MAXIMUM REDSHIFT AND MINIMUM ROTATION PERIOD OF NEUTRON STARS

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Abstract. The lower bound on the period of uniform rotation of neutron stars (NS) with causal equation of state (EOS), \( P_{\text{min}} \), is shown to be determined, with a high precision, by the upper bound on the surface redshift of static neutron star models, \( z_{\text{max}} \). We obtain \( z_{\text{max}} = 0.851 \), which turns out to be about 5\% lower than upper bound obtained by Lindblom (1984). Using then the empirical formula for the maximum rotation frequency of uniform rotation of neutron star models with realistic causal EOS, derived by Lasota et al. (1996), we establish an approximate relation between \( z_{\text{max}} \) and \( P_{\text{min}} \), in which \( P_{\text{min}} \) is proportional to the maximum mass of static NS, \( M_{\text{max}}^s \). In this way we reproduce, within 2\%, the exact formula of Koranda et al. (1997) based on extensive exact numerical calculations. Replacing \( M_{\text{max}}^s \) by the maximum measured NS mass, 1.442 \( M_\odot \), we get a lower bound \( P_{\text{min}} \simeq 0.288 \text{ ms} \).

1. Introduction

The lower limit on the period of a uniformly rotating neutron star depends on the largely unknown equation of state (EOS) of dense matter above the nuclear density. Therefore, an uncertainty in the high density EOS implies a large uncertainty in the minimum period of uniform rotation, \( P_{\text{min}} \) (see, e.g. [1], [2], [3], [4], [5]). Hence, it is, therefore, of interest to find a lower limit on \( P_{\text{min}} \), that is independent of the EOS. This limit results from the condition of causality, combined with the requirement that EOS yields neutron stars with masses compatible with observed ones (currently the highest accurately measured neutron star mass is \( M_{\text{obs}} = 1.442 M_\odot \) [6]). It will be hereafter referred to as \( P_{\text{min}}^{\text{CL}} \).

The first calculation of \( P_{\text{min}}^{\text{CL}} \) was done by Glendenning [7], who found the value of 0.33 ms. Glendenning [7], however, used a rather imprecise empirical formula, to calculate a lowest \( P_{\text{min}} \) by using the parameters (mass and radius) of the maximum mass configurations of a family of non-rotating neutron star models. His result, therefore, should be considered only as an estimate of \( P_{\text{min}}^{\text{CL}} \). Recently, Koranda et al. [8] extracted the value of \( P_{\text{min}}^{\text{CL}} \) from extensive exact calculations of uniformly rotating neutron star models. They have shown, that the method of Glendenning [7]...
overestimated the value of $P_{\text{min}}^{\text{CL}}$ by 6%. The result of Koranda et al. [8] calculations can be summarized in a formula

$$P_{\text{min}}^{\text{CL}} = 0.196 \frac{M_{\text{obs}}^{\text{max}}}{M_\odot} \text{ ms},$$

which combined with measured mass of PSR B1913+16 yields today’s lower bound for $P_{\text{min}} = 0.282$ ms. This absolute bound on the minimum period was obtained for the “causality limit (CL) EOS” $p = (\rho - \rho_0)c^2$, which yields neutron star models of the surface density $\rho_0$ and is maximally stiff ($dp/d\rho = c^2$) everywhere within the star; it does not depend on the value of $\rho_0$. In the present contribution we show that Eq. (1) can be reproduced using an empirical formula for $P_{\text{min}}$ derived for realistic causal EOS, combined with an upper bound on the relativistic surface redshift for static neutron stars with causal EOS.

2. Relation between $x_s$ and $P_{\text{min}}$

Numerical results for maximum frequency of stable uniform rotation of neutron stars, obtained by Salgado et al. [3], [4] for a broad set of realistic, causal EOS of dense matter, can be reproduced within better than 5% by an empirical formula [9]

$$\Omega_{\text{max}} \simeq 0.673 \left( \frac{GM_s}{R_s^3} \right)^{1/2} \text{ s}^{-1},$$

where $M_s$ and $R_s$ are, respectively, mass and radius of static configuration with maximum allowable mass. Numerical prefactor in Eq.(2) is independent of the EOS from the set considered in [3], [4].

Defining a relativistic compactness parameter

$$x_s \equiv \frac{2GM_s}{R_s c^2},$$

related to the surface redshift of static configuration with maximum allowable mass by

$$z_s = (1 - x_s)^{-1/2} - 1,$$

we can transform Eq. (2) into an approximate empirical formula, expressing the minimum period of rotation of neutron star, $P_{\text{min}} = 2\pi/\Omega_{\text{max}}$, in terms of $M_s$ and $x_s$,

$$P_{\text{min}} \simeq \frac{0.130 M_s}{x_s^{3/2} \frac{M}{M_\odot}} \text{ ms}.$$  

Clearly, for a given maximum mass of static configurations, $M_s$, the lower bound on $P_{\text{min}}$ will be obtained by replacing $x_s$ by its upper bound, $x_{s,\text{max}}$.

The formula for $P_{\text{min}}$ can be improved by using more precise empirical formula for $\Omega_{\text{max}}$. Such a formula was constructed by Lasota et al. [10]. As shown in [10], numerical results of [3], [4] for the maximum frequency of uniform stable rotation can be reproduced (within better than 2%), for a broad set of realistic causal EOS of dense matter, by an improved empirical formula

$$\Omega_{\text{max}} \simeq C(x_s) \left( \frac{GM_s}{R_s^3} \right)^{1/2} \text{ s}^{-1},$$

where $C(x_s)$ is a constant that depends on $x_s$.
where $C(x_s)$ is a universal (i.e. independent of the EOS) function of the compactness parameter $x_s$,

$$C(x_s) = 0.468 + 0.378x_s.$$  \hspace{1cm} (7)

Combining Eq. (6) and Eq. (7) we get

$$\text{improved empirical formula} \implies P_{\text{min}} \simeq \frac{8.754 \times 10^{-2} M_s}{C(x_s)x_s^{3/2} M_{\odot}^2} \text{ms.} \hspace{1cm} (8)$$

Both “empirical formulae” for $P_{\text{min}}$, Eq.(5) and Eq.(8), imply, that at given maximum mass of a spherical configuration, the minimum rotation period is obtained for the maximum value of $x_s$. At fixed $x_s$, the value of $P_{\text{min}}$ is proportional to $M_s$. Neutron stars for which masses have been measured, rotate so slowly that their structure can be very well approximated by that of a spherical star. Observations impose thus a condition $M_s \geq M_{\text{max,obs}}$.

3. Upper bounds on surface redshift

To minimize $P_{\text{min}}$ for given $M_{\text{max,obs}}$, we have to look for an EOS which yields maximum $x_s$ at $M_s = M_{\text{max}}$. It is well known, that if one relaxes the condition of causality, the absolute upper bound on $x_s$ for stable neutron star models is reached for an incompressible fluid (i.e., $\rho = \text{const.}$) EOS; the value of $x_s$ is then independent of $M_s$ and equal 8/9; it corresponds to an upper bound on $z_s$, equal 2 (see, e.g., [11]).

It is therefore rather natural to expect that in order to maximize $z_s$ under the condition of causality, one has to maximize the sound velocity throughout the star. Together with condition of density continuity in the stellar interior this points out at the CL EOS, $p = (\rho - \rho_0)c^2$, where $p_s$ and $\rho_b$ are, respectively, pressure and mass density at the bottom of the crust [13]. The case of $p_b = 0$ corresponds to stellar models with no normal crust. Numerical calculations show, that adding a crust onto a CL EOS core implies an increase of $R_s$, which is linear in $p_b/\rho_b c^2$; for a solid crust we have typically $p_b/\rho_b c^2 \sim 10^{-2}$. The change (increase) in $M_s$ is negligibly small; it turns out to be quadratic in $p_b/\rho_b c^2$. This implies, that the decrease of $x_{s,\text{max}}$, and of the maximum surface redshift $z_{s,\text{max}}$, due to the presence of a crust, is proportional to $p_b/\rho_b c^2$. This is consistent with Table 1 of Lindblom [13]. However, the extrapolation of his results to $p_b = 0$ yields $z_{s,\text{max}} = 0.891$, which is nearly 5% higher than our value of $z_{s,\text{max}}$. This might reflect a lack of precision of the variational method used by Lindblom [13], which led to an overestimate of the value of $z_{s,\text{max}}$. It should be stressed that while a precise determination of $M_{\text{max}} \equiv M_s$ for static neutron star models is rather easy, determination of the precise value of the radius of the maximum mass
configuration, $R_s$, (with the same relative precision as $M_s$) and consequently of the value of $x_s$ (with, say, four significant digits), is much more difficult and requires a rather high precision of numerical integration of the TOV equations.

4. Lower bound on $P_{\text{min}}$

We restrict ourselves to the case of the absolute lower bound on $P_{\text{min}}$, obtained for neutron star models with no crust. Inserting the value of $x_{s,\text{max}}$ into Eq. (5) we get

$$\text{simple empirical formula } \Rightarrow P_{\text{min}}^{\text{CL}} \simeq 0.22 \frac{M_{\text{max}}^{\text{obs}}}{M_\odot} \text{ ms}.$$ (9)

The current lower bound on $P$, resulting from the above equation, is 0.32 ms, which should be compared with the lower bound of 0.282 ms, obtained in extensive exact numerical calculations by Koranda et al. [8].

Using the improved empirical formula, Eq. (8), we get

$$\text{improved empirical formula } \Rightarrow P_{\text{min}}^{\text{CL}} \simeq 0.1997 \frac{M_{\text{max}}^{\text{obs}}}{M_\odot} \text{ ms},$$ (10)

which implies a current lower bound of 0.288 ms, only 2% higher than the exact result of Koranda et al. [8].

The formula (10) deserves an additional comment. In numerical calculations, of a family of stable uniformly rotating stellar models, for a given EOS of dense matter, one has to distinguish between the rotating configuration of maximum mass, which corresponds to the rotation frequency $\Omega_{M_{\text{max}}}(\text{EOS})$, and the maximally rotating one, which rotates at $\Omega_{\text{max}}(\text{EOS})$ [5], [14]. Notice, that determination of a maximum mass rotating configuration (and therefore of $\Omega_{M_{\text{max}}}$) is a much simpler task than the calculation of exact value of $\Omega_{\text{max}}$, which is time consuming and very demanding as far as the precision of numerical calculations is concerned. Usually, both configurations are very close to each other, and $\Omega_{\text{max}}$ is typically only 1-2% higher than $\Omega_{M_{\text{max}}}$; such a small difference is within the typical precision of the empirical formula for $\Omega_{\text{max}}$. Actually, the formula for $C(x_s)$, Eq. (6), was fitted to the values of $\Omega_{M_{\text{max}}}(\text{EOS})$ calculated in [3, 4]. Therefore, Eq.(10) should in principle be used to evaluate the causal lower bound to $P_{\text{min},M_{\text{max}}}$; it actually reproduces, within 0.2%, the exact formula for this quantity, obtained by Koranda et al. [8] [see their Eq. (8)].

General experience shows that - in contrast to interpolation - extrapolation is a risky procedure. The fact that in our case extrapolation of an empirical formula yields - within 2% - the value of $P_{\text{min}}$, of Koranda et al. [8] (and reproduces their value of $P_{\text{min},M_{\text{max}}}$), proves the usefulness of compact “empirical expressions” which might summarize, in a quantitative way, a relevant content of extensive numerical calculations of uniformly rotating neutron star models.

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