Critical Phenomena Inside Global Monopoles

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I. INTRODUCTION

Quite an industry has developed around the study of nonlinear phenomena occurring at the threshold of black hole formation first discovered by Choptuik [1]. A number of reviews detail the substantial progress in understanding these phenomena [2,3]. By specifying a form of initial data parameterized by some parameter \( p \), numerical evolutions can determine whether the value of \( p \) is sufficient to form a black hole. Such a solution is considered super-critical, while for small \( p \), no black hole forms and the solution is sub-critical. Separating super-critical solutions from sub-critical solutions is the critical solution parameterized by the critical value \( p^* \). In analogy with statistical mechanics where the black hole mass represents the order parameter, cases where infinitesimal mass black holes are formed are called Type II, and cases in which only finite mass black holes are formed are called Type I.

Here, I present a new Type II critical solution occurring within a model with a triplet scalar field. By assuming a hedgehog ansatz for the triplet, I find a critical solution which is, in a strict sense, non-spherically symmetric. Adding a potential allows the study of critical phenomena within a global monopole. No Type I critical behavior around the static monopole solutions is observed.

As a generalization of Choptuik’s original model, the triplet scalar field contains the single scalar field model under consideration here. First, I choose a triplet scalar field \( \Phi^a \) accompanied by the usual symmetry-breaking potential with vacuum value \( |\Phi^a| = \eta \) and coupling \( \lambda \). For certain evolutions discussed later, I also introduce a free, massless scalar field \( \psi(r,t) \) which couples to \( \Phi^a \) only through gravity.

With the most general form of the equations of motion, Section III then discusses various aspects of regions of the parameter space. In particular, Section III A presents a new critical solution obtained with the hedgehog ansatz, and Section III B discusses the stability of this solution with respect to the addition of \( \psi \), a free scalar field. Section III C addresses the results obtained in the interior of a monopole. I then conclude in Section IV.

II. THE MODEL

Letting \( \Phi^a \) represent an \( SO(3) \)-valued triplet scalar field, and \( \psi \) a single-valued, free, massless scalar field, the Lagrangian for the model is

\[
L = -\frac{1}{2} \psi_{\mu} \psi^\mu - \frac{1}{2} \Phi^a_{\mu} \Phi^a_{\mu} - \frac{1}{4} \lambda \left[(\Phi^a)^2 - \eta^2\right]^2 ,
\]

where \( \lambda \) is the coupling to the symmetry-breaking potential, and \( \eta \) is the scale of symmetry-breaking. The stress-energy tensor is then

\[
T_{\mu\nu} = \psi_{\mu} \psi_{\nu} - \frac{1}{2} g_{\mu\nu} \psi^\rho \psi_\rho + \Phi^a_{\mu} \Phi^a_{\nu} - \frac{g_{\mu\nu}}{2} \left(\Phi^a_{\mu} \Phi^a_{\nu} + \frac{1}{2} \lambda \left[(\Phi^a)^2 - \eta^2\right]^2\right)
\]

and the equations of motion for the matter fields are

\[
\Box \psi = 0
\]

\[
\Box \Phi^a = \lambda \Phi^a \left[(\Phi^a)^2 - \eta^2\right] .
\]
The hedgehog ansatz is assumed for the triplet scalar field

\[ \Phi^a = f(r, t) \hat{r} = f(r, t) \begin{pmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{pmatrix}, \] (5)

such that the triplet field is defined to be a vector in internal space of magnitude \( f(r, t) \) pointing everywhere in the radial direction, \( \hat{r} \). None of the fields \( \Phi^a \) is a function only of \( r \) and \( t \), but the magnitude \( f \) is. Thus, while highly symmetric, the model is not spherically symmetric in the strictest sense. With this hedgehog ansatz, a monopole of unit charge is obtained by enforcing the boundary condition \( f(\infty, t) = \eta \). When \( f(\infty, t) = 0 \), no monopole charge exists.

I choose the spherically symmetric metric

\[ ds^2 = -\alpha^2(r, t) dt^2 + \alpha^2(r, t) dr^2 + r^2 d\Omega^2, \] (6)

and introduce auxiliary variables

\[ \Pi_\psi \equiv \frac{a}{a} \dot{\psi}, \quad \Phi_\psi \equiv \psi', \] (7)
\[ \Pi_f \equiv \frac{a}{a} \dot{f}, \quad \Phi_f \equiv f', \] (8)

where an overdot denotes \( \partial/\partial t \) and a prime denotes \( \partial/\partial r \).

The full system of equations then becomes

\[ \ddot{\psi} = \frac{a}{a} \Pi_\psi, \quad \dot{\Phi}_\psi = \left( \frac{a}{a} \Pi_\psi \right)', \] (9)
\[ \ddot{\Pi}_\psi = \frac{1}{r^2} \left( \frac{a}{a} r^2 \Phi_\psi \right)' \] (10)
\[ \ddot{\Pi}_f = \frac{1}{r^2} \left( \frac{a}{a} r^2 \Phi_f \right)' - \alpha a \lambda f \left( f^2 - \eta^2 \right) - \frac{2 a \alpha f}{r^2} \] (11)
\[ \frac{a'}{a} = \frac{1}{2r} - 2 \pi \left( \Pi_f^2 + \Phi_f^2 + \Pi_\psi^2 + \Phi_\psi^2 \right) \] (12)
\[ + 4 \pi a^2 \left[ \frac{f^2}{r} + \frac{r}{4} \lambda (f^2 - \eta^2)^2 \right] \] (13)
\[ \frac{a'}{a} = \frac{2 - 1}{2r} - 2 \pi r \left( \Pi_f^2 + \Phi_f^2 + \Pi_\psi^2 + \Phi_\psi^2 \right) \] (14)
\[ - 4 \pi a^2 \left[ \frac{f^2}{r} + \frac{r}{4} \lambda (f^2 - \eta^2)^2 \right] \]
\[ \dot{a} = 4 \pi r \alpha \left( \Phi_f \Pi_f + \Phi_\psi \Pi_\psi \right). \] (15)

In addition to these equations of motion, appropriate boundary conditions are needed. At the origin of coordinates, regularity is enforced with the conditions

\[ \Phi_\psi(0, t) = 0 \]
\[ f(0, t) = 0 \quad \Phi_f(0, t) = 0 \]
\[ a(0, t) = 1 \quad a'(0, t) = 0 \] (16)
\[ \Phi_f(0, t) = 0. \]

Further, a mass aspect ratio \( m(r, t) \) is defined by associating the \( g_{rr} \) metric components of (6) and Schwarzschild

\[ a^2 = \left( 1 - \frac{2m}{r} \right)^{-1}. \] (17)

The quantity \( m(r, t) \) then describes the amount of mass contained within the radius \( r \) at time \( t \).

At large radius, an outgoing radiation condition is applied to the scalar field \( \psi \). For non-vanishing potential, the field \( f(r, t) \) has an effective mass which makes finding a good outer boundary condition difficult. While the use of an outgoing radiation boundary condition worked well in some situations, in general the value of \( f \) at the outer boundary was simply held constant which sufficed whether or not a monopole was present. Reflection off this boundary was often significant so that the maximum radius was kept relatively large. This prevented reflections from affecting the dynamics near the origin.

### III. RESULTS

The field equations (9-15) are then integrated with a second-order accurate, Crank-Nicholson finite difference code. The generated solutions are found to be second-order convergent, stable, and mass conserving. Because either \( \lambda \) or \( \eta \) can be rescaled away, I choose either \( \lambda = 0.1 \) or \( \lambda = 0 \) for all that follows. As almost all other studies of Type II critical behavior have, I have made use of an adaptive mesh. In particular, I incorporated finite difference approximations to the equations (9-15) into the adaptive infrastructure developed by Choptuik [1].

#### A. A New DSS Critical Solution

As the Lagrangian presented in (1) is quite general, I first discuss results obtained with vanishing scalar field \( \psi \). Specifically, if \( \psi(r, t) \) is initialized initially to zero with vanishing time derivative it will remain vanishing. This simplification allows the study of the collapse of the triplet field alone.

I also postpone discussion of collapse within a monopole until Section III C. Hence, in this section no monopole is present and \( f(\infty, t) = 0 \).

I find as the threshold of black hole formation is approached, a discretely self-similar solution is found with period \( \Delta = 0.46 \). A solution which approaches this precisely critical solution is shown in Fig. 1. In the limit of precise criticality, the solution is expected to exhibit an infinite number of echoes and to be singular at the origin at the time of collapse, occurring at a finite proper time.

The discrete nature of the self-similarity is demonstrated in Fig. 2. This solution also exhibits the usual mass scaling relationship in the super-critical regime, demonstrated in Fig. 3.
FIG. 1. Marginally sub-critical solution at late time for η = 0. The top frame displays the hedgehog profile f, the middle frame shows the quantity dm/dr explained in the text, and the last frame shows the value of 2m/r.

FIG. 2. Demonstration of scale-periodicity of the new critical solution with ∆ = 0.46. Here, the field f is shown at four different times for a marginally sub-critical solution. The times and the value of ∆ were chosen to minimize the difference f(ln(r),T₀) − f(ln(r) + ∆,T₁). Independently, the value of ∆ was also computed by ∆ = ln((T' − T_n)/(T' − T_{n+1})) where T represents the central proper time and T' the estimated critical time of collapse. The other two profiles are then shown shifted in ln(r) by n∆. The agreement demonstrates the scale periodicity.

FIG. 3. Demonstration of black hole mass scaling for η = 0. The circles indicate the mass of black holes formed versus a normalized distance from criticality. The line indicates a least-squares fit for slope γ = 0.119 where M_{BH} ∝ |p − p'|^γ.

Numerical results indicate that variations of η in the interval [0, 0.5] do not alter the nature of the critical solution. In other words, the discrete nature remains with ∆ = 0.46. The fact that the triplet field’s value for large radius vanishes while the true vacuum lies at η implies that the mass of the spacetime is divergent with increasing r. Hence, the profiles of m' and 2m/r as shown in Fig. 1 are not the same as when η = 0, differing for large r. However, the phenomena associated with approaching criticality occur for ever decreasing r, and in this regime, the solution remains discretely self-similar with period ∆ = 0.46. This result is consistent with the potential not playing a role in the kinetic-energy dominated critical behavior. In the language of [3], the potential becomes “asymptotically irrelevant.”

This divergence of the mass is not as problematic as it might initially appear because the numerical grid is artificially limited to some finite value of r. The mass is therefore limited, but this cut-off should not affect results for large r because of the distance needed for any effect to travel from the outer boundary. By comparing otherwise identical evolutions with different maximum radius, no such effect was seen.

It is interesting to note however that when η ≠ 0 and f(∞, t) = 0, the value of 2m/r was seen to be divergent. This situation represents an infinite mass spacetime (essentially an infinity of false vacuum energy) and should have a horizon at some radius where 2m/r → 1. Hence, it would seem that the critical behavior observed here is occurring within this horizon.
B. Addition of a regular scalar field

An interesting result found by Choptuik in his study of collapse of a Yang-Mills field, is that the presence of even an infinitesimal amount of scalar field added to the Einstein Yang-Mills model drives the solution to the scalar field’s critical solution at criticality \[8\]. Thus, the Yang-Mills field is unstable to the growth of the scalar field at criticality.

To test for a similar effect here, I allow \(\psi\) to be non-vanishing and look for the critical solution. Specifically, allowing the fields \(f(r,t)\) and \(\psi(r,t)\) to be arbitrary Gaussian pulses, I can “mix” the amplitudes with a parameter \(\sigma\) such that for \(\sigma = 0\), the field \(\psi\) vanishes and for \(\sigma = 1\), the field \(f\) vanishes.

![FIG. 4. Transition of critical solution from \(\Delta = 0\) to \(\Delta = 3.44\). A series of near-critical solutions are shown for \(\sigma = 0.4\) (solid), \(\sigma = 0.5\) (dotted), and \(\sigma = 0.6\) (dashed). The critical solution for small \(\sigma\) is the new DSS, and as \(\sigma\) increases, the critical solution becomes the original DSS. In the transition region, near-critical solutions appear to demonstrate structure from both critical solutions.](image1)

For the EYM case, any initial data for which \(\sigma \neq 0\) results in the scalar field critical solution being found \[8\]. Here, no such sensitivity is observed. Instead, within some finite range of \(\sigma\) the critical solution is observed to transition between the DSS of the scalar field (\(\Delta = 3.44\)) and the new DSS presented here (\(\Delta = 0.46\)).

As \(\sigma\) is varied across the transition from the original DSS to the new one, the fields \(f\) and \(\psi\) exchange dominance, both echoing at their respective value of \(\Delta\). In the transition region, the near-critical solution, as represented by the field \(m'(r,t)\), displays this dual echoing and hence has a non-trivial structure. This transition is shown in Fig. 4.

C. Critical Phenomena Inside a Monopole

In contrast to the cases presented above in which \(f(\infty,t) = 0\), I consider initial data which have unit monopole charge as indicated by \(f(r,t)\) asymptoting to \(\eta\). Here, collapse within a monopole is modeled by first solving for the static monopole and then adding an ingoing Gaussian perturbation to the profile \(f(r,t)\). This solution has the benefit of not being dynamic near the outer boundary. The Gaussian pulse is then parameterized by an amplitude \(p\), and a search is conducted in \(p\) for a near critical solution.

As stated, these evolutions should have the same critical solution as those lacking the potential. This is verified numerically for \(\eta \in [0,0.2]\). For \(\eta\) large (\(\eta \gtrsim 0.20\)), no horizon-less, static solutions exist \[9\] and hence the code is able to model these monopoles only within their horizons. A near-critical solution for \(\eta = 0.15\) is shown in Fig. 5.

![FIG. 5. Near-critical solution obtained within a monopole for \(\eta = 0.15\). Notice that the field \(f\) asymptotes to \(\eta\) for large \(r\) indicating the unit charge of the configuration. The discrete nature with \(\Delta = 0.46\) is observed to be the same as in the vanishing potential case. In comparison to the \(\eta = 0\) case (see Fig. 1), in the scaling regime (small \(r\)) the echoing behavior is evident, and in the large \(r\) regime the solutions are different.](image2)

That static monopoles exist in this model naturally leads one to question whether any Type I critical phenomena exists. However, evolutions appear to indicate that the static monopoles with no horizon are not unstable to collapse to a black hole. Hence, no such Type I critical phenomena has been observed.
A new critical solution is found by assuming a hedgehog ansatz for the triplet scalar field. The critical solution exhibits discrete self-similarity with echoing period $\Delta = 0.46$. The solution also exhibits power-law mass scaling with exponent $\gamma = 0.119$. Results verify that the potential does not affect the critical behavior as expected.

Furthermore, the threshold of black hole formation is examined within a global monopole. Though the potential is expected to play no role in Type II critical behavior, it does provide for the existence of static solutions, namely global monopoles. These static solutions then provide for the possibility that Type I behavior is present. The static solutions would have to be unstable to black hole formation for this Type I behavior to exist. However, evolutions do not show any instability of the monopole to collapse, and no Type I critical behavior is seen. A linear perturbation analysis should be sufficient to settle the question of their stability. If the static solutions are indeed stable, then no Type I behavior would be expected.

Finally, as discussed in [10] symmetries of the initial data can determine which critical solution is attracting. For the spherically symmetric complex scalar field, in addition to Choptuik’s original critical solution, another critical solution exhibiting a continuous self-similarity (CSS) exists [11]. An arbitrary family of initial data will find Choptuik’s solution, not the CSS. However, certain types of initial data which maximize the $U(1)$ charge density are unable to shed their charge density and find instead the CSS.

Here, instead of a complex scalar field (i.e. a doublet field), the single scalar field is extended to a triplet field. Within the triplet scalar field model before making any assumptions on the triplet, at least two critical solutions exist, both DSS. The appropriate question is then will arbitrary families of initial data find Choptuik’s original solution, the one presented here, or yet some other solution. To answer this with dynamical evolutions would require a general, three-dimensional collapse code.

As a first step towards the goal of answering this question, Martin-Garcia and Gundlach [12] have shown that non-spherical perturbations to Choptuik’s DSS decay and hence that at least some families of non-symmetric initial data will find Choptuik’s DSS. However this work considered only a single scalar field, and does not address the case where a triplet is present. The question of the relative stability of the original DSS to this new DSS essentially reduces to determining whether a family of generic, three-dimensional initial data for a triplet scalar field will shed its “in-phase” component to become the hedgehog DSS, or instead shed all its “out-of-phase” (i.e. non-zero monopole charge density) component to become the scalar field DSS.

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