The magnetic effects, direct or indirect, become less important after matter-radiation equality, due to the decrease of the Alfvén speed. The damping of the growing mode is greater on smaller scales. Indeed there is a minimum scale, below which the solution in Eq. (66) oscillates, since the magnetic pressure balances gravitational infall. This magnetic Jeans scale follows from Eq. (67):

$$\lambda_{m,1}(t_0) = \frac{4}{5} \pi \sqrt{6} \lambda_a(t_0), \quad (68)$$

where

$$\lambda_a = c_a t = \frac{2}{3} c_a \lambda_h$$

is the Alfvén horizon, with $\lambda_h = H^{-1}$ the Hubble scale. In fact, given the weakness of the magnetic field, it is likely that kinetic pressure cannot be ignored near the magnetic Jeans scale. In this case, a more sophisticated analysis is necessary, to incorporate nonrelativistic pressure effects in baryonic matter. On scales well above the Alfvén horizon (equivalently, magnetic Jeans scale) but well within the Hubble horizon, i.e., for
\[ k_{h0} \ll k \ll k_{a0} \quad \text{where} \quad k_a = \frac{3}{2c_a} k_h. \]

we find that the magnetized corrections \( \alpha_{\pm} \) of the adiabatic exponents are

\[
\alpha_{\pm} = \begin{cases} 
\frac{2}{3} - \frac{2}{5} (k/k_{a0})^2, \\
-1 + \frac{2}{5} (k/k_{a0})^2.
\end{cases}
\]

The way in which magnetic effects act to increase the adiabatic Jeans length may be qualitatively understood as follows. Consider a tube of magnetic force-lines with instantaneous cross-sectional area \( \delta S \).

In a perfectly conducting medium the field remains frozen into the fluid, i.e., the magnetic force-lines always connect the same particles [16]. More precisely, the induction equation (19) shows that \( a^3 B^a \) is a connecting vector. Thus the volume of the tube is given by

\[ \delta V = \delta \ell \delta S \propto a^3 B \delta S. \]

However, we also have that in general, \( \delta V \propto a^3 \). It follows that

\[ (B \delta S)' = 0. \]

The conservation law in Eq. (20) shows that \( B \propto a^{-1} \); thus

\[ \delta S \propto a^2, \]

so that the cross section of the flux tube increases as the expansion redshifts the energy density of the field. Thus the field acts against gravitational infall.

C. Pure-magnetic and magnetized isocurvature perturbations

We have seen that the magnetic field introduces nonadiabatic modes in the density perturbations. This means that the field itself can generate fluctuations in the density, even when there are no primordial density fluctuations. Thus, if

\[ \Delta(t_0) = 0 = \dot{\Delta}(t_0), \quad (69) \]

where \( t_0 \) is the epoch of magnetogenesis in the early radiation era, then nonzero \( \Delta \) will arise purely from the magnetic field; in the absence of magnetogenesis, Eq. (69) would imply \( \Delta = 0 \) for \( t > t_0 \). These nonadiabatic pure magnetic density perturbations can be found explicitly from the solutions given above. On superhorizon scales (assuming that the field is created on these scales at \( t_0 \)), the general solution in Eq. (55) implies with the initial conditions in Eq. (69) that the pure-magnetic nonadiabatic mode is (to lowest order in \( c_a^2 \))

\[
\Delta_{pm} = -\frac{1}{4} \left[ C(0) + C(2-) \right] \left( \frac{a}{a_0} \right)^2 \left[ 1 + 2 \left( \frac{a}{a_0} \right)^{-3 + c_a^2} \right] + C(0) + C(2-) \left( \frac{a}{a_0} \right)^{-c_a^2}.
\]

(70)

The pure-magnetic density perturbations have a dominant growing mode of the same strength as in the non-magnetized adiabatic case. The decaying modes are in fact the isocurvature part of the pure-magnetic density perturbations, as we now show.

Equation (69) is often taken to characterize isocurvature perturbations, but it does so only in specific cases [28]. For magnetized perturbations, this is not the isocurvature condition. Isocurvature density perturbations are those for which the curvature perturbation of the initial hypersurface orthogonal to the fluid flow is spatially constant, i.e., \( (\alpha \Delta_K)(t_0) = 0 \). (There is also the implicit condition that \( \omega_a = 0 \), which is necessary for the existence of the spatial hypersurface.) Taking the the comoving divergence of Eqs. (37) and (40), we find the condition for magnetized isocurvature perturbations:

\[ \dot{\Delta} + \frac{3}{2} (1 - w) H \Delta = \frac{3}{4} (1 - w) c_a^2 H B - c_a^2 H K \quad \text{at} \quad t = t_0. \]

(71)
In the non-magnetized case $c_a = 0$, it is clear that this condition is satisfied by Eq. (69), but when $c_a > 0$, then Eq. (69) does not characterize isocurvature perturbations.

As an example, consider the implication of the magnetized isocurvature condition in the dust era, on scales well above the Alfvén horizon but well within the Hubble horizon, i.e., $k_{h0} \ll k \ll k_{a0}$. Then Eqs. (71) and (66) give, to lowest order,

$$C_{(+)} = \frac{2}{5} \left( c_a^2 \right)_0 \left[ C_{(-)} + C_B \right] - \frac{1}{5} C_K.$$ 

On superhorizon scales, Eq. (71) holds for all $t$ by virtue of Eq. (38), which implies $(aD_a K) = 0$. The magnetized isocurvature condition then selects a sub-class of the general superhorizon solutions found above. In the radiation era, we find (to lowest order in $c_a^2$)

$$\Delta_{iso} = C_{(1-)} \left( \frac{t}{t_0} \right)^{-\frac{1}{2} + \frac{1}{2} c_a^2} + C_{(2-)} \left( \frac{t}{t_0} \right)^{-\frac{1}{2} + \frac{1}{2} c_a^2}. \tag{72}$$

Equation (72) arises from the general superhorizon solution Eq. (55) by eliminating the non-decaying modes. In the dust era,

$$\Delta_{iso} = C_{(1-)} \left( \frac{t}{t_0} \right)^{-1} + C_{(2-)} \left( \frac{t}{t_0} \right)^{-\frac{3}{2}}, \tag{73}$$

which is a special case of the general superhorizon dust solution Eq. (65), once again without the growing mode.

It is clear from Eqs. (72) and (73) that magnetized isocurvature perturbations on superhorizon scales are purely decaying. They are very different from the pure-magnetic solution Eq. (70) that is based on the initial conditions in Eq. (69). The latter has a constant and a growing mode. Magnetized nonadiabatic perturbations on superhorizon scales can contribute to the growing mode and generate a constant mode, whereas the magnetized isocurvature perturbations are purely decaying.

**IV. MAGNETIZED COSMIC VORTICES AND ALFVÉN WAVES**

In the previous section, we generalized the results given in [15], which itself provided a relativistic extension of previous work on magnetized density perturbations. A general inhomogeneous perturbation is characterized not only by its magnitude, i.e. the density perturbation $\Delta$, but also by its rotation and deformation properties, as described in general terms in [18,19]. Recently, these properties were investigated in CDM, and it was shown how the small stresses (isotropic and anisotropic) from residual velocity dispersion can have an important effect on rotation and deformation, even though the effect on density perturbations is effectively negligible [27].

The evolution equations for rotational and deformation variables in an imperfect fluid were derived in [19]. The evolution equations for inhomogeneities were coupled to causal transport equations for viscosity and heat conduction. By Eq. (2), a magnetized perfect fluid can be considered as an imperfect fluid with anisotropic stress, and the equations of [19] may be specialized to this case. However, the system needs to be completed by evolution equations for the magnetic stress, which are determined by Maxwell’s equations. Here we investigate the coupled equations governing rotational and deformational inhomogeneity in the fluid and magnetic field.

The comoving gradient of the density inhomogeneity $\Delta_a$ splits irreducibly as

$$aD_b \Delta_a = \frac{1}{3} \Delta h_{ab} + \varepsilon_{abc} W^c + \xi_{ab}.$$  

The density perturbation is the comoving divergence, the rotational part is given by the comoving curl and the deformation part is the comoving PSTF derivative:

$$\Delta = aD^a \Delta_a, \quad W_a = -\frac{1}{2} a \text{curl} \Delta_a, \quad \xi_{ab} = aD_{(a} \Delta_{b)}.$$ 

The vector $W_a$ governs rotational instabilities in the density distribution of the matter, and $\text{div} W = 0$. On the other hand, $\xi_{ab}$ determines the volume-true anisotropic distortion, with $h^{ab} \xi_{ab} = 0$. Both quantities describe differential, i.e., infinitesimal, properties. Here we focus on rotation, and in the next section we look at anisotropic deformations.

A fundamental property of rotational perturbations is that they are proportional to the vorticity vector. This arises from the identity Eq. (7) which ensures that the curl of any gradient field, such as $\Delta_a$, derives from the vorticity. It follows that
\[ W_a = -3a^2 H (1 + w) \omega_a . \] (74)

Not only \( W_a \), but also rotational perturbations in magnetic density and expansion inhomogeneities, are parallel to the vorticity:

\[ \text{curl} \mathbf{B}_a \equiv \text{curl} \left( a \mathbf{D}_a \right) \mathbf{B}^2 = \frac{4}{3(1 + w)} W_a , \text{curl} \Theta_a \equiv \text{curl} a \mathbf{D}_a \Theta = -\frac{\dot{H}}{(1 + w)H} W_a . \]

Thus all rotational instability in the magnetized medium arises from vorticity. As we have seen from the vorticity propagation equation (25), magnetic inhomogeneities can source vorticity. Rewriting Eq. (25) using Eq. (74), we have

\[ \dot{W}_a + \frac{3}{2}(1 - w)HW_a = \left( \frac{3a^2H}{2\rho} \right) B^b D_b \text{curl} B_a . \] (75)

Thus the field is a source of vorticity provided that its curl varies along its force lines. Furthermore, the effect of the field is to induce precession of the rotational vector \( W_a \). In the absence of the field, \( \dot{W}_a \) remains parallel to \( W_a \), so that the initial direction is preserved along the fluid flow. By contrast, in the magnetized case, \( W_a \) is no longer parallel to \( \dot{W}_a \), and the initial direction changes along the fluid flow. The rate of precession is

\[ \nu_{\text{prec}} = \frac{|B^b D_b \text{curl} B_a|}{2(1 + w)\rho \omega_a} . \] (76)

Equation (75) shows how the magnetic field can become a source of density vortices. However, the field effect upon pre-existing rotational perturbations is not clear yet. To quantify the magnetic influence on \( W_a \) we need to go one step further and obtain a decoupled equation for the evolution of \( W_a \). We take the curl of Eq. (43), using the above results and the identities in Eqs. (7), (10) and (14), and we arrive at the required evolution equation (with \( \Lambda = 0 \)):

\[ \ddot{W}_a + (4 - 3w) HW_a + \frac{1}{2} \left[ 1 - 7w + 3c_s^2(1 + w) \right] \rho W_a = \left[ \frac{c_s^2}{3(1 + w)} \right] \mathbf{D}^2 W_a . \] (77)

This is a wave equation for \( W_a \), with signal speed \( v_a \) given by

\[ v_a^2 = \frac{c_s^2}{3(1 + w)} . \]

Propagating solutions of this equation are Alfvén waves (compare [8]), i.e., incompressible, vector waves, as opposed to the compressible, scalar magneto-sonic waves. In the non-magnetized case, the signal speed vanishes, and no wave solutions exist. We note also that the only scale-dependence arising in the wave equation is via the magnetic \( c_s^2 \) term. Thus magnetized vortices are scale-dependent, unlike the non-magnetized case.

Decomposing the solenoidal vector \( W_a \) into Fourier modes \( W \), Eq. (77) gives

\[ \dddot{W} + (4 - 3w) HW\dot{W} + \left\{ \frac{1}{2} \left[ 1 - 7w + 3c_s^2(1 + w) \right] \rho + \frac{c_s^2 k^2}{3(1 + w)a^2} \right\} W = 0 . \] (78)

Clearly, \( W_a \) can only grow if the term in square brackets becomes negative. If \( \dot{w} = 0 \), as in the radiation and dust eras, then the quantity \( 1 - 7w + 3(1 + w)c_s^2 = (3w - 1)(w - 1) \) becomes negative when \( \frac{1}{3} < w < 1 \). Thus it is only when matter stiffer than radiation dominates the universe (and is coupled to the magnetic field), that vortices in the density distribution can grow (in the linear regime). Furthermore, the presence of the magnetic field ensures that such growth occurs only on scales larger than the critical “rotational Jeans” wavelength

\[ \lambda_{\text{J}} = 2\pi c_s \left[ \frac{2}{3(3w - 1)(1 - w^2)\rho} \right]^{1/2} , \]

which is small due to the weakness of the field.

---

3 A similar equation was derived in [19] for a fluid with shear viscosity; in that case \( v^2 = \eta/\tau \rho (1 + w) \), where \( \eta \) is the viscosity and \( \tau \) is the causal relaxation time.

14
In the radiation era, Eq. (78) becomes

\[ \mathcal{W}'' + 2 \left( \frac{a_0}{a} \right) \mathcal{W}' + \left( \frac{k}{k_{a0}} \right)^2 \mathcal{W} = 0, \]

where a prime denotes \(d/d(a/a_0)\), and \(k_a = 2k_h/c_a\) is the wavenumber of the Alfvén horizon \(\lambda_a = \frac{1}{2}c_a\lambda_h\). This has the general solution

\[ \mathcal{W} = \frac{a_0}{a} \left[ C_1 \cos \left( \frac{k}{k_{a0}} a_0 \right) + C_2 \sin \left( \frac{k}{k_{a0}} a_0 \right) \right], \quad (79) \]

which describes Alfvén waves. The Alfvén frequency

\[ \nu_a = \frac{H_0}{\pi} \left( \frac{k}{k_{a0}} \right)^2 = \frac{1}{2} \delta \nu_{ac}, \quad (80) \]

where \(\delta \nu_{ac} = \nu_{ac,\text{mag}} - \nu_{ac}\) is the excess magnetic acoustic frequency given in Eq. (58). Thus, local differential vortices in the density distribution are “flip-flopping” in concert with the acoustic oscillations in the density perturbations. Alfvén waves are a purely magnetic effect, arising from the fluctuations in the magnetic field direction. These waves have decaying amplitude, in common with non-magnetized (and non-propagating) vector perturbations.

On scales beyond the Alfvén horizon, \(k \ll k_{a0}\), the oscillatory behavior is not felt, and Eq. (79) gives, to lowest order,

\[ \mathcal{W} = C_1 \left( \frac{a_0}{a} \right) - \frac{1}{2} \left( \frac{k}{k_{a0}} \right)^2 \left( \frac{a}{a_0} \right) + C_2 \left( \frac{k}{k_{a0}} \right). \]

On superhorizon scales, the oscillations disappear: \(\mathcal{W} \rightarrow C_1(a_0/a)\), regaining the standard non-magnetized result. Thus, before matter-radiation equality, and on scales much larger than the Alfvén horizon, density vortices evolve unaffected by the presence of a cosmological magnetic field.

After equality, in the matter-dominated dust era, Eq. (78) becomes

\[ 3 \left( \frac{t}{t_0} \right)^2 \mathcal{W}'' + 8 \left( \frac{t}{t_0} \right) \mathcal{W}' + \left[ 2 + \left( \frac{k}{k_{a0}} \right)^2 \right] \mathcal{W} = 0, \quad (81) \]

where a prime denotes \(d/d(t/t_0)\). The solution is

\[ \mathcal{W} = C_{(+)} \left( \frac{t}{t_0} \right)^{\alpha_+} + C_{(-)} \left( \frac{t}{t_0} \right)^{\alpha_-}, \quad (82) \]

where

\[ \alpha_{\pm} = \frac{1}{6} \left[ -5 \pm \sqrt{1 - 12 \left( \frac{k}{k_{a0}} \right)^2} \right]. \quad (83) \]

Therefore, any rotational instabilities present in the density distribution of the dust die away with time, as they do in non-magnetized cosmologies. The field effect on a given mode \(k\) is to reduce the depletion rate of \(W_a\) by an amount proportional to the initial Alfvén speed squared, \((c_a^2)_0\). Thus, magnetized dust universes will contain more residual vortices than magnetic-free ones. However, the effect is confined within a narrow wavelength band beyond the Alfvén horizon \(\lambda_a\). On much larger scales, the field influence becomes negligible, and \(W_a \propto t_0/t\) as in non-magnetized models.

On scales with \(k > k_{a0}/\sqrt{12}\), i.e. within a few times the Alfvén scale, Eq. (83) shows that the density vortices oscillate as Alfvén waves.

V. MAGNETIZED SHAPE-DISTORTION

We monitor anisotropic deformation (shape distortion) in the density distribution of the medium through the PSTF tensor \(\xi_{ab} = aD_{(a}\Delta_{b)}\). This is associated with density variations that do not represent matter aggregations, since
the associated divergence of $\Delta_a$ is zero, but rather describe changes in the local anisotropy pattern of the density gradients.

Distortion in the density is coupled to distortion in the expansion and the magnetic energy density, defined via the PSTF tensors

$$\dot{\vartheta}_{ab} = a^2 D_{(a}D_{b)}\Theta, \quad \beta_{ab} = \frac{a^2}{B^2} D_{(a}D_{b)}B^2, \quad \beta_{ab}$$

which vanish in the background and are thus gauge-invariant. The propagation equations for $\xi_{ab}$, $\vartheta_{ab}$ and $\beta_{ab}$ follow from the comoving PSTF derivatives of Eqs. (40)–(42): with $\Lambda = 0$, we have

$$\dot{\xi}_{ab} = 3w H \xi_{ab} - \frac{\rho}{\sqrt{\kappa}} \frac{\rho}{\kappa} \xi_{ab} + \frac{1}{2} c_s^2 H \kappa_{ab} - c_s^2 H \kappa_{ab} - 3H \mu_{ab}$$

$$\dot{\vartheta}_{ab} = -2H \vartheta_{ab} + \frac{1}{2} \rho \xi_{ab} - \frac{c_s^2}{(1 + w)} D^2 \xi_{ab} + \frac{1}{2} c_s^2 \rho \beta_{ab} - \frac{c_s^2}{2(1 + w)} D^2 \beta_{ab}$$

$$- \frac{1}{2} c_s^2 \rho \kappa_{ab} - \frac{3}{2} \rho \mu_{ab} - \left( 6c_s^2 + \frac{4c_s^2}{1 + w} \right) H \varpi_{ab}$$

$$\dot{\beta}_{ab} = \frac{4}{3(1 + w)} \xi_{ab} + \frac{4c_s^2 H}{(1 + w)} \xi_{ab}$$

The additional gauge-invariant PSTF tensors

$$\kappa_{ab} = a^2 R_{(a}B_{b)}$$

$$\varpi_{ab} = a^2 D_{(a}B_{b)c}\omega_{b)}$$

$$\mu_{ab} = \frac{a^2}{B^2} D_{(a}B_{b)c}B_{(ab)}$$

respectively describe distortions caused by projected curvature, rotation and by anisotropies in the distribution of the magnetic field gradients. The first is due to the natural coupling of the field to the curvature and is given by

$$\kappa_{ab} = a^2 \left( \frac{1}{2} \pi_{ab} - H \sigma_{ab} + E_{ab} \right)$$

obtained from Eq. (34) by means of the shear propagation equation (26). The second arises from the fluid flow, which generally is not hypersurface orthogonal. It has no impact on deformation if the rather special condition $D_{(a}B_{b)c}\omega_{b)} = 0$ holds. Finally, the effect of $\mu_{ab}$ vanishes when any anisotropies present in the distribution of $B_{ab} \equiv a_D B_{a}$ remain invariant along the magnetic force-lines, that is when $B^c D_c B_{(ab)} = 0$. Note that both the scalar and vector aspects of the field contribute to shape distortion.

Equations (85) and (86) combine to provide a second order differential equation, also obtained by taking the comoving PSTF derivative of Eq. (43). With $\Lambda = 0$, we have

$$\dot{\xi}_{ab} = - \frac{1}{(1 + w)} \xi_{ab} + \frac{c_s^2 + 3w}{(1 + w)} \xi_{ab} + \frac{c_s^2 H}{(1 + w)} \beta_{ab} - \frac{4H}{(1 + w)} \mu_{ab} - 2 \varpi_{ab} - D^2 \sigma_{ab}$$

$$\dot{\vartheta}_{ab} = - [4 - 3(1 + w)] H \mu_{ab} + \frac{c_s^2 H}{3(1 + w)} \xi_{ab} + \frac{(c_s^2 - 2w) c_s^2 H}{(1 + w)} \xi_{ab}$$

$$- \frac{c_s^2 H}{2(1 + w)} \beta_{ab} + \frac{c_s^2 H}{3(1 + w)} \kappa_{ab} + \frac{1}{2} c_s^2 a^2 D^2 \sigma_{ab}$$

$$\dot{\sigma}_{ab} = - \frac{2 - 3c_s^2 H}{2(1 + w)} H \varpi_{ab} + \frac{a}{2(1 + w)} D_{(a}D_{b)c}B_{(ab)} [B_{b)c} - B^c_{b)}$$

We used the propagation equations (30) and (21) for $E_{ab}$ and $\pi_{ab}$, which imply

$$\dot{E}_{ab} = -3H E_{ab} + \frac{3}{2} H \pi_{ab} - \frac{1}{2} (1 + w) \rho \beta_{ab} - D^2 \sigma_{ab} + \frac{1}{a^2} (\vartheta_{ab} + 2 \varpi_{ab})$$
on using the constraint equations (27) and (29). Equation (90) requires Eqs. (19) and (85). Finally, to obtain the evolution formula of $\xi_{ab}$ we have successively taken the comoving curl and the comoving PSTF derivative of Eq. (25).

The system of equations (87)–(91) provides in principle a complete description of linear infinitesimal shape distortion generated by magnetic effects, provided we have a prescription for the $\mathbf{D}^2\sigma_{ab}$ terms in Eqs. (89) and (90), and for the last term on the right of Eq. (91). Even without these terms, the system is too complicated to analyze in general. However, it is clear in general terms how magnetic effects will actively generate distortion. We can illustrate this by comparing with the non-magnetized case in a simple example.

For simplicity, consider superhorizon scales in the dust era (neglecting vorticity). Suppose that at a given event $(t_0, \vec{x}_0)$, we have no initial distortion or rate of distortion:

$$\langle \xi_{ab} \rangle_0 = 0 = \langle \dot{\xi}_{ab} \rangle_0. \quad (92)$$

In the non-magnetized case, the distortion system collapses to the single equation

$$\ddot{\xi}_{ab} = -2H\dot{\xi}_{ab} + \frac{1}{2}H^2\xi_{ab}.$$  

It follows that along the fluid flow line through $\vec{x}_0$, no distortion is generated:

$$\xi_{ab}(t, \vec{x}_0) = 0.$$  

The evolution is purely passive, or inertial, i.e., distortion can only develop if it is there a priori. In the magnetized case, by contrast, distortion is actively and nonadiabatically generated by magnetic effects. Equation (88) shows that $\mathbf{H}_0^{-2} \langle \xi_{ab} \rangle_0 = (c^2_\alpha)_0 \left[ -\frac{3}{2} (\kappa_{ab})_0 + 2 (\kappa_{ab})_0 \right] + 6 (\mu_{ab})_0$, so that $\langle \dot{\xi}_{ab} \rangle_0 \neq 0$. Distortion is immediately generated along the flow line. In fact, the distortion has a growing mode, as we now show.

The superhorizon scalar modes of the distortion tensors satisfy a system that follows from Eqs. (87) and (88)–(90):

$$\xi'' = -\frac{1}{3}(t_0 \ell) \xi' + \frac{4}{3} (t_0 \ell)^2 \xi - \frac{2}{3} (c^2_\alpha)_0 \left( \frac{t_0}{t} \right)^{8/3} [3\beta - 4\kappa] + \frac{8}{3} \left( \frac{t_0}{t} \right)^2 \mu, \quad (93)$$

$$\beta' = \frac{4}{3}\xi', \quad (94)$$

$$\kappa' = -\xi' + \frac{4}{3} (c^2_\alpha)_0 \left( \frac{t_0}{t} \right)^{5/3} \beta - \frac{8}{3} \left( \frac{t_0}{t} \right) \mu, \quad (95)$$

$$\mu' = -\frac{2}{3} \left( \frac{t_0}{t} \right) \mu + \frac{1}{3} (c^2_\alpha)_0 \left( \frac{t_0}{t} \right)^{2/3} \xi'. \quad (96)$$

Equations (94) and (96) integrate to

$$\beta = \frac{4}{3} \left( \xi + \Gamma_{\beta} \right), \quad \mu = \frac{4}{3} (c^2_\alpha)_0 \left( \frac{t_0}{t} \right)^{2/3} \left[ \xi + \Gamma_{\mu} \right], \quad (97)$$

where $\dot{\Gamma}_{\beta} = 0 = \dot{\Gamma}_{\mu}$. Then Eq. (97) transforms Eq. (95) into

$$\kappa' = -\xi' + \frac{8}{3} (c^2_\alpha)_0 \left( \frac{t_0}{t} \right)^{2/3} \left[ \xi + 2\Gamma_{\beta} - \Gamma_{\mu} \right]. \quad (98)$$

According to Eq. (97), the effect of any anisotropies present in the distribution of $B_{ab} = a D_a B_0$ on $\xi$ decreases after matter-radiation equilibrium. Since $\mu$ is a key source of shape-distortion, we expect the evolution of $\xi$ to approach that of $\Delta$ as the universe expands. Equation (93) gives

$$9 \left( \frac{t}{t_0} \right)^3 \xi'' + 36 \left( \frac{t}{t_0} \right)^2 \xi'' + 14 \left( \frac{t}{t_0} \right) \xi' - 4\xi = 0, \quad (99)$$

on using Eqs. (97) and (98). This has the same form as the corresponding density perturbation equation (64), and the solution is thus of the form Eq. (65). Imposing the initial conditions in Eq. (92), we find that

$$\xi(t, \vec{x}_0) = \Gamma \left[ \left( \frac{t}{t_0} \right)^{2/3} + 4 \left( \frac{t}{t_0} \right)^{-1} - 5 \left( \frac{t}{t_0} \right)^{-2/3} \right],$$

where $\Gamma$ is a constant. Thus the shape distortion has a growing mode along the fluid flow line, due purely to magnetic effects; in the absence of the magnetic field, $\Gamma = 0$. (A similar situation arises in the simpler case of distortion generated by velocity dispersion [27].)
VI. CONCLUSION

We have given a fully general relativistic treatment of the scalar and vector effects of a weak large-scale magnetic field on cosmological density inhomogeneity. This refines the results of [15] on magnetized density perturbations, and extends that work to analyze magnetized vortices and shape distortion in the density distribution. Our covariant Lagrangian approach allows us to derive gauge-invariant evolution equations for all these aspects of density inhomogeneity in the general case, i.e., incorporating all fluctuations of, and couplings between, the field, the fluid and the curvature. In summary, magnetized density perturbations are governed by Eqs. (44)–(46), magnetized density vortices are governed by Eq. (77), and magnetized shape distortion is governed by Eqs. (87)–(91).

We give the solutions in closed form for magnetized density perturbations and vortices, in the radiation and dust eras. Some of the scalar solutions and all of the vector solutions are new. For magnetized shape distortion, we found a special solution with a growing mode. Our main results are:

1. The magneto-curvature coupling, which is a direct consequence of the field’s vectorial nature, first identified in [15], has an important influence. In particular, on superhorizon scales in the radiation era, this coupling slightly enhances the growing mode of density perturbations, as shown in Eqs. (50) and (54):

\[
\Delta = C_{(+)} \left( \frac{a}{a_0} \right)^{2+\frac{4}{3}c_s^2} + C_{(-)} \left( \frac{a}{a_0} \right)^{-1+c_s^2} + C_{(0)} + C_{(2-)} \left( \frac{a}{a_0} \right)^{-c_s^2}.
\]

When the coupling is neglected, the growing mode is incorrectly found to be damped relative to the non-magnetized case.

In the dust era, the coupling produces a new nonadiabatic decaying mode, but does not affect the non-magnetized adiabatic modes [see Eq. (65)].

2. Magneto-sonic waves in the radiation era are given in exact form in Eq. (56),

\[
\Delta = \left[ C_{(1)} - C_K \text{Si} \left( \frac{\beta k}{k_{h0} a_0} \right) \sin \left( \frac{\beta k}{k_{h0} a_0} \right) \right. \\
\left. + \left[ C_{(2)} - C_K \text{Ci} \left( \frac{\beta k}{k_{h0} a_0} \right) \right] \cos \left( \frac{\beta k}{k_{h0} a_0} \right) \right] - C_B e_c.
\]

This shows the nonadiabatic modulation of the amplitude and increase in the frequency of acoustic oscillations. These effects, together with the nonzero average value implied by the \( C_B \) term, have potentially important implications for the CMB acoustic peaks, some of which have been investigated in [7].

3. In the dust era, subhorizon magnetized density perturbations are given exactly in Eq. (66), which leads to the magnetized Jeans scale in Eq. (68):

\[
\lambda_{mJ}(t_0) = \frac{8}{3\pi} \sqrt{6} (c_a)_0 \lambda_{h0}.
\]

On scales such that \( \lambda_{mJ} \ll \lambda \ll \lambda_h \), the density perturbations are

\[
\Delta = C_{(+)} \left( \frac{t}{t_0} \right)^{\frac{2}{3} - \epsilon} + C_{(-)} \left( \frac{t}{t_0} \right)^{-1+\epsilon} + C_C \left( \frac{t}{t_0} \right)^{-2/3} - \frac{5}{2} e C_B,
\]

where \( \epsilon = \frac{2}{3}(k/k_{\text{min}})^2 \). This shows the small damping effect on the adiabatic growing mode, as well as the new nonadiabatic modes. These results imply small modifications to structure formation in the linear regime. However, they are limited by the fact that we have neglected any non-baryonic matter or cosmological term.

4. Pure-magnetic density fluctuations, which are induced in an initially smooth fluid by magnetogenesis, are given on superhorizon scales by Eq. (70). This solution would be important in any attempt to model large-scale structure formation as seeded by magnetogenesis.

5. Magnetized isocurvature perturbations are characterized by Eq. (71). On superhorizon scales, these modes are purely decaying.
6. The field is a source of incompressible rotational instabilities, and the condition for this to happen is given via Eq. (75). Magnetized density vortices are shown be scale-dependent and to precess, at a rate given by Eq. (76). The general propagation equation for these vortices (i.e., incorporating all relevant effects) is given by Eq. (77):

\[ \ddot{W}_a + (4 - 3w) H \dot{W}_a + \frac{1}{2} \left[ 1 - 7w + 3c_s^2(1 + w) \right] \rho W_a = \left[ \frac{c_a^2}{3(1 + w)} \right] D^2 W_a. \]

In the radiation era, the Alfvén wave solutions are given exactly in Eq. (79). The Alfvén frequency and wave-speed are

\[ \nu_a = \frac{H_0}{\pi} \left( \frac{k}{k_a} \right)^2, \quad v_a = \frac{1}{2} c_a. \]

These results generalize some of the theoretical results of Durrer et al. [9], who then go further and apply the results to determine the effect of Alfvén wave modes on CMB anisotropies.

7. After recombination, magnetized density vortices are given exactly in Eq. (82). They decay like their adiabatic counterparts, but at a slower rate, so that rotational instability persists for longer in a magnetic universe. This will have a small effect on structure formation in the linear regime.

8. Finally, we have investigated for the first time magnetic effects on infinitesimal shape distortion in the density distribution. The magnetic influence is manifold. Anisotropies in the field energy density, together with those in the distribution of the magnetic vector itself are direct sources of density deformation. The field’s coupling to curvature and rotation also acts as an indirect source of magnetically induced shape distortions. Following the evolution of shape-distortion along the worldline of a fluid element, we showed that the field is an active source of distortion. On superhorizon scales, we showed via a special solution of the shape-distortion system that there is a growing mode of magnetized shape-distortion [see Eq. (99)].

Unlike the magnetic effects on density and rotational perturbations, which are small corrections of the non-magnetized results, magnetic effects on shape-distortion constitute a significant change from the non-magnetized (and passive) case. (A similar statement applies in the case of velocity dispersion effects in CDM [27].) The results on magnetized shape-distortion have potentially important implications for (linear) structure formation. Not only is distortion actively generated once scales re-enter the Hubble horizon and begin to collapse, but it is also actively generated while the scales are beyond the horizon. Of course, the shape distortion in the linear regime will be overwhelmed by effects that arise during the nonlinear stages of collapse.

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