Motivated by recent work involving the graviton-graviton tree scattering amplitude, and its twin descriptions as the square of the Bel–Robinson tensor, $B_{\mu\nu\alpha\beta}$, and as the “current-current interaction” square of gravitational energy pseudo-tensors $t_{\alpha\beta}$, we find an exact tensor-square root equality $B_{\mu\nu\alpha\beta} = \partial_{\mu\nu} t_{\alpha\beta}$, for a combination of Einstein and Landau–Lifschitz $t_{\alpha\beta}$, in Riemann normal coordinates. In the process, we relate, on-shell, the usual superpotential basis for classifying pseudo-tensors with one spanned by polynomials in the curvature.

1. Introduction

This paper revisits and relates the three ancient subjects of our title in a novel way, suggested to us in a very different context, the construction of invariants of D=11 supergravity [1]. The graviton-graviton tree level amplitude $M$, particularly in the present exclusively D=4 context, is remarkably simple, namely the square of the famous Bel–Robinson tensor [2],

$$L \sim B_{\mu\nu\alpha\beta}^2.$$  

To be sure, there is some work involved here: First, since $M$ is generated by intermediate graviton exchange, it is nonlocal, but when multiplied by the Mandelstam variables $stu$, it reduces to a local invariant $L$. The latter must, by power counting and (abelian) gauge invariance to this lowest $(k h_{\mu\nu})^4 \equiv (g_{\mu\nu} - \eta_{\mu\nu})^4$ order, be a scalar proportional to $R^{4}_{\mu\nu\alpha\beta}$. [The external gravitons are all on-shell $(R_{\mu\nu} = 0)$, so $R_{\mu\nu\alpha\beta}$ really means the Weyl tensor throughout.] Here the power of using a suitable basis to classify scalar or tensorial powers of curvature [3] first emerges. For example, there are seven algebraically independent $R^4$ scalar monomials in generic dimension, but these can be shown to reduce to just two in D=4, owing to identities such as $R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} \equiv \frac{1}{4} \delta_{\mu}^{\alpha\beta} R^{2}_{\alpha\beta\gamma}$.\[valid only there. This basis is spanned by $(E^2, P^2)$, where $(E, P)$ are the D=4 Euler and Pontryagin invariants. Since graviton scattering must be maximally helicity conserving (as follows directly from the supersymmetrizability of Einstein theory), this singles out $L \sim (E - P)(E + P)$; but another deep D=4 fact is that this is in turn just $B^2$.\[\]
The final objects in our title, the gravitational energy pseudo-tensors, enter in a very physical way as the currents that generate the scattering through the cubic vertex \( \sim \kappa \int \varpi^\alpha_\alpha \) in the expansion of the Einstein action, which can be expressed as the coupling of the field \( h_{\alpha\beta} \) to a current \( t_{\alpha\beta} \sim \kappa (\partial h \partial h)^{\alpha\beta} \). The amplitude can then be expressed in the usual current-current form \( M \sim \int \varpi^\alpha_\alpha \) of the intermediate graviton’s propagator. Although gauge invariance further requires inclusion of the quartic contact term \( \kappa^2 \int d^4 x h \partial h \partial h \partial h \), it is plausible that this characterization of \( M \) yields another version of \( L \),

\[
L \sim (\partial^2_{\mu\nu} t_{\alpha\beta})^2 \tag{1b}
\]

upon keeping track of momentum dimensions. Hence the temptation to equate the “tensor square roots” of (1a) and (1b),

\[
B_{\mu\nu\alpha\beta} = \partial^2_{\mu\nu} t_{\alpha\beta} . \tag{2}
\]

Let us next recall some properties of \( t_{\alpha\beta} \) and earlier attempts to relate \( B \) and \( t \). First, we note that gauge systems with spin > 1 do not even possess gauge invariant stress tensors, but only integrated Poincaré generators [4]. While this local defect is irrelevant in the absence of gravity, it becomes critical when the stress tensor is to be its source, not least for (self-coupled) general relativity itself, as witness the history of its countless energy pseudotensors (see e.g., [5]). The problem is of course the clash between the second derivative order required to build the (linearized) in- or (generically) co-variant curvatures and the first derivative order building blocks of the \( t_{\alpha\beta} \). Indeed, the Bel–Robinson tensor arose as an attempt at constructing a “more covariant” tensorial quantity modelled on the Maxwell stress tensor. However, the price is high: \( B \) has too many derivatives and indices, it is only covariantly conserved and it is not a physical current [6]. Clearly any putative relation like (2), even at linearized level, must be a non-invariant one. Nevertheless, there exists a textbook exercise [7] proposing the equality \( B_{\mu\nu\alpha\beta} = \partial^2_{\mu\nu} t_{\alpha\beta} + S_{\mu\nu\alpha\beta} \) in Riemann normal coordinates (RNC) for a particular (Einstein) \( t_{\alpha\beta} \), but with a rather mysterious remainder. In achieving (2), we will have to consider all possible pseudo-tensors; as we have mentioned, there is an infinity of them and being identically ordinarily (rather than covariantly) conserved on shell, they can all be expressed as superpotentials there. This is also the case for \( B \); on shell, it reduces to the identically (covariantly) conserved Sachs tensor [8]. In establishing (2), in the particular RNC frame, we will in fact translate the superpotential “basis” for the \( t_{\alpha\beta} \) and for \( B \) into one for four-index curvature quadratics. This will enable us not only to verify that it can be done there, but also how uniquely.

Despite our arguments in favor of the existence of an exact connection (2) for some \( t_{\alpha\beta} \), its validity is far from obvious; \( B \) is totally symmetric and traceless, while the \( \partial^2_{\mu\nu} t_{\alpha\beta} \) do not even display \( (\alpha\beta) \) symmetry in general, let alone the other invariances of \( B \). It is therefore amusing that this correspondence, even if highly gauge variant, can be established at all!

2. Ingredients

A. Bel–Robinson. As mentioned, \( B \) first appeared in the endless search for a covariant version of gravitational energy density; the analogy with the Maxwell stress tensor \( T_{\mu\nu} = F_{\mu\alpha} F^\alpha_{\nu} + *F_{\mu\alpha} *F^\alpha_{\nu} \) \((*F \text{ is the dual field strength})\) is striking in either of the two equivalent expressions,

\[
B_{\mu\nu\alpha\beta} = R^\rho_\mu \sigma_\alpha R_{\rho\nu\sigma\beta} + *R^\rho_\mu \sigma_\alpha *R_{\rho\nu\sigma\beta} , \tag{3}
\]

\[
B_{\mu\nu\alpha\beta} = R^\rho_\mu \sigma_\alpha R_{\rho\nu\sigma\beta} + R^\rho_\mu \sigma_\beta R_{\rho\mu\sigma\alpha} - \frac{1}{2} g_{\mu\nu} R^\rho_\sigma R_{\rho\sigma\alpha} . \tag{4}
\]
Here the dual curvature is $^\ast R^{\mu\nu}{}_{\lambda\sigma} = \frac{1}{2} \epsilon^{\mu\nu}{}_{\alpha\beta} R^{\alpha\beta}{}_{\lambda\sigma}$. The interplay between these two expressions underlies the various special properties that $B$ enjoys in four dimensions, due to on-shell identities that (like the dual itself) are only valid there. These include, besides the $(\mu \leftrightarrow \nu)$ and $(\alpha \leftrightarrow \beta)$ symmetries, two further ones: (a) $(\mu \nu) \leftrightarrow (\alpha \beta)$ symmetry. This is transparent in (3); requiring it in (4) demands that

$$g_{\mu\nu} R^{\rho\sigma\tau}_{\alpha} R_{\beta\rho\sigma\tau} = g_{\alpha\beta} R^{\rho\sigma\tau}_{\mu} R_{\nu\rho\sigma\tau} ,$$

i.e., that $R^{\rho\sigma\tau}_{\alpha} R_{\beta\rho\sigma\tau}$ be a pure trace:

$$R^{\rho\sigma\tau}_{\alpha} R_{\beta\rho\sigma\tau} - \frac{1}{4} g_{\alpha\beta} R^{\mu\rho\sigma\tau} R_{\mu\rho\sigma\tau} = 0 .$$

Thus, this well-known D=4 identity is encoded in $B$. (b) $(\mu \leftrightarrow \nu)$ symmetry: “dualizing” $B$ with $\epsilon^{\gamma\tau\mu\nu}$ in (3) obviously annihilates it by the "$^\ast^\ast = -1$" property. In terms of (4), this then implies the more general 4-index relation

$$\frac{1}{2} R^{\rho\sigma\tau}_{\mu\alpha} R_{\nu\beta\rho\sigma\tau} + R^{\rho\sigma\tau}_{\mu} R_{\nu\rho\sigma\tau} - \frac{1}{4} (g_{\mu\nu} g_{\alpha\beta} - g_{\mu\beta} g_{\nu\alpha}) R^{\lambda\rho\sigma\tau} R_{\lambda\rho\sigma\tau} = 0$$

of which (6) is the $(\alpha\beta)$ trace; indeed these identities “explain” why $B$ is necessarily 4-index, rather than 2-index like $t^{\alpha\beta}$. [For D>4 the identities (6),(7) cease to hold and a three-parameter family of conserved $B$-like generalization (each with partial properties) can be constructed; for more about $B$’s and the curvature quartics $BB$ in D>4, see [1].]

B. Riemann Normal Coordinates (RNC). We recall that at any one point in a Riemann space, coordinate invariance can be exploited in order to simultaneously (i) rotate the metric to Minkowski form (local flatness), (ii) annihilate all affinities (free fall) and (iii) remove 80 of the 100 $\partial^2_{\mu\nu} g_{\alpha\beta}$ by use of the 80 independent $\partial^3_{\alpha\beta\gamma} \xi_\delta$ components of the gauge functions $\xi_\delta$ in the curvature, leaving only 10 $\partial^2 g$ combinations to represent the 10 Weyl curvatures $R^{\mu\nu\alpha\beta}$ of interest on shell. In addition to (i) and (ii),

$$g_{\mu\nu} = \eta_{\mu\nu} , \quad g^{\mu\nu} = \eta^{\mu\nu} , \quad g_{\mu\nu,\alpha} = 0 ,$$

the choices (iii) defining RNC are summarized as follows:

$$-3 g_{\mu\nu,\alpha\beta} = R_{\mu\nu,\beta\alpha} + R_{\mu\beta,\nu\alpha} , \quad -3 \Gamma^{\alpha}_{\mu\nu,\beta} = R^{\alpha}_{\mu\nu\beta} + R^{\alpha}_{\nu\mu\beta} .$$

Note that raising and lowering indices “passes through derivatives”, here denoted by commas. On-shell, this means in addition, that all traces of (9) vanish:

$$g^{\mu\nu} g_{\mu\nu,\alpha\beta} = 0 , \quad g^{\mu\nu} \Gamma^{\alpha}_{\mu\nu,\beta} = 0 , \quad \Gamma^{\alpha}_{\alpha\beta,\mu} = 0 = \Gamma^{\alpha}_{\mu\nu,\alpha} .$$

C. Pseudo-tensors. As we have stated, these can be parametrized either by all possible independent superpotentials (of second derivative order), or more usefully in RNC by the appropriate curvature basis. Before proceeding to the general case, however, we write directly the two specific $t^{\alpha\beta}$ that will enter in (2), namely the non-symmetric Einstein $E^{\alpha}_{\beta}$ and symmetric Landau–Lifschitz $L^{\alpha\beta}$. This will also give an idea of how RNC simplifies matters. We only keep those terms in each that will not vanish on-shell because of (10) after taking the two further, $\partial^2_{\mu\nu}$, derivatives:

$$E^{\alpha}_{\beta} = -2 \Gamma^{\alpha}_{\lambda\nu} \Gamma^{\lambda}_{\beta\sigma} + \delta^{\alpha}_{\beta} \Gamma^{\lambda}_{\sigma\tau} \Gamma^{\tau}_{\lambda\sigma} ,$$

$$L^{\alpha\beta} = -\Gamma^{\lambda}_{\alpha\sigma} \Gamma^{\beta}_{\lambda\sigma} + \Gamma^{\alpha}_{\lambda\sigma} \Gamma^{\beta}_{\sigma\lambda} - (\Gamma^{\alpha}_{\lambda\beta} \Gamma^{\beta}_{\sigma\lambda} + \Gamma^{\alpha}_{\beta\lambda} \Gamma^{\beta}_{\sigma\lambda}) + g^{\alpha\beta} \Gamma^{\lambda}_{\sigma\tau} \Gamma^{\tau}_{\lambda\sigma} .$$
In writing these expressions, we have used the economy of notation permitted by RNC: first, since both $E$ and $L$ are special in being bilinear in $\Gamma$’s (i.e., not involving terms like $\partial^2 g$), then both of the $\partial^2_{\mu \nu}$ must act on these $\Gamma$’s and not on any undifferentiated metrics; we have therefore set all the latter to their Minkowski values, so that summation and moving indices in (11) is to be understood in light of (8). Note also that, unlike $L^{\alpha \beta}$, $E^{\alpha \beta}$ (here expressible as $E^{\alpha \beta}$) is not yet $(\alpha \beta)$-symmetric; this will, however, come about after differentiation.

D. Quadratic Curvature Basis. All 4-index curvature bilinears are algebraically equivalent to the double contraction

$$Q_{\mu \nu \alpha \beta} \equiv R_{\mu \alpha \beta} R_{\nu \alpha \beta} = Q_{\nu \beta \mu \alpha} = Q_{\alpha \beta \nu \mu} \equiv Q_{\alpha \beta \mu \nu} \equiv Q_{\mu \nu \alpha \beta} \equiv Q_{\nu \beta \mu \alpha} \equiv Q_{\alpha \beta \nu \mu},$$

(12)

where we have indicated the symmetries of $Q$; there are no further ones, either within a pair or under index exchange between pairs. The three basis members we need are representable by

$$X_{\mu \nu \alpha \beta} \equiv Q_{\alpha \mu \nu \beta} \quad Y_{\mu \nu \alpha \beta} \equiv Q_{\alpha \beta \mu \nu} \quad Z_{\mu \nu \alpha \beta} \equiv Q_{\alpha \nu \beta \mu}$$

(13a)

where $(+\nu \mu)$ means that each quantity is to include the $(\nu \mu)$-interchanged form in its definition. These three linearly independent quantities are a subset of the complete (6-member) non-symmetrized basis and agree with the enumerations in [3]. It is convenient, in addition, to define a single trace object, which we take to be

$$T_{\mu \nu \alpha \beta} \equiv -\frac{1}{6} g_{\mu \nu} Q_{\sigma \alpha \sigma \beta} = -\frac{1}{24} g_{\mu \nu} g_{\alpha \beta} R_{\lambda \sigma \gamma \delta}^2.$$

(13b)

All the $(X, Y, Z, T)$ are uniformly labelled by the indices $(\mu \nu \alpha \beta)$ in that order; we will avoid index proliferation below by (usually) leaving them off altogether. In terms of this basis, $B_{\mu \nu \alpha \beta}$ is the combination

$$B = Z + 3T.$$

(14)

It is also convenient to define the tensor $S_{\mu \nu \alpha \beta} \equiv R_{\mu \alpha}^{\rho \sigma} R_{\nu \beta \rho \sigma} + R_{\nu \alpha}^{\rho \sigma} R_{\mu \beta \rho \sigma} + \frac{1}{4} g_{\mu \nu} g_{\alpha \beta} R_{abcd}^2$ which is expressible as

$$S = 2B - 2X - 12T.$$

(15)

It also follows by (7) that

$$S = 2Y - 2B - 12T + 6(T_{\mu \alpha \nu \beta} + T_{\mu \beta \alpha \nu})$$

(16)

3. The Relation

In RNC, any $\partial^2_{\mu \nu} t_{\alpha \beta}$ will be a sum of products $\sim (\partial^2 g \partial^2 g)_{\alpha \beta \mu \nu}$, with manifest $(\mu \nu)$, but not necessarily $(\alpha \beta)$, symmetry. There is thus a small class of possible terms; all are expressible in terms of the basis $(X, Y, Z, T)$ of (13). We start with the Einstein contribution (11a); using the prescription (9), we obtain

$$\frac{1}{2} \partial^2_{\mu \nu} E_{\alpha \beta} = \frac{1}{2} X + \frac{1}{2} Z + T.$$

(17)

In terms of $B$ and $S$ this can be rewritten as

$$\frac{1}{2} \partial^2_{\mu \nu} E_{\alpha \beta} = \frac{2}{7} (B - \frac{4}{7} S),$$

(18)

the original relation proposed in [7]. The $\partial^2_{\mu \nu} L_{\alpha \beta}$ of (11b) contains more terms but is equally straightforward. We find for it

$$\partial^2_{\mu \nu} L_{\alpha \beta} = -\frac{1}{8} X + \frac{8}{7} Z + 2T.$$

(19)
or equivalently
\[ \partial^2_{\mu \nu} L_{\alpha \beta} = + \frac{1}{9} (7B + \frac{1}{2} S). \] (20)

Adding (17) and (19) yields (14); hence the promised formula,
\[ B_{\alpha \beta \mu \nu} = \partial^2_{\mu \nu} \left( L_{\alpha \beta} + \frac{1}{2} E_{\alpha \beta} \right). \] (21)

In contrast to (18), there is no remainder here. Incidentally, although (21) has been derived for D=4, where there is a unique B, it can be extended to any D using the B of (4).

There are two immediate questions about (21): How unique is our result, and if it is, why just this combination of \( t_{\alpha \beta} \)? We have no wisdom as to the latter: the invariant amplitude provides no hints about why precisely \( (L_{\alpha \beta} + \frac{1}{2} E_{\alpha \beta}) \) requires no remainder in this frame. To quantify the uniqueness aspect, we will write down all superpotentials in RNC, since these cover all \( t_{\alpha \beta} \). There are both symmetric and non-symmetric \( \Delta_{\alpha \beta} \) (relevant because we used \( E_{\alpha \beta} \neq E_{\beta \alpha} \)). Symmetric ones are of the form \( S^{\alpha \beta} \equiv \partial^2_{\lambda \sigma} H^{\alpha \lambda \beta \sigma} \), where \( H \) has the algebraic symmetries of the Riemann tensor; this ensures both identical conservation and symmetry of \( S^{\alpha \beta} \). Nonsymmetric ones will be conserved on only one index, \( A^{\alpha \beta} \equiv \partial^1_{\lambda} H^{\alpha \lambda \beta} \) where \( H^{\alpha \lambda \beta} \) is only antisymmetric in \((\alpha \lambda)\). Since we are concerned in RNC with terms of the form \( \partial^2 g \partial^2 g \) and in particular with \( \partial^2_{\mu \nu} S^{\alpha \beta} \), \( H^{\alpha \lambda \beta} \) depends only on the metric, while \( H^{\alpha \lambda \beta} \) will be \( \sim g \partial g \). As shown in the Appendix, there just are enough independent superpotentials to span the complete \((X, Y, Z, T)\) basis and thus to express \( B \) uniquely. Hence, there is just one effective \( t_{\alpha \beta} \) – in RNC – that fulfills our relation, though \( t_{\alpha \beta} \)’s that are intrinsically different in an arbitrary frame may degenerate to a single one in RNC.

4. Summary

Our modest result is that on shell, at the origin of the RNC frame, there is an exact local equality between the Bel–Robinson tensor and the double gradient of a particular energy pseudotensor. We have conjectured this relation to be the “tensor square root” of a more physical one in tree level graviton-graviton scattering, whose amplitude is simultaneously proportional to the square of \( B_{\mu \nu \alpha \beta} \) and (essentially, if not quite gauge invariantly) to that of a pseudotensor. Despite all this fine print, one cannot help but wonder if there is more to be learned from (21).

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References


Appendix: Superpotentials

We enumerate here the independent superpotentials in RNC. Our purpose is to show that they in fact constitute a basis equivalent to the \((X,Y,Z,T)\), in terms of which we know that our class of tensors is expandable. Consider first the symmetric ones; we are actually concerned with

\[
\Delta^s_{\mu\nu\alpha\beta} \equiv \partial^2_{\mu\nu} S_{\alpha\beta} = \partial^4_{\mu\nu\lambda\sigma} H^{\alpha\lambda\beta\sigma}(g) .
\] (A.1)

As explained in text, \(H\) contains no derivatives. It is then of the form

\[
H^{\alpha\lambda\beta\sigma} \equiv (g^{\alpha\beta} g_{\lambda\sigma} - g_{\alpha\sigma} g^{\beta\lambda}) K(g) \] (A.2)

where \(K\) is a “scalar” like 1 or \(\sqrt{-g}\). The four derivatives will then all fall on the \((gg-\text{gg})\) term, or all on \(K(g)\) to form the surviving \(\partial^2 \partial^2 g\) combinations, but not two on each: a \(\partial^2 K(g)\) is excluded because it would necessarily be proportional to \(g^{\lambda\sigma} \partial^2 g_{\lambda\sigma}\) which vanishes on shell, irrespective of the \(\partial^2\) indices. When all four derivatives fall on the \((gg-\text{gg})\), the undifferentiated \(K\) is a constant, \(K(g) = \eta_{\mu\nu}\), and it is easy to see that this gives rise to the contribution

\[
9 \Delta^s_1 = (-5X + 2Y + 4Z)K(\eta) .
\] (A.3)

When all derivatives act on \(K\) alone, we have the usual transverse projector in \((\alpha\beta)\),

\[
\Delta^s_2 = \left(\eta_{\alpha\beta} \Box - \partial^2_{\alpha\beta}\right) \partial^2_{\mu\nu} \left(g^{\sigma\lambda} g_{\lambda\sigma}\right)
\] (A.4)

where we have kept the relevant, quadratic in metric, part of \(K\); two derivatives are to be distributed on each metric, neglecting terms such as \(\Box g^{\lambda\sigma} \partial^2 g_{\lambda\sigma}\) that vanish on shell. The significant point for us is the presence of a (unique) trace term here, \(i.e.,\) that \(\Delta^s_2\) includes a part

\[
\eta_{\alpha\beta} g^{\sigma\tau,\lambda\mu} g_{\sigma\tau,\lambda\nu} \sim T ,
\] (A.5)

using (9), the identity (6) and the definition (13b). As we shall next see, there are enough independent antisymmetric contributions to span the remaining (non-trace) basis members \((X,Y,Z)\) between them. The three possible forms are given by

\[
A^1_{\alpha\lambda\beta} = g_{\lambda\sigma} g_{\alpha\beta,\sigma} - g_{\alpha\sigma} g_{\lambda\beta,\sigma} , \quad A^2_{\alpha\lambda\beta} = (g_{\lambda\sigma, \alpha} - g_{\alpha\sigma, \lambda}) g_{\beta\sigma} , \quad A^3_{\alpha\lambda\beta} = g_{\alpha\sigma} g_{\beta\sigma, \lambda} - g_{\lambda\sigma} g_{\beta\sigma, \alpha} .
\] (A.6)

In principle there can again be coefficients \(K(g)\) here, but in fact they will not contribute: We are interested in expressions \(\sim \partial^3_{\mu\nu\alpha} (g \partial g K)\). Only the \(\partial^3(g \partial g) K(\eta)\) part fails to vanish: But as in the symmetric case, \(\partial^2 K(g) \sim \partial^2 (g^{\lambda\sigma} g_{\lambda\sigma})\) always vanishes on shell. Hence we have the three possible terms \(\Delta^i_A \equiv \partial^2_{\mu\nu} A^i_{\alpha\beta}\) with the \(A^i\) of (A.6). The result is as follows

\[
9 \Delta^1_A = -5X + 2Y + 4Z = 9 \Delta^s_1 , \quad 9 \Delta^2_A = 3X - 6Z , \quad 9 \Delta^3_A = -4X + Y + 5Z .
\] (A.7)

Being linearly independent, these \(\Delta^i\) are equivalent to \((X,Y,Z)\); they contain no trace, which is instead carried by \(\Delta^s_2\), as explained above. Thus, in RNC the set of four possible superpotentials, like the \((XYZT)\), span all (double gradients of) the pseudotensors \(t_{\alpha\beta}\) as well as \(B\).