In this chapter we will study the relation between gravity theories (string theories) on \(AdS_3\) and two dimensional conformal field theories. First we are going to describe some generalities which are valid for any \(AdS_3\) quantum gravity theory, and then we will discuss in more detail IIB string theory compactified on \(AdS_3 \times S^3 \times M^4\) with \(M^4 = K3\) or \(T^4\).

\(AdS_3\) quantum gravity is conjectured to be dual to a two dimensional conformal field theory which can be thought of as living on the boundary of \(AdS_3\). The boundary of \(AdS_3\) (in global coordinates) is a cylinder, so the conformal field theory is defined on this cylinder. We choose the cylinder to have radius one, which is the usual convention for conformal field theories. Of course, all circles are equivalent since this is a conformal field theory, but we have to rescale energies accordingly. If the spacetime theory or the conformal field theory contain fermions then they have anti-periodic boundary conditions on the circle. The reason is that the circle is contractible in \(AdS_3\), and close to the “center” of \(AdS_4\) a translation by \(2\pi\) on the circle looks like a rotation by \(2\pi\), and fermions get a minus sign. So, the dual conformal field theory is in the NS-NS sector. Note that we will not sum over sectors as we do in string theory, since in this case the conformal field theory describes string theory on the given spacetime and all its finite energy excitations, and we do not have to second-quantize it.

The Virasoro Algebra

The isometry group of \(AdS_3\) is \(SL(2,\mathbb{C}) \times SL(2,\mathbb{C})\), or \(SO(2,2)\). The conformal group in two dimensions is infinite. This seems to be, at first sight, a contradiction, since in our previous discussion we identified the conformal group with the isometry group of \(AdS\). However, out of the infinite set of generators only an \(SL(2,\mathbb{C}) \times SL(2,\mathbb{C})\) subgroup leaves the vacuum invariant. The vacuum corresponds to empty \(AdS_3\), and this subgroup corresponds to the group of isometries of \(AdS_3\). The other generators map the vacuum into some excited states. So, we expect to find that the other generators of the conformal group map empty \(AdS_3\) into \(AdS_3\) with (for instance) a graviton inside. These other generators are associated to reparametrizations that leave the asymptotic form of \(AdS_3\) invariant at infinity. This problem was analyzed in detail in Brown:1986nw and we will just sketch the argument here. The metric on \(AdS_3\) can be written as metastds \(ds^2 = R^2(-\cosh^2 \rho d\tau^2 + \sinh^2 \rho d\phi^2 + d\rho^2)\). When \(\rho\) is large (close to the boundary) this is approximately metastds \(ds^2 \sim R^2 [-e^{2\rho}d\tau^+ d\tau^- + d\rho^2]\), where \(\tau^\pm \equiv \tau \pm \phi\). An infinitesimal reparametrization generated by a general vector field \(\xi^\alpha(\tau,\phi,\rho)\) changes the metric by \(g_{\alpha\beta} \rightarrow g_{\alpha\beta} + \nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha\). If we want to preserve the asymptotic form of the metric, we require that Brown:1986nw

\[
\xi^+ = f(\tau^+) + \frac{e^{-2\rho}}{2} g''(\tau^-) + O(e^{-4\rho}) ,
\]

where

\[
\xi^- = g(\tau^-) + \frac{e^{-2\rho}}{2} g''(\tau^+) + O(e^{-4\rho}) .
\]

and \(\xi^0 = \frac{1}{2} f'/(\tau^+) + g'/2 + O(e^{-2\rho})\).

functions \(f = \sum L_n e^{\rho^+}, g = \sum \bar{L}_n e^{\rho^-}\), we recognize the Virasoro generators \(L_n, \bar{L}_n\). For the cases \(n = 0, \pm 1\) one can find some isometries that reduce to \(\tau \rightarrow \text{at infinity}, \text{are globally defined, and leave the metric invariant. There are two such cases, but it does not leave the metric invariant.} \)

It is possible to calculate the classical Poisson brackets among these generators, and one finds that this classical algebra has a central charge which is equal to Brown:1986nw brown c = \(3\ R \sum 2G_N^{(3)}\), where \(G_N^{(3)}\) is the three dimensional Newton constant. So, this should also be the central charge of the dual conformal field theory, since \(\Rightarrow \text{implies that these Virasorogenerators are acting on the boundary as the Virasoro generators of a 1 dimensional conformal field theory.} \)

A simple calculation of the central charge term was given in Balasubramanian:1999re. Under a diffeomorphism of the form \(\Rightarrow, \text{the metric near the boundary changes to} \Rightarrow, \text{the stress tensor changes as} \Rightarrow, \text{and the Virasoro generators act} \Rightarrow,\text{where} \Rightarrow, \text{so comparing with we get} \Rightarrow.\)

It is also possible to show that if we have boundary conditions on the metric at infinity that in the dual conformal field theory correspond to considering the theory on a curved geometry, then we get the right conformal anomaly Henriksen:1998gx (generalizing the discussion in section anomalies).

The BTZ Black Hole

Three dimensional gravity has no propagating degrees of freedom. But, if we have a negative cosmological constant, we can have black hole solutions. They are given by Banados:1992wn, Banados:1993gq lorio
\[ ds^2 = -\frac{(r^2-r_-^2)(r^2-r_+^2)}{r^2}dt^2 + \frac{r^2r_-^2}{(r^2-r_-^2)(r^2-r_+^2)}dr^2 + r^2(d\phi + \frac{r_+r_-}{r^2}dt)^2, \] with \( \phi \equiv \phi + 2\pi \). We can combine the temperature \( T \) and the angular momentum potential \( \Omega \) into the temperature \( T_\pm \equiv \frac{T_+ \pm T_-}{2} \) and the relation to the parameters in is

\[ r_\pm = \pi R(T_\pm \pm T_-). \]

The mass and angular momentum are massng \( 8 G_N^3 M = R + \frac{(r_+^2+r_-^2)}{R} \), \( J = \frac{r_+r_-}{4G_N^3R} \), where we are measuring the mass relativeto the AdS \( 3 \) space, which we define to have \( M = 0 \) (the scale of the mass is set by the radius of the circle in the dual CFT). This is not the usual convention, but it is much more natural in this context since we are measuring energies with respect to the NS-NS vacuum. Note that the mass of a black hole is always at least massmin \( M_{\text{min}} \)

\[ r_+ = r_- = 0. \]

All these black holes are locally the same as AdS \( 3 \) but they differ by some global identifications Banados:1992wn, Banados:1993gq, i.e., they are quotients of AdS \( 3 \). In theories that have supersymmetry it can be checked that the zero mass black hole preserves some supersymmetries provided that we make the fermions periodic as we go around the circle Coussaert:1994jp, which is something we have the freedom to do once the circle is not contractible in the gravity geometry. These supersymmetries commute with the Hamiltonian conjugate to \( t \). Furthermore, we will see below that if we consider the near horizon geometry of branes wrapped on a circle with periodic boundary conditions for the spinors, we naturally obtain the BTZ black hole with mass \( M_{\text{min}} \). This leads us to identify the \( M = M_{\text{min}} \) BTZ black hole with the RR vacuum of the conformal field theory Coussaert:1994jp. The energy \( M_{\text{min}} \) is precisely the energy difference between the NS-NS vacuum and the RR vacuum. Of course, we could still have the \( M = M_{\text{min}} \) BTZ black hole with anti-periodic boundary conditions as an excited state in the NS-NS sector.

Next, let us calculate the black hole entropy. The Bekenstein-Hawking entropy formula gives

\[ S = \frac{\text{Area}}{4G_N^3} = \frac{\pi^2}{4} (T_+ + T_-), \]

where we used. We can also calculate this in the conformal field theory. All we need is the central charge of the conformal field theory.

\[ T \gg 1. \]

When is the result valid? In principle we would say that it is valid as long as the area of the horizon is much bigger than the Planck length, \( r_+ \gg G_N^3 \). This gives \( T \gg 1/c \), which is a much weaker bound on the temperature for large \( c \). So, we see that the corresponding conformal field theory has to be quite special, since the number of states should grow as determined by the asymptotics for energies that are much smaller than one would expect for a generic conformal field theory.

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